COMPUTATION OF MAASS WAVEFORMS WITH NON-TRIVIAL MULTIPLIER SYSTEMS

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ABSTRACT. The aim of this paper is to describe efficient algorithms for computing Maass waveforms on subgroups of the modular group $PSL(2,\mathbb{Z})$ with general multiplier systems and real weight. A selection of numerical results obtained with these algorithms is also presented. Certain operators acting on the spaces of interest are also discussed. The specific phenomena that were investigated include the Shimura correspondence for Maass waveforms and the behavior of the weight-k Laplace spectra for the modular surface as the weight approaches 0.

1. INTRODUCTION AND NOTATION

The purpose of the present paper is to present computational methods and experimental results for general real weight and general multiplier systems.

The classical theory of holomorphic automorphic forms was developed in the setting of (even) integer weight. This was motivated both from a geometrical point of view and by number theoretical applications (e.g. the study of modular forms related to the modular invariant j and the discriminant Δ). The problem of finding the number of representations of an integer as a sum of a fixed number of squares (cf. e.g. [45] and [19]) was successfully treated by using the theory of half-integral weight forms (e.g. θ-series). This motivated Petersson to develop a theory of automorphic forms and multiplier systems of arbitrary real weight [52] and later also complex weight [54, I-IV].

Many applications of modular forms use Hecke operators as a principal tool and Wohlfahrt developed a theory of Hecke-like operators in arbitrary real weight [76] (cf. also [70]) .

Now recall the definition of a Maass waveform as a real-analytic square-integrable eigenfunction of the Laplacian on a Riemann surface of finite volume with constant negative curvature -1. These were introduced by Maass for zero weight in [40]. In [41] the theory of waveforms for general weights was developed by using the so called lowering and raising operators, which send a waveform of a given weight to one with a smaller or larger weight.

In a slightly different setting Selberg [58, pp. 82–83] observed that considering the invariant differential operators on the space $\mathcal{H}\times S^1$ with representation χ and separating variables lead to real analytic eigenfunctions of the form $f(z,\phi)=y^{\frac{k}{2}}F(z)e^{-ik\phi}$ where $F(z)$ is a holomorphic modular form of integer weight k and character χ.

For an overview of the spectral theory of real analytic modular forms with arbitrary real weights and multiplier systems see e.g. Maass [42] and Roelcke [56]. More recent work can be found in e.g. [49, 50, 51], [22, 23] and [8, 9, 10].

It is also worth mentioning that the recent interest in Maass waveforms as representative objects for studying quantum chaos also applies to real weights. If a weight zero waveform corresponds to a quantum mechanical particle moving freely on a Riemann surface then a weight k waveform represents a similar particle moving in a constant magnetic field with field strength proportional to k.

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1.1. **Algorithms**

Previously published algorithms for computing Maass waveforms on cofinite Fuchsian groups have been restricted to groups with one cusp, e.g. the full modular group or Hecke triangle groups (see e.g. [63, 75, 25, 26, 27, 46]). By adjoining certain elements it is also possible to bring certain Hecke congruence subgroups to the one-cusp case, cf. e.g. [16, 17, 4].

In addition, with the exception of the (somewhat crude) computations in [46], only trivial multiplier system and zero (or even integer for the holomorphic case in [26]) weight has been considered.

The most stable of the algorithms cited above is the one based on "implicit automorphy" by Hejhal (as detailed in e.g. [27]) which admits generalizations first of all to remarkably large spectral parameter (cf. [69]) and a further advantage is that it does not depend on any underlying arithmetical properties (i.e. Hecke operators).

In [65] this algorithm was generalized to groups with several cusps, e.g. Hecke congruence subgroups $\Gamma_0(N)$ with non-trivial Dirichlet characters and in and [64, ch. 3] we also considered general subgroups of the modular group. Recently, in [6], this algorithm in combination with other theoretical methods was used to show the existence of certain Maass waveforms close to the tentative waveforms produced by the algorithm.

The aim of the present paper (which is based on [64, ch. 2]) is to demonstrate how to extend the algorithm to general multiplier systems and arbitrary real weights.

The first section contains a brief review of the basic theory of multiplier systems and then we will introduce the notion of Maass waveforms in this context. We will also provide some details on the different operators that act on the space of Maass waveforms with non-trivial multiplier system. While being of interest in themselves, these operators can also be used in combination with other tests of reliability and accuracy of the algorithm.

In section 7 we will give the specifics of how the algorithm is modified and in the last section we present a selection of the results which has been obtained with the described method.

1.2. **Summary of notation**

We will use the notation $e(x) = e^{2\pi i x}$ and for a complex number z we always use the principal branch of the argument, $-\pi < \mathbf{Arg} z \leq \pi$.

Let $\mathcal{H} = \{z = x+iy \,|\, y > 0\}$ be the upper half-plane equipped with the hyperbolic line- and area-elements $ds^2 = \frac{|dz|^2}{y^2}$ and $d\mu = \frac{dxdy}{y^2}$ respectively. The boundary of $\mathcal{H}$ is $\partial\mathcal{H} = \mathbb{R} \cup \{\infty\}$. The isometry group of $\mathcal{H}$ is identified with $PGL(2,\mathbb{R}) = GL(2,\mathbb{R})/\{\pm \mathrm{Id}\}$, where $GL(2,\mathbb{R})$ is the group of invertible two-by-two matrices with real elements and $\mathrm{Id} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{R})$ and $z \in \mathcal{H}$ we define an action by

$$\gamma z = \begin{cases} \frac{az+b}{cz+d}, & \text{if} \quad ad-bc > 0, \\ \\ \overline{\frac{az+b}{cz+d}}, & \text{if} \quad ad-bc < 0. \end{cases}$$

The subgroup of orientation-preserving isometries of $\mathcal{H}$ is given by $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm \mathrm{Id}\}$ where $SL(2,\mathbb{R})$ is the subgroup of $GL(2,\mathbb{R})$ consisting of matrices with determinant 1. For any subgroup $\Gamma \subseteq \mathrm{PGL}(2,\mathbb{R})$, we use $\overline{\Gamma}$ to denote the inverse image of Γ in $GL(2,\mathbb{R})$. Note that this forces $-\mathrm{Id} \in \overline{\Gamma}$. We are mainly interested in *Fuchsian groups,* i.e. discrete subgroups of $PSL(2,\mathbb{R})$. Of particular interest is the subgroup consisting of matrices with integer entries, the *modular group,* $PSL(2,\mathbb{Z})$. We are also interested in the so-called Hecke congruence subgroups, $\Gamma_0(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2,\mathbb{Z}) \,|\, c \equiv 0 \bmod N\right\}$, defined for any positive integer N (note that $\Gamma_0(1) = PSL(2,\mathbb{Z})$).

We say that an element γ of $PSL(2,\mathbb{R})$ is *elliptic*, *parabolic* or *hyperbolic* if the absolute value of the trace of the associated matrix is smaller than, equal to or greater than 2 respectively, or equivalently, if γ has one fixed point in $\mathcal{H}$, one (double) fixed point in $\partial\mathcal{H}$ or two (different) fixed points in $\partial\mathcal{H}$. Fixed points of parabolic elements are called *cusps*.

If $\Gamma \subset PSL(2,\mathbb{R})$ is a finitely generated Fuchsian group we identify the set of Γ-orbits with a connected subset of $\mathcal{H}$, $\mathcal{F} = \Gamma\backslash\mathcal{H}$, a *fundamental domain* of Γ. If Γ has a set $p_1,\ldots,p_\kappa$ of inequivalent cusps then the set $\mathcal{F}$ will meet $\partial\mathcal{H}$ at κ inequivalent points which we will also denote by $p_1,\ldots,p_\kappa$ and we usually abuse the notation and call these points the cusps of Γ (or of $\mathcal{F}$). By conjugation we may always assume $p_1 = i\infty$. Corresponding to a cusp p_j of Γ we use S_j to denote a parabolic generator of Γ_{p_j} - the subgroup of Γ which fixes p_j. We also choose a *cusp normalizer,* $\sigma_j \in PSL(2,\mathbb{R})$, with the property that $\sigma_j(\infty) = p_j$ and $\sigma_j S \sigma_j^{-1} = S_j$, where $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is the parabolic generator of $PSL(2,\mathbb{Z})$ (the other generator being $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The map σ_j is uniquely determined up to a translation S^k. If $\mathcal{F}$ meet $\partial\mathcal{H}$ at the points q_i, $1 \le j \le \kappa_0$ we fix a set of maps $U_i \in \Gamma$ such that $U_i q_i = p_j$ where p_j is the unique cusp equivalent to q_i.

2. Multiplier systems

2.1. **Introduction**

We will give a brief introduction to multiplier systems, for more extensive treatments see [22, pp. 331-338], [53], or [55, pp. 70-87].

Let Γ be a Fuchsian group and m an even integer. Classically, a function φ, meromorphic on the upper half-plane $\mathcal{H}$, which satisfies

$$\varphi(Az) = \Theta_A(z;m)\varphi(z) = (cz+d)^m \varphi(z), \quad \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{2.1}$$

is called an *automorphic form* of weight m for Γ. The function

$$\Theta_A(z;m) = (cz+d)^m, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

is said to be an *automorphy factor* on Γ. The classical theory of automorphic forms is well-known; for instance, if $m = 2$, then the automorphic forms can be identified with the meromorphic differential forms of degree 1 on the orbifold (classical Riemann surface) $\Gamma\backslash\mathcal{H}$.

We observe that, for even m, the number $(cz+d)^m$ is uniquely defined and the automorphy factor $\Theta_A(z;m)$ in (2.1) clearly satisfies

$$\Theta_A(Bz;m)\Theta_B(z;m) = \Theta_{AB}(z;m). \tag{*}$$

To generalize these notions to arbitrary real m, there needs to be a choice of branch of the argument, and to make certain everything is well-defined, we have to introduce the notion of a multiplier system.

Definition 2.1. For any real number m define

$$j_A(z;m) = e^{im\mathbf{Arg}(cz+d)} = \frac{(cz+d)^m}{|cz+d|^m} = \left(\frac{cz+d}{c\bar{z}+d}\right)^{\frac{m}{2}}, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}).$$

To adapt the relation (*), we also write

$$\sigma_m(A,B) = j_A(Bz;m) j_B(z;m) j_{AB}(z;m)^{-1}. \tag{2.2}$$

It is clear that for integer m, $\sigma_m(A,B) = 1$, but it can also be shown (cf. [53, §2, pp. 42–50]) that the only values which σ_m can take are 1 and $e^{\pm 2\pi i m}$.

Definition 2.2. $v : \overline{\Gamma} \to S^1 = \{z \mid |z| = 1\}$ is said to be a *multiplier system* of weight m on $\overline{\Gamma}$ if

- $v(-I) = e^{-\pi i m}$, and
- $v(AB) = \sigma_m(A,B) v(A) v(B)$, $\forall A,B \in \overline{\Gamma}$.

Observe that v can be regarded equally well as a multiplier system of any weight $m' \equiv m \bmod 2$. The question of whether there exist multiplier systems of a given weight and on a given group is most easily answered by the following proposition (cf. [22, Prop. 2.1, p. 333]).

Proposition 2.3. *Given $v : \overline{\Gamma} \to S^1$ and $m \in \mathbb{R}$. The following are equivalent:*

- *$v(T)$ is a multiplier system of weight m on $\overline{\Gamma}$.*
- *There exists a function $\varphi \not\equiv 0$ on $\mathcal{H}$ which is either C^∞ or meromorphic such that*

$$\varphi(Az) = v(A)\varphi(z)(cz+d)^m, \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}.$$

A φ as above is said to be an automorphic form of weight m and multiplier system v on Γ.

Definition 2.4. Given a multiplier system v on $\overline{\Gamma}$ and an element $\alpha \in GL(2,\mathbb{R})$ we define a multiplier system, v^α, on the group $\alpha^{-1}\overline{\Gamma}\alpha$ by

$$v^\alpha(A) = v\left(\alpha A \alpha^{-1}\right) \frac{\sigma_m\left(\alpha A \alpha^{-1}, \alpha\right)}{\sigma_m\left(\alpha, A\right)}, A \in \alpha^{-1}\overline{\Gamma}\alpha.$$

That this indeed gives a multiplier system on $\alpha^{-1}\overline{\Gamma}\alpha$ is shown in [42, p. 138].

Using Prop. 2.3, we will construct the two most widely used multiplier systems in the following sections. Compare: [22, pp. 334-337].

2.2. The η multiplier system

2.2.1. *The η-function.* The Dedekind $\eta-$function is a holomorphic function on $\mathcal{H}$, defined by

$$\eta(z) = e\left(\frac{z}{24}\right) \prod_{n=1}^{\infty} (1 - e(nz)).$$

It is clear from the definition that $\eta(z) \neq 0$ for $z \in \mathcal{H}$ and that, for each $k \in \mathbb{R}$, η^{2k} can be defined as a holomorphic function on $\mathcal{H}$. (Cf. [55, p. 205].) Note that $\eta(z)^{24}$ is the famous Discriminant function $\Delta(z)$. Cf. [55, pp. 196-197]. It is clear that $\eta(z+1) = e\left(\frac{1}{24}\right)$ and as for $\Delta(z)$ it is also possible to express $\eta(z)$ as a lacunary Fourier series (cf. [46, p. 18]).

2.2.2. *The multiplier system.* It can be proved (cf. Thm. 3.1 and Thm. 3.4 [1, p. 48 and p. 52]) that η satisfies the following functional equations

$$\begin{aligned} \eta\left(\frac{-1}{z}\right) &= (-iz)^{\frac{1}{2}}\eta(z), \text{ and in general,} \\ \eta(Az) &= v_\eta(A)(cz+d)^{\frac{1}{2}}\eta(z), \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \end{aligned} \tag{2.3}$$

This functional equation expresses the fact that η is an $SL(2,\mathbb{Z})-$automorphic form of weight $\frac{1}{2}$ and multiplier system given by v_η. Accordingly the function η^{2k} is an $SL(2,\mathbb{Z})$-automorphic form of weight k and multiplier system given by v_η^{2k}, and we can use η^{2k} in the context of Proposition 2.3 to assure the existence of the multiplier system, $v_{\eta,k} = v_\eta^{2k}$, of weight k on $SL(2,\mathbb{Z})$ (and any of its subgroups, e.g. $\Gamma_0(N)$). We have the following explicit formula for $v = v_\eta^{2k}$:

$$\frac{1}{2\pi i}\log v\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \frac{kb}{12}, & a=d=1, c=0, \\ k\left(\frac{a+d-3c}{12c} - s(d,c)\right), & c>0, \end{cases} \tag{2.4}$$

and for $c<0$ we use that $v(-A) = e^{-k\pi i}v(A)$ (cf. Def. 2.2). Here $s(d,c)$ is the Dedekind sum,

$$s(d,c) = \sum_{n=1}^{c-1} \frac{n}{c}\left(\left(\frac{dn}{c}\right)\right),$$

where $((x))$ is the saw-tooth function

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$$

and $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Note that if x is not an integer, then $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$ so $((-x)) = -((x))$, and hence $s(-d,c) = -s(d,c)$ if $gcd(d,c) = 1$.

Remark 2.5. It is also possible to express the eta multiplier explicitly without Dedekind sums but using extended quadratic residue symbols instead. We have the following formulas from Knopp [31, p. 51] or van Lint [71, Thm. 3]:

$$v_\eta\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right) = \begin{cases} \left(\frac{c}{d}\right) e\left(\frac{1}{24}\left[(a+d)c - bd\left(c^2-1\right) + 3d - 3 - 3cd\right]\right), & c > 0, \text{ even}, \\ \left(\frac{d}{c}\right) e\left(\frac{1}{24}\left[(a+d)c - bd\left(c^2-1\right) - 3c\right]\right), & c > 0, \text{ odd}. \end{cases} \tag{2.5}$$

(Note that the symbols $\left(\frac{c}{d}\right)_*$ and $\left(\frac{d}{c}\right)^*$ of [31, 71] agree with our symbols in these two cases.)

Remark 2.6. It is known that, for each $k \in \mathbb{R}$, there exist exactly 6 different multiplier systems of weight k on $PSL(2,\mathbb{Z})$ (cf. [55, §3.4, pp. 83, 206] or [42, Thm. 19, p. 132]). We will denote these by $v_{\eta,k}^{(r)} = v_\eta^{2(k+r)}$, where $r \in \{0,2,4,6,8,10\}$. Compare [55, eq. (6.4.7)]; one knows, of course, that $v_\eta^{24} = 1$.

When dealing with the modular group and weight k, it is sufficient to consider only the multiplier system $v_{\eta,k}^{(0)} = v_{\eta,k} = v_\eta^{2k}$ (for reasons to be discussed later in Section 4.4.1).

2.3. The θ multiplier system

On any subgroup of $PSL(2,\mathbb{Z})$, we can always use the η-multiplier system, but in general, on subgroups of $PSL(2,\mathbb{Z})$, there are also other multiplier systems available. In particular, on $\Gamma_0(4)$, there is a multiplier system of weight $\frac{1}{2}$ which is interesting from an arithmetical point of view.

It is well-known (cf. [60] or [28]) that the Jacobi theta function

$$\theta(z) = \sum_{-\infty}^{\infty} e(n^2 z), z \in \mathcal{H},$$

is automorphic on $\Gamma_0(4)$ with weight $k = \frac{1}{2}$ and can be used to define a multiplier system on $\Gamma_0(4)$. Using the Poisson summation formula one can prove (cf. [18, pp. 72–75] or [28, pp. 167–168]) that the theta function satisfies:

$$\theta\left(\frac{-1}{2z}\right) = (-iz)^{\frac{1}{2}} \theta\left(\frac{z}{2}\right), \tag{2.6}$$

and one can also prove the general formula (cf. [28, Thm. 10.10, p. 177] or [60, p. 447]):

$$\theta(Az) = v_\theta(A)(cz+d)^{\frac{1}{2}} \theta(z), A = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma_0(4). \tag{2.7}$$

The multiplier $v_\theta(A)$ can be expressed explicitly as

$$v_\theta(A) = \bar{\varepsilon}_d \left(\frac{c}{d}\right),$$

where $\varepsilon_d = 1$ if $d \equiv 1 \bmod 4$ and $\varepsilon_d = i$ if $d \equiv -1 \bmod 4$, and $\left(\frac{c}{d}\right)$ denotes the extended quadratic residue symbol defined as the traditional Jacobi symbol if $0 < d \equiv 1 \bmod 2$ and extended by

$$\left(\frac{c}{d}\right) = \frac{c}{|c|}\left(\frac{c}{-d}\right), c \neq 0,$$

and

$$\left(\frac{0}{d}\right) = \begin{cases} 1 & \text{if } d = \pm 1, \\ 0 & \text{otherwise}. \end{cases}$$

For the sake of completeness we also use the traditional Kronecker extension, i.e. we define

$$\left(\frac{c}{2}\right) = \left(\frac{2}{c}\right). \tag{2.8}$$

One can verify that our symbol $\left(\frac{\cdot}{\cdot}\right)$ satisfies reciprocity relations similar to the usual ones:

Proposition 2.7. *Suppose that $c,d\in\mathbb{Z}$ are odd and $c\neq 0$. Then we have:*

$$\begin{aligned}\left(\frac{-1}{d}\right) &= (-1)^{\left(\frac{d-1}{2}\right)},\\ \left(\frac{c}{d}\right) &= \begin{cases}\left(\frac{d}{c}\right)(-1)^{\left(\frac{d-1}{2}\right)\left(\frac{c-1}{2}\right)}, & d,\ or\ c>0,\\ -\left(\frac{d}{c}\right)(-1)^{\left(\frac{d+1}{2}\right)\left(\frac{c+1}{2}\right)}, & d,\ and\ c<0.\end{cases}\end{aligned}$$

Remark 2.8. Relations (2.6) and (2.7) can also be proved using the corresponding relations (2.3) for η and the following relation between the η and the θ functions (cf. [28, p. 177] or [31, Thm. 12, p. 46]): $\theta(z)=\frac{\eta\left(\frac{z+1}{2}\right)}{\eta(z+1)}$.

3. Maass waveforms

The slash-operator $f_{|A}(z)=f(Az)$ can be extended to an operator of weight k as:

$$f_{|[k,A]}(z)=f(Az)j_A(z;k)^{-1},$$

and the natural analog of the Laplace-Beltrami operator, Δ, which is invariant under this action is the weight-k Laplacian:

$$\Delta_k=\Delta-iyk\frac{\partial}{\partial x}=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-iyk\frac{\partial}{\partial x}.$$

If Γ is a Fuchsian group we define the space $\mathcal{M}(\Gamma,v,k,\lambda)$ consisting of Maass waveforms on Γ, of weight k, multiplier system v and eigenvalue λ, as the space of functions which satisfy the following conditions:

1) $\quad f_{|[A,k]}(z)=v(A)f(z),\ \forall A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\overline{\Gamma},$

2) $\quad \Delta_k f+\lambda f=0$, and

3) $\quad \int_{\mathcal{F}}|f|^2d\mu<\infty.$

Observe that condition 1) is equivalent to

1') $\quad f(Az)=v(A)j_A(z;k)f(z),\ \forall A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\overline{\Gamma}.$

For purposes of the computational work to be described in this paper, we shall be content to restrict ourselves to cases where $\lambda>\frac{1}{4}$. (Cf. also here para. 4 of sect. 8.1 below.)

Instead of the Bessel equation in the case of weight 0, condition 2) above gives us the Whittaker equation, and using the method of separation of variables gives us Whittaker functions instead of the K-Bessel functions at weight 0 (for complete details see [22, Chap. 9]). Since $f(x+iy)$ is no longer periodic in x, but instead satisfies $f(z+1)=v(S)f(z)=e(\alpha)f(z)$, with $\alpha\in[0,1)$, the Fourier series of f can ([22, pp. 26, 348, 420(19)]) be written as

$$f(z) = \sum_{\substack{-\infty\\ n+\alpha\neq 0}}^{\infty}\frac{c(n)}{\sqrt{|n+\alpha|}}W_{sgn(n+\alpha)\frac{k}{2},iR}(4\pi|n+\alpha|y)e((n+\alpha)x), \tag{3.1}$$

where $W_{l,\mu}(x)$ is the Whittaker function in standard notation (cf. [15, vol. I, p. 264]) and R is the usual spectral parameter, $\lambda=\frac{1}{4}+R^2$. One notes here that $W_{0,iR}(x)=\pi^{-\frac{1}{2}}x^{\frac{1}{2}}K_{iR}\left(\frac{x}{2}\right)$. For $k=0$, the expansion above thus reduces to usual Fourier expansion with $2y^{\frac{1}{2}}K_{iR}(2\pi|n+\alpha|y)$ as in [22, p. 26, prop. 4.12].

If we have more than one cusp we define functions f_j related to f at each cusp, p_j, of Γ by using the cusp normalizing maps σ_j from section 1.2 and setting $f_j(z)=f_{|[\sigma_j,k]}(z)=j_{\sigma_j}(z;k)^{-1}f(\sigma_j z)$. It is easy to see that

$$f_j(z+1)=v(S_j)f_j(z)=e(\alpha_j)f_j(z),$$

with $\alpha_j \in [0,1)$ (cf. [28, p. 41]). Thus the Fourier series of f at the cusp j can be written as

$$f_j(z) \quad = \quad \sum_{n=-\infty}^{\infty} \frac{c_j(n)}{\sqrt{|n+\alpha_j|}} W_{sgn(n+\alpha_j)\frac{k}{2},iR}(4\pi|n+\alpha_j|y)e\left((n+\alpha_j)x\right). \tag{3.2}$$

As in the case of weight 0 and Dirichlet character, we say that the cusp number j is *open* or *singular* if $\alpha_j = 0$ and *closed* if $\alpha_j \neq 0$. If all cusps of Γ are singular for the multiplier system v we say that v is a *singular multiplier system for* Γ.

Remark 3.1. Observe that, for the eta-multiplier on $PSL(2,\mathbb{Z})$ and weight k, we have $\alpha = \alpha_1 = \frac{k}{12}$.

3.1. Decomposition of the discrete spectrum

It is known (see for example [11] or [22]) that closed cusps (i.e. $v(S_j) \neq 1$) do not contribute to the continuous spectrum, and if all cusps are closed there is only the discrete part of the spectrum left and this is spanned by the Maass waveforms. We also know (see [22, p. 385]) that on the modular group with weight k the smallest eigenvalue is

$$\lambda_{min} = \frac{|k|}{2}\left(1 - \frac{|k|}{2}\right),$$

or larger. In the case of $PSL(2,\mathbb{Z})$ and $k \geq 0$, $F(z) = y^{\frac{k}{2}}\eta(z)^{2k}$ has eigenvalue equal to λ_{min}.

In this paper, any eigenvalues $\lambda \in \left[\lambda_{min}, \frac{1}{4}\right]$ will be regarded as exceptional. The non-exceptional eigenvalues thus satisfy $\frac{1}{4} < \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \to \infty$. One can obtain lower bounds for the eigenvalue λ_0 (see [11, p. 183]), but in light of the numerical experiments in Section 8.1 they are not very effective (cf. Figure 2).

4. Operators

4.1. Conjugation and reflection

Let J and K denote the reflection, $Jz = -\bar{z}$ and conjugation, $Kz = \bar{z}$. Then J and K act as involutions on the space of Maass waveforms via the operations

$$\begin{aligned} Kf &= f_{|K}(z) = \overline{f(z)}, \\ Jf &= f_{|J}(z) = f(-\bar{z}). \end{aligned}$$

It follows from the definition of the action of $GL(2,\mathbb{R})$ on $\mathcal{H}$ (cf. page 2) that we can use the matrix $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in $GL(2,\mathbb{R})$ to represent the operator J. For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we define

$$A^* = JAJ^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

and then $A^{**} = A$, and $A(z)_{|K} = -A^*(z)$, meaning that also $-\overline{A(z)} = A^*(-\bar{z})$.

Remark 4.1. It is easy to verify that if $f \in \mathcal{M}(\Gamma_0(N), v, k, \lambda)$, then $Kf \in \mathcal{M}(\Gamma_0(N), \bar{v}, -k, \lambda)$ and $Jf \in \mathcal{M}(\Gamma_0(N), v^*, -k, \lambda)$, where v^* is the multiplier system determined by

$$v^*(A) = v(A^*) \cdot \begin{cases} 1, & c \neq 0, \\ e^{\pi i k(1-sgn(d))}, & c = 0, \end{cases}, \text{for} A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Of particular interest is the involution obtained by combining J and K, i.e.

$$KJf(z) = f_{|JK}(z) = \overline{f(-\bar{z})}.$$

It is easily seen that if f has Fourier coefficients $c_j(n)$, then (by [15, p. 265(8)]) $f_{|JK}$ has Fourier coefficients $\overline{c_j(n)}$, and we thus would like to have f and $f_{|JK}$ belonging to the same space (i.e. transform according to the same multiplier system), since then we can assume that the Fourier coefficients are real.

It is clear that if $f \in \mathcal{M}(\Gamma_0(N), v, k, \lambda)$, then $KJf \in \mathcal{M}(\Gamma_0(N), \overline{v^*}, k, \lambda)$ so we are left to see whether $\overline{v^*} = v$ or not. By using the explicit formulas one can verify that indeed $\overline{v^*} = v$ for both v_θ and v_η (see [64, pp. 66-68] for details) and we arrive at the following proposition.

Proposition 4.2. *If v is either the η- or the θ-multiplier system (in the latter case $4|N$) then a basis $\{g_1, \ldots, g_m\}$ of $\mathcal{M}(\Gamma_0(N), v, k, \lambda)$ can be chosen so that each g_j can be expanded in a Fourier series at ∞ with real coefficients.*

Proof. We have seen that for both the theta and the eta multiplier systems the product KJ is a conjugate-linear involution of the space $\mathcal{M}(\Gamma_0(N), v, k, \lambda)$, and hence we can assume that any $f \in \mathcal{M}(\Gamma_0(N), v, k, \lambda)$ is an eigenfunction of KJ with eigenvalue ε, where $|\varepsilon| = 1$. Note that if $f(z)$ has a Fourier series expansion as above with Fourier coefficients $c(n)$ then $f_{|KJ}$ has Fourier coefficients $\overline{c(n)}$ and hence $\overline{c(n)} = \varepsilon c(n)$. Finally we observe that if $\varepsilon = e^{i\theta}$we can look at the function $g = e^{i\frac{\theta}{2}} f$ which then satisfies $KJg = e^{-i\frac{\theta}{2}} KJf = e^{-i\frac{\theta}{2}} e^{i\theta} f = g$. After proper normalization it is thus no restriction to assume that the eigenvalue of KJ is $\varepsilon = 1$, and that the Fourier coefficients are real. □

Remark 4.3. For the sake of completeness it should be remarked that in general one can not simultaneously take Fourier coefficients at cusps other than ∞ to be real (cf. the next subsection where we introduce the map ω_N, which is a cusp normalizing map for the cusp at 0 and which has eigenvalues $\pm i^{-k}$).

4.2. **The involution τ_N**

As in the case of zero weight (e.g. [65]) we define $\omega_N z = \frac{-1}{Nz}$, or equivalently $\omega_N = \begin{pmatrix} 0 & \frac{-1}{\sqrt{N}} \\ \sqrt{N} & 0 \end{pmatrix}$. We know that ω_N is an involution of $\Gamma_0(N)$, i.e. $\Gamma_0(N) = \omega_N \Gamma_0(N) \omega_n^{-1}$, but the question is how it relates to the weight and multiplier system.

If $f \in \mathcal{M}(\Gamma_0(N), v, k, \lambda)$ it is easy to see that $f_{|[k,\omega_N]} \in \mathcal{M}(\Gamma_0(N), v^{\omega_N}, k, \lambda)$, and it is also easy to verify that $v^{\omega_N}(T) = v\left(\omega_N T \omega_N^{-1}\right)$. We also have

$$\begin{aligned} f_{|[k,\omega_N]|[k,\omega_N]}(z) &= j_{\omega_N}(z;k)^{-1} f(\omega_N z)_{|[k,\omega_N]} = j_{\omega_N}(z;k)^{-1} j_{\omega_N}(\omega_N z;k)^{-1} f\left(\omega_N^2 z\right) \\ &= e^{-ikArg(\sqrt{N}z)} e^{-ikArg(-1/\sqrt{N}z)} f(z) = e^{-i\pi k} f(z), \end{aligned}$$

and hence if we define the operator τ_N by

$$\tau_N f(z) = e^{ik\frac{\pi}{2}} f_{|[k,\omega_N]}(z),$$

we have that

$$\tau_N^2 = Id.$$

To show that τ_N is a linear involution, we also have to verify that $v^{\omega_N} = v$. This is easily done in the two cases $N = 1$ together with $v = v_\eta$ and $N = 4$ together with $v = v_\theta$. And we conclude:

Proposition 4.4. *If $N = 1$ and $v = v_\eta$, or $N = 4$ and $v = v_\theta$ the operator*

$$\tau_N : \mathcal{M}(\Gamma_0(N), v, k, \lambda) \to \mathcal{M}(\Gamma_0(N), v, k, \lambda),$$

defined by

$$\tau_N f(z) = e^{ik\frac{\pi}{2}} f_{|[k,\omega_N]} = e^{-ik(Argz - \frac{\pi}{2})} f\left(\frac{-1}{Nz}\right),$$

is a linear involution, i.e. $\tau_N(af) = a\tau_N f$ for all $a \in \mathbb{C}$ and $\tau_N^2 f = f$. Hence it has eigenvalues ± 1.

Remark 4.5. Note that in the case of $\Gamma_0(4)$ (i.e. $N = 4$) if $\tau_N f(z) = \pm f(z)$, then

$$f_2 = f_{|\omega_N} = e^{-ik\frac{\pi}{2}} \tau_N f = \pm e^{-ik\frac{\pi}{2}} f,$$

which means that the Fourier coefficients at the cusp at 0 are proportional to the coefficients at $i\infty$:

$$c_2(n) = \pm e^{-i\frac{\pi}{2}k} c_1(n) = \pm i^{-k} c_1(n). \tag{4.1}$$

4.3. **The operator L**

Definition 4.6. For $N=4$ and the θ-multiplier system at weight $\frac{1}{2}$, following Kohnen [32] or Katok-Sarnak [29] we define the operator L acting on $\mathcal{M}(\Gamma_0(4), v_\theta, \frac{1}{2}, R)$ by

$$L = \frac{1}{\sqrt{2}}\tau_4 T^{v_\theta}_{4,\frac{1}{2}},$$

where $T^{v_\theta}_{4,\frac{1}{2}}$ is the following (exceptional) Hecke operator

$$T^{v_\theta}_{4,\frac{1}{2}} f(z) = \frac{1}{2}\sum_{j \bmod 4} f\left(\frac{z+j}{4}\right). \tag{4.2}$$

Cf. Subsection 4.5.1. It can be shown that the operator L is self-adjoint, commutes with $\Delta_{\frac{1}{2}}$ and all Hecke operators $T^{v_\theta}_{p^2,\frac{1}{2}}$, $p \neq 2$ (cf. (4.9)) and have the eigenvalues 1 and $-\frac{1}{2}$. Cf. [32] and [48]. Suppose that $f(z)$ has Fourier expansions à la (3.2) at the cusps $p_1=\infty$, $p_2=0$ and $p_3=-\frac{1}{2}$ with Fourier coefficients $a_j(n)$ respectively. By calculations similar to [32] or [32, 5] it can be shown that Lf has Fourier coefficients $b(n)$ (with respect to ∞) given by

$$b(n) \quad = \quad \frac{1}{2}\begin{cases} a_1(n) + (1+i)a_2\left(\frac{n}{4}\right), & n \equiv 0 \bmod 4,\\ a_1(n) + \sqrt{2}a_3\left(\frac{n-1}{4}\right)(-1)^{\frac{n-1}{4}}, & n \equiv 1 \bmod 4,\\ -a_1(n), & n \equiv 2,3 \bmod 4.\end{cases} \tag{4.3}$$

Let $V^+ \subseteq \mathcal{M}(\Gamma_0(4), v_\theta, \frac{1}{2}, \lambda)$ denote the subspace introduced by Kohnen in [32], i.e. V^+consists of all f with Fourier coefficients $a_1(n)=0$ for $n \equiv 2,3 \bmod 4$. By using (4.3) it is easy to verify that V^+ is exactly the eigenspace of L corresponding to the eigenvalue 1. Unfortunately the eigenspace V^- corresponding to the eigenvalue $-\frac{1}{2}$ is not as simple to characterize. However, (4.3) can be used to identify V^- by means of certain relations between coefficients at the cusps at ∞, 0 and $-\frac{1}{2}$. E.g. if $a_1(1)=1$ then $a_3(0)=-\sqrt{2}$ and if $a_1(1)=0$ then $a_3(0)=0$.

Clearly $T^{v_\theta}_{4,\frac{1}{2}}$ does not in general commute with L but in case $Lf=\lambda f$, $T_{4,\frac{1}{2}}f=\lambda_4 f$ and $\tau_N f=\varepsilon f$ (with $\varepsilon=\pm 1$) then $a_1(4)=\varepsilon\sqrt{2}\lambda = -\frac{\varepsilon}{\sqrt{2}}, \sqrt{2}\varepsilon$. This should be compared with the results on newforms at weight zero, e.g. [3, p. 147] and [64, p. 31].

4.4. **Maass operators**

So far, the operators we have seen act on spaces of Maass waveforms of a given weight and multiplier system.

We will show that we may limit the range of weights k to investigate to $k \in [0,6]$. For this we will use the Maass lowering and raising operators, $E_k^\pm$,which raise or lower the weight of a Maass waveform by units of 2. They are defined by

$$E_k^\pm \quad = \quad iy\frac{\partial}{\partial x} \pm y\frac{\partial}{\partial y} + \frac{k}{2},$$

and using the relation between the Whittaker function and the confluent hypergeometric function together with the transformation formulas [15, p. 258, (10)] (see also: [42, p. 183 (middle)] and [43, p. 302 lines

-3 and -1]) we see that for $k>0$ (here we set $Y=4\pi|n+\alpha|y$ and $n_\alpha=n+\alpha$)

$$E_k^+\left(W_{\frac{k}{2},iR}(Y)e(n_\alpha x)\right) = -W_{\frac{k+2}{2},iR}(Y)\,e(n_\alpha x), \qquad n_\alpha>0, \tag{4.4}$$

$$E_k^-\left(W_{\frac{k}{2},iR}(Y)e(n_\alpha x)\right) = -\left(\frac{k(k-2)}{4}+\frac{1}{4}+R^2\right)W_{\frac{k-2}{2},iR}(Y)\,e(n_\alpha x), n_\alpha>0, \tag{4.5}$$

$$E_k^+\left(W_{-\frac{k}{2},iR}(Y)e(n_\alpha x)\right)=\left(\frac{k(k+2)}{4}+\frac{1}{4}+R^2\right)W_{-\frac{k+2}{2},iR}(Y)\,e(n_\alpha x),\ n_\alpha<0, \tag{4.6}$$

$$E_k^-\left(W_{-\frac{k}{2},iR}(Y)e(n_\alpha x)\right)=W_{-\frac{k-2}{2},iR}(Y)\,e(n_\alpha x), \qquad n_\alpha<0. \tag{4.7}$$

To verify that they respect the weight k-action it is easiest to proceed straight forward but to use the following form of the operators

$$\begin{aligned} E_k^+ &= (z-\bar{z})\frac{\partial}{\partial z}+\frac{k}{2}, \\ E_k^- &= -(z-\bar{z})\frac{\partial}{\partial \bar{z}}+\frac{k}{2}, \end{aligned}$$

and write $e^{-iArg(cz+d)}=\left(\frac{c\bar{z}+d}{cz+d}\right)^{\frac{1}{2}}$. Cf. [42, p. 178].

One can then show that $E_k^\pm$ maps $\mathcal{M}(\Gamma_0(N),v,k,\lambda)$ into $\mathcal{M}(\Gamma_0(N),v,k\pm 2,\lambda)$, and that the composition

$$E_{k\pm 2}^{\mp}E_k^{\pm}:\mathcal{M}(\Gamma_0(N),v,k,\lambda)\mapsto\mathcal{M}(\Gamma_0(N),v,k,\lambda)$$

is just multiplication by a constant, which is readily seen to be nonzero anytime $\lambda>\frac{1}{4}$. Hence $E_k^\pm$ acts bijectively on the spaces corresponding to non-exceptional eigenvalues, i.e. they are always bijections for $\lambda>\frac{1}{4}$.

4.4.1. *Maass operators and the symmetry about $k=6$ 4.4.1.* First of all, observe that $E_k^\pm$ only change the weight and not the multiplier system v, but in view of the remark after Def. 2.2 it is clear that $v=v_\eta^{2k}$ is also a multiplier system of weight $k+r$ for any $r\in 2\mathbb{Z}$, and with the notation $v_{\eta,k}^{(r)}=v_\eta^{2(k+r)}$ it is clear that

$$E_k^+:\mathcal{M}(\Gamma_0(N),v_{\eta,k}^{(r)},k,\lambda)\to\mathcal{M}(\Gamma_0(N),v_{\eta,k+2}^{(r-2)},k+2,\lambda),$$

and

$$E_k^-:\mathcal{M}(\Gamma_0(N),v_{\eta,k}^{(r)},k,\lambda)\to\mathcal{M}(\Gamma_0(N),v_{\eta,k-2}^{(r+2)},k-2,\lambda).$$

Our main purpose is to investigate the eigenvalues of Maass waveforms on the modular group when the weight and multiplier system are varied. That is, we would like to investigate the space $\mathcal{M}(\Gamma_0(1),v_{\eta,k}^{(r)},k,\lambda)$ for all $k\in\mathbb{R}$ and $r\in\{0,2,4,6,8,10\}$. Suppose that $\lambda>\frac{1}{4}$ so the the lowering and raising operators act bijectively, then using the (k,r) to denote that space $\mathcal{M}(\Gamma_0(1),v_{\eta,k}^{(r)},k,\lambda)$ and using $\approx$ to denote bijectively corresponding spaces we have:

- Since all $v_{\eta,k}^{(r)}$ are 24th-roots of unity we have trivially: $(k,r+12)=(k,r)$.
- A composition of Maass operators which raises or lowers the weight by 12 will preserve the multiplier system. Hence $(k+12,r)\approx(k,r)$ and we may assume that $k\in[0,12]$.
- K is a bijection from (k,r) to $(-k,-r)=(-k,12-r)\approx(12-k,12-r)$ so there is no restriction to assume $k\in[0,6]$ and $r\in\{0,2,4,6,8,10\}$.
- By using the raising operator we see that $(k,0)\approx(k+2,-2)\approx(k+2,10)$ and by repeated use we see that with out loss of generality we can also assume $r=0$.

We are thus justified in our choice of restricting the investigation to the spaces $\mathcal{M}(\Gamma_0(1),v,k,\lambda)$ for $k\in[0,6]$ and $v=v_{\eta,k}^{(0)}=v_\eta^{2k}$.

4.5. **Hecke operators for non-trivial multiplier systems**

We know that Hecke operators play an important role in the understanding of the theory of both modular forms and Maass waveforms at integer weights.

For general real weights, the Hecke operators are not important (and may well be non-existent), but we will begin with a general definition anyway, and then we will consider two special cases: $\Gamma_0(4)$ with the theta multiplier system and weight $\frac{1}{2}$ and $\Gamma_0(1)$ with the eta multiplier system and weight 1.

The general introductory discussion is based on [67] but the specific example of integer weights (as worked out in detail in [66]) is based on ideas from [76] and [70].

Let $\Gamma \subset PSL(2,\mathbb{R})$ be cofinite and suppose $v : \overline{\Gamma} \to S^1$ is a multiplier system of weight $k \in \mathbb{R}$. The commensurator of Γ, $\mathrm{comm}(\Gamma)$, in $PSL(2,\mathbb{R})$ is defined as

$$\mathrm{comm}(\Gamma) = \left\{\alpha \in PSL(2,\mathbb{R}) \,|\, \alpha\Gamma\alpha^{-1} \cap \Gamma \text{ has finite index in } \Gamma \text{ and } \alpha\Gamma\alpha^{-1}\right\}.$$

We know that the Hecke operators are associated with the members of the commensurator, or actually with the double cosets, $\overline{\Gamma}\alpha\overline{\Gamma}$, for $\alpha \in \mathrm{comm}(\Gamma)$. Fix $\alpha \in \mathrm{comm}(\Gamma)$. It can be shown that if the multiplier system v satisfies

$$v(g) = v^{\alpha}(g), \forall g \in \overline{\Gamma} \cap \alpha^{-1}\overline{\Gamma}\alpha, \tag{4.8}$$

then we can define an associated multiplier system, v_α, on the double coset:

$$v_\alpha : \overline{\Gamma}\alpha\overline{\Gamma} \to S^1,$$

by setting

$$v_\alpha (g_1\alpha g_2) = \sigma_k (g_1\alpha, g_2)\, \sigma_k (g_1, \alpha)\, v (g_1)\, v (g_2),$$

for all $g_1, g_2 \in \overline{\Gamma}$. It might be the case that there exists an associated multiplier system of $W = \chi v$, where χ is a character on $\overline{\Gamma}\alpha\overline{\Gamma}$, even though there does not exist an associated multiplier system of v itself. Suppose that v_α exists as above, and that we have $\overline{\Gamma}\alpha\overline{\Gamma} = \cup_{i=1}^{d} \overline{\Gamma}\alpha_i$. We then define the Hecke operator $T^v_{\alpha,k} : \mathcal{M}\left(\overline{\Gamma}, v, k, \lambda\right) \to \mathcal{M}\left(\overline{\Gamma}, v, k, \lambda\right)$ by

$$\left(T^v_{\alpha,k} f\right)(z) = \sum_{i=1}^{d} \overline{v_\alpha(\alpha_i)} f_{|[\alpha_i,k]}(z).$$

When $\Gamma = \Gamma_0(N)$ (or any congruence subgroup of level N), one usually constructs Hecke operators $T^v_{n,k}$ corresponding to positive integers n in which case $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right)$ (cf. e.g. [3], [18, ch. 5], [55, ch. 9], [44] or [59] for more details).

4.5.1. *Hecke Operators for the Theta multiplier System.* Consider the case $\Gamma = \Gamma_0(4)$, $k = \frac{1}{2}$ and $v = v_\theta$. Let n be a positive integer and $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right)$. Then $g \in \overline{\Gamma_0(4)} \cap \alpha^{-1}\overline{\Gamma_0(4)}\alpha$ can be written $g = \left(\begin{smallmatrix} a & nb \\ c/n & d \end{smallmatrix}\right)$, and $\alpha g \alpha^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with $ad - bc = 1$ and $c \equiv 0 \bmod 4n$. It is easy to verify that $v^{\alpha}(g) = v(\alpha g \alpha^{-1})$ and $v(g) = v(\alpha g \alpha^{-1}) \left(\frac{n}{d}\right)^{-1}$, and hence $v^{\alpha}(g) = v(g)$ if and only if $\left(\frac{n}{d}\right) = 1$. By (4.8) this relation must hold for all odd integers d, and hence it is clear that the extension v_α exists if and only if n is a perfect square. It is also easy to verify that in this case the multiplier system is given by $v_\alpha(g_1\alpha g_2) = v(g_1)\, v(g_2)$.

Suppose now for simplicity that $n = p^2$, with $p \neq 2$ a prime number. The $p^2 + p$ different coset representatives of $\overline{\Gamma}$ in $\overline{\Gamma}\alpha\overline{\Gamma}$ are given by $\alpha_b = \left(\begin{smallmatrix} 1 & b \\ 0 & p^2 \end{smallmatrix}\right)$, $b = 0, \ldots, p^2 - 1$, $\sigma = \left(\begin{smallmatrix} p^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $\beta_b = \left(\begin{smallmatrix} p & b \\ 0 & p \end{smallmatrix}\right)$, $b = 1, \ldots, p-1$. By factorization of these representatives we see that $v_\alpha(\sigma) = v_\alpha(\alpha_b) = 1$, $b = 0, \ldots, p^2 - 1$

and $v_\alpha(\beta_b) = \varepsilon_p \left(\frac{b}{p}\right)$ for $b = 1, \ldots, p-1$. Hence, for $p \neq 2$

$$\begin{aligned} T^{v_\theta}_{p^2,\frac{1}{2}} f(z) &= \frac{1}{p}\left\{\sum_{b=0}^{p^2-1} \overline{v_\alpha}(\alpha_b) f(\alpha_b z) + \overline{v_\alpha}(\sigma) f(\sigma z) + \sum_{b=1}^{p-1} \overline{v_\alpha}(\beta_b) f(\beta_b z)\right\} \\ &= \frac{1}{p}\left\{\sum_{b=0}^{p^2-1} f\left(\frac{z+b}{p^2}\right) + f\left(p^2 z\right) + \overline{\varepsilon_p} \sum_{b=0}^{p-1} \left(\frac{b}{p}\right) f\left(z + \frac{b}{p}\right)\right\}. \end{aligned} \tag{4.9}$$

For $p = 2$ the 4 coset representatives are given by $\alpha_b = \left(\begin{smallmatrix} 1 & b \\ 0 & 4 \end{smallmatrix}\right)$, $b = 0, \ldots, 3$ and we obtain precisely the operator in (4.2). The above construction is analogous to [60, pp. 450–451, thm. 1.7]. Suppose that $f \in \mathcal{M}\left(\Gamma_0(4), v_\theta, \frac{1}{2}, \lambda\right)$ has the following Fourier series at infinity

$$f(z) = \sum_{n \neq 0} \frac{a(n)}{\sqrt{|n|}} W_{\frac{1}{4}sgn(n),iR}(4\pi|n|y)e(nx) \tag{4.10}$$

and that $T^{v_\theta}_{p^2,\frac{1}{2}} f(z)$ has a similar Fourier expansion but with coefficients $b^{(p)}(n)$. Using the formula for the standard Gauss sum, $\sum_{b=0}^{p^2-1} e\left(\frac{nb}{p^2}\right) = p^2$ if $p^2|n$ else 0, and the twisted version $\sum_{b=1}^{p-1}\left(\frac{b}{p}\right) e\left(\frac{nb}{p}\right) = \sqrt{p}\left(\frac{n}{p}\right)\varepsilon_p$ (cf. [7, Satz 7, p. 375] or [39, pp. 83-87]) we get that for all non-zero integers n

$$\begin{aligned} b^{(p)}(n) &= \left\{a(p^2 n) + a\left(\frac{n}{p^2}\right) + p^{-\frac{1}{2}}\left(\frac{n}{p}\right) a(n)\right\}, \; p \neq 2, \text{ and} \\ b^{(2)}(n) &= a(4n). \end{aligned}$$

(We use the standard convention that $a(x) = 0$ if $x \notin \mathbb{Z}$.) It is now obvious that our Hecke operator $T^{v_\theta}_{p^2,\frac{1}{2}}$ is equal to the corresponding Hecke operator defined in [29, p. 199]. Observe that our Fourier coefficients $a(n) = \sqrt{n} \times$Katok-Sarnak's Fourier coefficients $b(n)$.

As usual, we consider Hecke eigenforms in $\mathcal{M}\left(\Gamma_0(4), v_\theta, \frac{1}{2}, \lambda\right)$ which are simultaneous eigenfunctions of all $T^{v_\theta}_{p^2,\frac{1}{2}}$ with $p \neq 2$. An additional commuting normal operator can be chosen as either $T^{v_\theta}_{4,\frac{1}{2}}$ *or* L (these two operators does not commute in general).

As it turns out, the operator L is particularly useful in connection with the Shimura correspondence on the Kohnen space, V^+, where L has eigenvalue 1 (cf. Section 6).

Observe that the Hecke eigenvalues in this case does not equal to the Fourier coefficients. In fact, suppose that f as in (4.10) is an eigenfunction of all Hecke operators with $T^{v_\theta}_{p^2,\frac{1}{2}} f = \lambda_p f$ and that $a(t) \neq 0$ for some integer t. It is then easy to see that

$$\begin{aligned} \lambda_p &= \left\{\frac{a\left(tp^2\right)}{a(t)} + \frac{\left(\frac{t}{p}\right)}{\sqrt{p}}\right\}, \; p \neq 2, \text{ and} \\ \lambda_2 &= \frac{a(4t)}{a(t)}. \end{aligned} \tag{4.11}$$

Using multiplicative relations of the Hecke operators one can prove (cf. [60, p. 453]) that if t is square free, then

$$a(tm^2)a(tn^2) = a(t)a(tm^2n^2), \text{ for } (m,n) = 1.$$

Furthermore, if f is also an eigenfunction of $T^{v_\theta}_{4,\frac{1}{2}}$ then

$$a(m)a(4n) = a(4m)a(n), \; m,n \in \mathbb{Z}.$$

4.5.2. *Hecke operators at integer weights and Fourier coefficients.* Consider the modular group, $\Gamma = PSL(2,\mathbb{Z})$, together with the integer weight $k \not\equiv 0 \bmod 12$ and multiplier $v = v_\eta^{2k}$.

It can be shown that for all positive integers n and m with $kn \equiv -km \equiv k \bmod 12$ we can construct Hecke operators $T^v_{n,k}$ and Hecke-like operators $\Theta^v_{m,k}$ acting on $\mathcal{M}(\Gamma, v, k, \lambda)$. Using these operators one can obtain multiplicative relations for Fourier coefficients similar to the weight zero case. This is shown in detail in [66] and here we will only state the theorem and give a brief outline of the ideas of the proof.

Theorem 4.7. *Let k be an integer $k \not\equiv 0 \bmod 12$ and $R > 0$ then there exist a basis of $\mathcal{M}(\Gamma, v, k, R)$ consisting of Maass wave forms f with Fourier expansions at infinity*

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{c(n)}{\sqrt{\left|n+\frac{k}{12}\right|}} W_{\frac{k}{2}sgn\left(n+\frac{k}{12}\right), iR}\left(4\pi\left|n+\frac{k}{12}\right|y\right) e\left(\left(n+\frac{k}{12}\right)x\right)$$

where the coefficients $c(n)$ satisfies the following multiplicative relations if $c(0) \neq 0$. For positive integers m, n with $12m \equiv 12n \equiv 0 \bmod k$ set $m_1 = \frac{12m}{k}$, $n_1 = \frac{12n}{k}$ and $D = \frac{k}{(12,k)}$. If $(m_1+1, D) = (n_1+1, D) = 1$

$$c(m)c(n) = c(0) \sum_{0<d|(m_1+1,n_1+1)} \chi_k(d)\, c\left(\frac{k}{12}\left(\frac{(m_1+1)(n_1+1)}{d^2} - 1\right)\right), \tag{4.12}$$

and if $(m_1-1, D) = (n_1-1, D) = 1$ then

$$c(-m)\,c(-n) = \Lambda_{k,R}\, c(0) \sum_{0<d|(m_1-1,n_1-1)} \chi_k(d)\, c\left(\frac{k}{12}\left(\frac{(m_1-1)(n_1-1)}{d^2} - 1\right)\right), \tag{4.13}$$

where $\chi_k(d) = i^{k(d-1)}$ $\left(= \left(\frac{-1}{d}\right)^k \text{ for odd } d\right)$ and

$$\Lambda_{k,R} = \begin{cases} \prod_{j=1}^{l}\left(j(j-1) + \frac{1}{4} + R^2\right)^2, & k = 2l, \\ -R^2 \prod_{j=1}^{l}\left(j^2 + R^2\right)^2, & k = 2l+1. \end{cases} \tag{4.14}$$

In particular, we see that if $k|12$ we have $D = 1$ and the multiplicative relations (4.12) and (4.13) are valid for all positive integers.

Remark 4.8. As in the weight zero case and the coefficient $c(1)$ one can show that if $D = 1$ and f is an eigenfunction of all Hecke operators (defined below) then $c(0) \neq 0$ unless $f(z)$ is identically 0. In the case $D > 0$, if $c(0) = 0$ and $f(z) \not\equiv 0$ we can choose an integer n_0 such that $c(n_0) \neq 0$ and obtain a similar set of multiplicative relations.

The proof of the "positive part" of the theorem, i.e. equation (4.12) relies on the construction of a family of Hecke operators $T^v_{m,k}$ with $km \equiv k \bmod 12$. It is shown that this family consist of self-adjoint operators commuting with each other and the weight k Laplacian. An explicit form of $T^v_{m,k}$ is

$$T^v_{m,k} f(z) = \frac{1}{\sqrt{m}} \sum_{ad=m, d>0} \chi_k(d) \sum_{b=0}^{d-1} \overline{v(T)^{bd}}\, f\left(\frac{az+b}{d}\right). \tag{4.15}$$

It is not hard to show that if $f(z)$ has Fourier coefficients $c(n)$ then $T^v_{m,k}$ has coefficients

$$b(n) = \sum_{0<d|\left(m, n - \frac{k(m-1)}{12}\right)} \chi_k(d)\, c\left(\frac{nm}{d^2} + \frac{k\left(m-d^2\right)}{12d^2}\right)$$

from which we see that if $T^v_{m,k} f = \lambda_m f$ and $c(0) \neq 0$ then

$$\lambda_m = \frac{1}{c(0)}\left[c\left(\frac{k(m-1)}{12}\right) + \chi_k(D)\, c\left(\frac{k\left(m-D^2\right)}{12D^2}\right)\right].$$

The proof of (4.12) is concluded with a proof of the following multiplication rule (using straight-forward calculations and induction). If $km \equiv kn \equiv k \bmod 12$ then

$$T^{v}_{m,k}T^{v}_{n,k} = \sum_{0<d|(m,n)} \chi_k(d)\, T^{v}_{\frac{mn}{d^2},k}.$$

To obtain the "negative part" of theorem, i.e. (4.13) we have to consider another family of operators, $\Theta^{v}_{m,k}$ with $km \equiv -k \bmod 12$. These operators are given as a combination of a bijection $\Theta = J \circ \mathcal{E}_k^-$ mapping $\mathcal{M}(\Gamma, v, k, \lambda)$ to $\mathcal{M}(\Gamma, \overline{v}, k, \lambda)$ and an Hecke operator $T^{\overline{v}}_{m,k}$ mapping $\mathcal{M}(\Gamma, \overline{v}, k, \lambda)$ back to $\mathcal{M}(\Gamma, v, k, \lambda)$. Here $\mathcal{E}_k^- = E^-_{2-k} \circ \cdots \circ E^-_{k-2} \circ E^-_k$ maps $\mathcal{M}(\Gamma, v, k, \lambda)$ to $\mathcal{M}(\Gamma, v, -k, \lambda)$ bijectively since $\lambda > \frac{1}{4}$ and J reflects this space back to $\mathcal{M}(\Gamma, \overline{v}, k, \lambda)$. $T^{\overline{v}}_{m,k}$ is given by (4.15) with v interchanged with $\overline{v}$. The operator Θ is similar to the operator defined by Maass in [42, p. 181]. Using (4.4)-(4.7) it is easy to show that $\Theta^2 f = \Lambda_{k,R} f$ for all $f \in \mathcal{M}(\Gamma, k, v, \lambda)$ and that if $f(z)$ has Fourier coefficients $c(n)$ then $\Theta^{v}_{m,k} f(z)$ has coefficients

$$d(n) = \sum_{0<d|\left(m,n-\frac{k(m+1)}{12}\right)} \chi_k(d)\, c'\left(\frac{nm}{d^2} + \frac{k\left(m+d^2\right)}{12d^2}\right)$$

where $c'(n) = c(-n)\begin{cases} 1, & n \geq 1, \\ \Lambda_{k,R}, & n \leq 0. \end{cases}$ If $\Theta^{v}_{m,k} f = \mu_m f$ and $c(0) \neq 0$ then

$$\mu_m = \frac{1}{c(0)}\left[c\left(\frac{k(m+1)}{12}\right) + \chi_k(D)\, c\left(\frac{k\left(m+D^2\right)}{12D^2}\right)\right].$$

The multiplication rule for the operators $\Theta^{v}_{m,k}$ is that for $km \equiv kn \equiv k \bmod 12$ we have

$$\Theta^{v}_{m,k}\Theta^{v}_{n,k} = \Lambda_{k,R} \sum_{0<d|(m,n)} \chi_k(d)\, \Theta^{v}_{\frac{mn}{d^2},k}.$$

This in combination with the expressions for μ_m and λ_m concludes the proof of (4.13).

Example 4.9. Look at the specific case $k = 1$ and a function $f \in \mathcal{M}(\Gamma, v, 1, \lambda)$ then by by setting $m = 1$ in (4.12) and using the normalization $c(1) = 1$ we see that

$$c(n) = \sum_{0<d|(12n+1,13)} \chi_1(d)\, c\left(\frac{\left(13(12n+1) - d^2\right)}{12d^2}\right)$$

and hence if $(12n+1, 13) = 1$ then we get a striking proportionality relation:

$$c(n) = c(0)c\left(\frac{13(12n+1) - 1}{12}\right) = c(0)c(13n+1). \tag{4.16}$$

For $12n+1 = 13l$ we get

$$c(n) = c(0)\left(c(13n+1) + c\left(\frac{n-1}{13}\right)\right), \tag{4.17}$$

and if $(l, 13) = 1$ we can combine these two equations and obtain:

$$c(n) = c(0)\left(c(13n+1) + c(0)c(n)\right) \tag{4.18}$$

and hence

$$c(n) = \frac{c(0)}{1 - c(0)^2} c(13n+1). \tag{4.19}$$

Consider now $m=-1$ in (4.13) and note that $\Lambda_{1,R}=-R^2$. If $(12n-1,11)=1$ then

$$c(-n)=\frac{-R^2c(0)}{c(-1)}c(11n-1), \tag{4.20}$$

and if $(12n-1,11)=11$ then

$$c(-n)=\frac{-R^2c(0)}{c(-1)}\left[c(11n-1)-c\left(\frac{n-1}{11}\right)\right]. \tag{4.21}$$

We have not seen relations of type (4.16)-(4.21) earlier in the literature. For some numerical examples see Tables 2 and 3. Examples of relations for higher weights can be found in [66].

Remark 4.10. An alternative approach to the above coefficient relations in the case of weight 1 is to identify $\mathcal{M}(\Gamma_0(1),v_\eta^2,1,\lambda)$ with a subspace of $\mathcal{M}(\Gamma_0(144),\left(\frac{-1}{d}\right),1,\lambda)$ via the map $f(z)\mapsto g(z)=f(12z)$. Cf. e.g. [64, sec. 2.4.7].

This identification also provides an explanation for the occurrence of CM-type forms found (numerically) in $\mathcal{M}(\Gamma_0(1),v_\eta^2,1,\lambda)$. These forms have eigenvalues in an arithmetic progression: $R_k=\frac{2\pi k}{\ln(7+2\sqrt{12})}$, for $k\in\mathbb{Z}$. See Table 4.

5. The Eisenstein series for $PSL(2,\mathbb{Z})$ at weight zero

In case there is a cusp p_j at which the multiplier system is singular (i.e. $v(S_j)=1$) we have a continuous spectrum: $[\frac{1}{4},\infty)$ (with multiplicity equal to the number of singular cusps), and in general we can not say much about the embedded discrete spectrum in $[\frac{1}{4},\infty)$.

Examples of singular cusps are the cusp at infinity for the eta multiplier and weight $k\equiv 0 \bmod 12$ on $PSL(2,\mathbb{Z})$ and the cusps at 0 and $i\infty$ for the theta multiplier and weight $\frac{1}{2}$ on $\Gamma_0(4)$.

Remember that Maass waveforms are part of the discrete spectrum, but as we continuously "turn off" the multiplier system, i.e. for $v=v_\eta^{2k}$ we let $k\longrightarrow 0$, the continuous spectrum will emerge in the limit. For this reason we want to review some details concerning the Eisenstein series on the modular group.

At weight 0 and singular character χ, the continuous spectrum of $\Delta=\Delta_0$ is the interval $[\frac{1}{4},\infty)$ and the eigenpacket is given by the Eisenstein series $E(z;s;\chi)$ defined by

$$E(z;s;\chi)=\sum_{T\in\Gamma_\infty\backslash\Gamma}\chi(T^{-1})\left(\Im(Tz)\right)^s,$$

where $\Gamma_\infty=[S]$. For the trivial character we have the Fourier series expansion (cf. [22, p. 65 and p. 76])

$$E(z;s;\chi)=y^s+\varphi(s)y^{1-s}\sum_{n\neq 0}\varphi_n(s)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e(nx),$$

where

$$\begin{aligned}\varphi(s)&=\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)},\text{ and}\\ \varphi_n(s)&=\frac{2\pi^s|n|^{s-\frac{1}{2}}}{\Gamma(s)}\frac{\sigma_{1-2s}(|n|)}{\zeta(2s)}.\end{aligned}$$

Hence we can see that for $s=\frac{1}{2}+iR$, the nth Fourier coefficient of $E(z;s)$ is given by

$$\begin{aligned}c(n)&=\varphi_n\left(\frac{1}{2}+iR\right)=K\cdot|n|^{iR}\sigma_{-2iR}(|n|)\\ &=K\cdot|n|^{iR}\sum_{d||n|,d>0}d^{-2iR},\end{aligned} \tag{5.1}$$

where $K = K(R)$ is a constant dependent on R. For a prime p we get

$$c(p) \quad = \quad K \cdot p^{iR}(1 + p^{-2iR}) = 2K \cdot \cos(R \ln p).$$

Using this formula we can compute quotients of various $c(p)$ (e.g. $\frac{c(2)}{c(3)}$) and compare this with corresponding quotients for the experimentally obtained forms of weight $k \approx 0$.

6. The Shimura correspondence

We know that the $\theta-$function is an automorphic form (not a cusp form) of weight $\frac{1}{2}$ on $\Gamma_0(4)$, hence we consider $\Gamma_0(4)$ together with the $\theta-$multiplier system (cf. section 2.3).

6.1. **Introduction – the holomorphic case**

We will consider the Shimura correspondence only in the particular case of trivial character and level a square-free multiple of 4. Let $S_k(N)$ denote the space of holomorphic cusp forms of weight $k \in \mathbb{Z}$ (and trivial multiplier) on $\Gamma_0(N)$, and let $S_{k+\frac{1}{2}}(4N)$ denote the space of holomorphic cusp forms of weight $k + \frac{1}{2}$, $k \in \mathbb{Z}$ and multiplier v_θ, on $\Gamma_0(4N)$. The *Shimura correspondence* is a correspondence between the space $S_{k+\frac{1}{2}}(4N)$ and spaces $S_{2k}(N')$ for certain integers $N'|4N$ (e.g. $N' = 2N$ or N).

The map from $S_{k+\frac{1}{2}}$ to S_{2k} was first constructed by Shimura [60] and later an adjoint map from S_{2k} to $S_{k+\frac{1}{2}}$ was constructed by Shintani [62]. The former uses a Dirichlet-series and the latter uses an integral against a theta-function. Both these maps commute with the Hecke operators that are acting on $S_{2k}(N)$ and $S_{k+\frac{1}{2}}(4N)$ respectively. Kohnen [32, 33] proved that for N odd and square-free, the correspondence is a bijection between the newforms on $S_{2k}(N)$ and a certain subspace, $V^+ \subseteq S_{k+\frac{1}{2}}(4N)$. The subspace V^+ is composed of Hecke eigenfunctions whose Fourier coefficients, $c(n)$, satisfy certain vanishing properties; namely, $c(n) = 0$ for $n \equiv 2, 3 \bmod 4$ (see also Section 4.3).

Following from the Shimura correspondence is also a connection between certain Fourier coefficients of the half integral weight forms and critical values of twisted L-series for the corresponding integral weight form. Cf. e.g. [73, 74], [35, 34] and [61].

6.2. **The Shimura correspondence for Maass waveforms**

The extension of the Shimura correspondence and Kohnen's result to spaces of Maass waveforms has been investigated by e.g. Sarnak [57], Hejhal [20], Duke [14], Katok-Sarnak [29], Khuri-Makdisi [30], Kojima [36, 37, 38], Biró [5] and Arakawa [2]. Of these, [30] and [37, 38] are written in the more general setting of Hilbert modular forms for number fields. Reading [30] together with [61] and [29] gives a good picture of the current state of affairs.

Throughout this section let $\mathcal{M}(N,R)^+$ denote the space of even (with respect to $J : z \mapsto -\overline{z}$) Hecke normalized Maass waveforms in $\mathcal{M}(\Gamma_0(N), 1, 0, R)$ and let $\mathcal{M}_{\frac{1}{2}}(4,R)$ denote the space of Hecke normalized (with respect to all T_{p^2}, $p \neq 2$) weight $\frac{1}{2}$ Maass waveforms in $\mathcal{M}\left(\Gamma_0(4), v_\theta, \frac{1}{2}, R\right)$. Also let $V^+ \subseteq \mathcal{M}_{\frac{1}{2}}(4,R)$ be defined as in Section 4.3.

For $f \in \mathcal{M}(N,R)^+$ and $\phi \in \mathcal{M}_{\frac{1}{2}}(4,R)$ we will use the following notation:

$$f(z) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} A(n)\sqrt{y} K_{iR}(2\pi|n|y)\, e(nx),$$

and

$$\phi(z) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{a(n)}{\sqrt{|n|}} W_{\frac{1}{4}sgn(n),iR}(4\pi|n|y)\, e(nx).$$

(Both expansions are given with respect to the cusp at ∞).

The existence of a Shimura correspondence and an inverse for Maass waveforms is expressed by the following proposition (essentially [30, thm. 5.1 and 5.2]):

Proposition 6.1.

a) *The Shimura correspondence gives a map* $\Phi : \mathcal{M}_{\frac{1}{2}}(4,R) \to \mathcal{M}(2,2R)^{+}$.

b) *Conversely, let* $f \in \mathcal{M}(2,2R)^{+}$. *Then there exists a* $\phi \in \Phi^{-1}(f) \in \mathcal{M}_{\frac{1}{2}}(4,R)$.

c) *If* $A(p)$ *is the Hecke eigenvalue of* f *with respect to the operator* T_p *and* λ_p *is the eigenvalue of* $\phi \in \Phi^{-1}(f)$ *with respect to* $T^{v_\theta}_{p^2,\frac{1}{2}}$ *then we actually have*

$$A(p) = \lambda_p.$$

Proof. See the proof of [30, thm. 5.1 and 5.2]. For c) observe the difference in normalization of the Hecke operators. Cf. also [36, thm. 2], Proposition 6.3 a) and (4.11). □

Remark 6.2. From [29, prop. 4.1 and 2.3] we know that the above correspondence Φ actually restricts to a map between $\mathcal{M}(1,2R)^{+}$ and the subspace V^{+}. And hence a new form in $\mathcal{M}(2,2R)^{+}$ will be mapped to V^{-}, the orthogonal complement of V^{+}. Vice versa, a form in V^{-} will be mapped to a new form in $\mathcal{M}(2,2R)^{+}$.

To generalize the results mentioned at the end of the previous subsection to Maass waveforms we need the following definition. For $f \in \mathcal{M}(2,R)^{+}$ with Fourier coefficients $\{A(n)\}$ and a given Dirichlet character χ we define the χ-twisted L-series of f by

$$L(f,\chi,s) = \sum_{n \neq 0}^{\infty} A(n)\chi(n)|n|^{-s-\frac{1}{2}}.$$

Proposition 6.3. *Let* $\phi \in \mathcal{M}_{\frac{1}{2}}(4,R)$ *have Fourier coefficients* $\{a(n)\}$ *and correspond (via Prop. 6.1) to* $f = \Phi(\phi) \in \mathcal{M}(N,2R)^{+}$ *(where* $N = 1$ *if and only if* $\phi \in V^{+}$, *otherwise* $N = 2$*) with Fourier coefficients* $\{A(n)\}$. *Let* $t \in \mathbb{Z}^{+}$ *be square free and let* χ'_t *be the quadratic residue symbol* $\left(\frac{t}{\cdot}\right)$ *considered mod* Nt *(i.e. we have* $\chi'_t(n) = \left(\frac{t}{n}\right)$ *if* $(n,N) = 1$ *and* $\chi'_t(n) = 0$ *otherwise.*

Then the following properties hold:

a) *If* $a(t) \neq 0$ *then* $A(n)$ *can be expressed by:*

$$A(n) \quad = \quad \sum_{\substack{kd=n \\ k>0}} \frac{\chi'_t(k)}{\sqrt{k}} \frac{a\left(td^2\right)}{a(t)}, n \in \mathbb{Z}^{+}. \tag{6.1}$$

To express $a(t)$ *in terms of* $A(n)$*'s we get three cases.*

b) *If* f *is an oldform. i.e.* $f \in \mathcal{M}(1,2R)^{+}$ *then* $\phi \in V^{+}$ *and hence*

$$a(t) = 0, t \equiv 2,3 \bmod 4.$$

c) *If* f *is a newform with eigenvalue* ε *with respect to the involution* $z \mapsto \frac{-1}{2z}$ *then*

$$a(t) = 0$$

for $t \equiv 5 \bmod 8$ *if* $\varepsilon = 1$ *and for* $t \equiv 1 \bmod 8$ *if* $\varepsilon = -1$.

d) *For all other (square free) values of* t *the following formula holds*

$$|a(t)|^2 = Q\frac{\langle\phi,\phi\rangle}{\langle f,f\rangle} L(f,\chi_t,0)$$

where Q *is a constant independent of* t.

e) *If* ϕ *is a normalized Hecke eigenform (for all* $T^{v_\theta}_{p^2,\frac{1}{2}}$, p *an odd prime) then* f *is also a normalized Hecke eigenform.*

Proof. Cf. [60, pp. 448(prop. 1.3), 458 (main theorem), 474 (line -11)], and [30, p. 422] for the choice of χ_t' (the $N=1$ case is simply to incorporate the results of [29]). Relation a) follows from [30, thm. 5.1 (2)], which in our case can be written as

$$\sum_{n=1}^{\infty} A(n)n^{-s} = c \cdot L\left(s+\frac{1}{2},\chi_t'\right) \sum_{n=1}^{\infty} a\left(tn^2\right) n^{-s}. \tag{6.2}$$

(Cf. e.g. also [60, p. 458], [47, p. 159], [61, prop. 3.1].) The relation b) follows from [29, prop. 2.3] (see also [32]) and relation c) is implicit in [30, thm. 8.1 (8.6)]. Simply observe that in our normalization the relevant factor in the product is given by

$$\left(\sqrt{2}A(2) - \left(\frac{2}{t}\right)\right) = -\left(\varepsilon + \left(\frac{2}{t}\right)\right)$$

since $A(2) = \frac{-\varepsilon}{\sqrt{2}}$ for a newform (analogous to [3, thm. 3]). (Alternatively consider the sign of the functional equation for $L(f,\chi_t,s)$.) Relation d) is the Maass waveform-analogue of [35, thm. 1] and follows from [30, thm. 8.1] (in the notation of [30] we have $a(n) = |n|^{\frac{1}{4}}\mu(n;\phi)$). □

Remark 6.4. That the map Φ is well-defined and surjective from $\mathcal{M}_{\frac{1}{2}}(4,R)$ onto $\mathcal{M}(2,2R)^+$ follows from [30, thm. 5.1 and 5.2] and "multiplicity one" for the Hecke operators T_p on $\mathcal{M}(2,2R)^+$ (cf. [68, thm. 4.6]). Experimentally we have observed that Φ restricted to V^+ also seems to be injective. Theoretically, this is still an open problem that might be possible to resolve using the trace formula for the Hecke operators T_{p^2} on V^+.

Remark 6.5. Note that the same argument as for 6.3 c) implies that to any $f \in \mathcal{M}(2,2R)^+$ with Fourier coefficient $A(2) = \frac{\pm 1}{\sqrt{2}}$ there corresponds a function $\phi \in \Phi^{-1}(f) \subseteq \mathcal{M}_{\frac{1}{2}}(4,R)$ with coefficients $a(n) = 0$ for either $n \equiv 1$ or $5 \mod 8$ respectively. Hence, since oldforms on $\Gamma_0(2)$ occur in pairs, we can choose two forms f_1, f_2 with $A_1(2) = \frac{1}{\sqrt{2}}$ respectively $A_2(2) = \frac{-1}{\sqrt{2}}$. These two functions thus correspond to two linearly independent non-zero functions in $\mathcal{M}_{\frac{1}{2}}(4,R)$ which hence is at least two dimensional when $2R$ is an even eigenvalue for $PSL(2,\mathbb{Z})$.

The relations a)–c) in Proposition 6.3 appeared in [64, ch. 2] as experimental observations (cf. 8.4, in particular Tables 5-8).

See also [12, §4.1] and [13, p. 633], and [61, pp. 502 (bottom) – 504 (top)] for some additional perspectives.

7. Some computational remarks

We recall the key ingredients in the standard Hejhal's algorithm (cf. e.g. [25, 26, 27], [65, 64] and [6]) to compute weight zero Maass waveforms on cofinite Fuchsian groups. First of all the asymptotic properties of the K-Bessel function are used to obtain an approximation the Maass waveform by a truncated Fourier series. By viewing this as a discrete Fourier transform one can use the inverse transform to express any coefficient as a linear combination of the other coefficients. Finally, by using the assumed automorphy properties of the function one obtains a non-trivial linear system that is satisfied by the Fourier coefficients.

To use this algorithm to also *locate* eigenvalues the most general method is to use two different sets of sampling points for the inverse transform and try to minimize the difference between the correspondingly computed solution vectors.

The following two modifications are needed in order to make the weight zero algorithm work in the general case:

- the K-Bessel function needs to be replaced with the Whittaker function, and
- the automorphy condition needs to incorporate the multiplier system.

The first modification, although trivial in theory is the most complex in terms of the numerics involved. There was no efficient algorithm for the Whittaker function in the literature and a new algorithm had to be developed. We used the integral representation (cf. [22, p. 375 (top)])

$$W_{k,iR}(2x) = \sqrt{\frac{2x}{\pi}} \int_0^{\infty} e^{-x\cosh t} \Psi\left[-k; \frac{1}{2}; x(1+\cosh t)\right] \cosh\left[iRt\right] dt,$$

where Ψ is a confluent hypergeometric function together with a stationary phase method to develop a robust and efficient algorithm. This algorithm is in essence similar to Hejhal's algorithm (cf. [25]) for the K-Bessel function, $K_{iR}(x)$, and the generalization of it made by Avelin (cf. [4, Avelin]) to a complex argument, $K_s(x)$, $s \in \mathbb{C}$. The details of this algorithm is described in [64, Ch. 4].

Let Γ be a Fuchsian group with fundamental domain $\mathcal{F}$ and let p_j, σ_j, $1 \le j \le \kappa$, q_i and U_i, $1 \le i \le \kappa_0$ be as in Section 1.2. Let $I(w) = i$ if q_i is the closest (in the hyperbolic metric) parabolic vertex to $w \in \mathcal{F}$.

Consider $z \in \mathcal{H}$ and let $T_j \in \Gamma$ be the pull-back map of $\sigma_j z$, i.e. $T_j \sigma_j z \in \mathcal{F}$. Observe that $f_j(z) = f_{|\sigma_j}(z) = j_{\sigma_j}(z;k)^{-1} f(\sigma_j z)$, and with $z_j^* = \sigma_{I(j)}^{-1} U_{I(j)} T_j \sigma_j z$ where we set $I(j) = I(T_j \sigma_j z)$ (cf. [65, p. 23]) the automorphy condition now becomes:

$$\begin{aligned} f_j(z) &= j_{\sigma_j}(z;k)^{-1} f(\sigma_j z) = j_{\sigma_j}(z;k)^{-1} f(T_j^{-1} U_{I(j)}^{-1} \sigma_{I(j)} z_j^*) \\ &= j_{\sigma_j}(z;k)^{-1} v(T_j^{-1} U_{I(j)}^{-1}, k) j_{T_j^{-1} U_{I(j)}^{-1}}(\sigma_{I(j)} z_j^*; k) f(\sigma_{I(j)} z_j^*) \\ &= j_{\sigma_j}(z;k)^{-1} j_{T_j^{-1} U_{I(j)}^{-1}}(\sigma_{I(j)} z_j^*; k) j_{\sigma_{I(j)}}(z_j^*; k) v(T_j^{-1} U_{I(j)}^{-1}, k) f_{I(j)}(z_j^*). \end{aligned} \tag{7.1}$$

The entire setup of the Phase 1 algorithm, i.e. locating eigenvalues and computing the first few Fourier coefficients, goes through exactly as in the case of weight 0 (cf. [65] or [64, Ch. 1]) with some trivial modifications. For the sake of completeness we will give an outline of the modified algorithm. Consider $f \in \mathcal{M}(\Gamma, v, k, \lambda)$ and using the notation $n_i = n + \alpha_i$ f has a Fourier series expansion at the cusp p_i:

$$f_i(z) = f_{|[k,\sigma_j]}(z) = \sum_{\substack{-\infty \\ n+\alpha_i \neq 0}}^{\infty} \frac{c_i(n)}{\sqrt{|n_i|}} W_{\frac{k}{2} sgn(n_i), iR}(4\pi |n_i| y) e(n_i x),$$

and since the Whittaker function is ultimately exponentially decaying, given an $\varepsilon > 0$, there exists a constant (depending on y and ε), $M(y)$, such that

$$f_i(z) = \hat{f}_i(z) + [[\varepsilon]],$$

where we use $[[\varepsilon]]$ to denote a constant of magnitude less than ε. The truncated Fourier series

$$\hat{f}_i(z) = \sum_{\substack{-M(Y) \\ n+\alpha_i \neq 0}}^{M(Y)} \frac{c_i(n)}{\sqrt{|n_i|}} W_{\frac{k}{2} sgn(n_i), iR}(4\pi |n_i| y) e(n_i x),$$

is now viewed as a discrete Fourier transform, and if we take the inverse transform over the horocyclic points: $z_m = x_m + iY$, $1 - Q \le m \le Q$, where $x_m = \frac{1}{2Q}(\frac{1}{2} - m)$ we get:

$$\begin{aligned} \frac{1}{2Q} \sum_{1-Q}^{Q} \hat{f}_i(z_m) e(-n_i x_m) &= \frac{1}{2Q} \sum_{m=1-Q}^{Q} \sum_{\substack{l=-M(Y) \\ l_i \neq 0}}^{M(Y)} \frac{c_i(l)}{\sqrt{|l_i|}} W_{\frac{k}{2} sgn(l_i), iR}(4\pi |l_i| Y) e(l_i x_m - n_i x_m) \\ &= \sum_{\substack{l=-M(Y) \\ l_i \neq 0}}^{M(Y)} \frac{c_i(l)}{\sqrt{|l_i|}} W_{\frac{k}{2} sgn(l_i), iR}(4\pi |l_i| Y) \frac{1}{2Q} \sum_{1-Q}^{Q} e(l_i x_m - n_i x_m) \\ &= \frac{c_i(n)}{\sqrt{|n_i|}} W_{\frac{k}{2} sgn(n_i), iR}(4\pi |n_i| Y), \end{aligned}$$

where we used the fact that

$$\frac{1}{2Q}\sum_{1-Q}^{Q} e\left(l_i x_m - n_i x_n\right) \quad = \quad e\left(\frac{l_i-n_i}{4Q}\right)\frac{1}{2Q}\sum_{1-Q}^{Q} e\left(-\left(l_i-n_i\right)\frac{m}{2Q}\right) = \delta_{nl}.$$

Now we also have $f_i(z_m) = \chi_{mi} f_{I(m,i)}(z^*_{mi})$, where (by (7.1))

$$\chi_{mi} = j_{\sigma_i}(z_m;k)^{-1} j_{T_i^{-1}U_{I(m,i)}^{-1}\sigma_{I(m,i)}}(z^*_{mi};k) w(T_i^{-1}U_{I(m,i)}^{-1},\sigma_{I(m,i)}) v(T_i^{-1}U_{I(m,i)}^{-1},k).$$

Hence

$$\begin{aligned}
\frac{c_i(n)}{\sqrt{|n_i|}} W_{sgn(n_i)\frac{k}{2},iR}(4\pi|n_i|y) \quad &= \quad \frac{1}{2Q}\sum_{1-Q}^{Q} f_i(z_m) e(-n_i x_m) + [[\varepsilon]] \\
&= \quad \frac{1}{2Q}\sum_{1-Q}^{Q} \chi_{mi} f_{I(m,i)}(z^*_{mi}) e(-n_i x_m) + [[\varepsilon]] \\
&= \quad \sum_{j=1}^{\kappa} \sum_{\substack{l=-M_0 \\ n+\alpha_i \neq 0}}^{M_0} c_j(l) V_{nl}^{ij} + 2[[\varepsilon]],
\end{aligned}$$

where

$$V_{nl}^{ij} \quad = \quad \frac{1}{\sqrt{|l_j|}} \frac{1}{2Q} \sum_{\substack{1-Q \\ I(m,i)=j}}^{Q} \chi_{mi} W_{sgn(l_j)\frac{k}{2},iR}(4\pi|l_j|y^*_{mj}) \times e\left(l_j x^*_{mj}\right) e(-n_i x_m).$$

We then define $\tilde{V}_{nl}^{ij}$ by

$$\tilde{V}_{nl}^{ij} = V_{nl}^{ij} - \delta_{nl}\delta_{ji}\frac{1}{\sqrt{|n_i|}} W_{sgn(n_i)\frac{k}{2},iR}(4\pi|n_i|Y),$$

and if we neglect the error of magnitude ε we finally obtain a (well-conditioned) linear system

$$CV = 0, \tag{*}$$

which must be satisfied by the Fourier coefficients of f. Here V is the matrix $\tilde{V}_{nl}^{ij}$ and C is the vector $c_i(n)$, both depending on R and Y. The basic idea of Phase 1 is now to locate eigenvalues R together with sets of corresponding Fourier coefficients, $c_i(n)$, by solving (*) repeatedly for different R's, and seeking those values of R for which the solution vectors $C = C(R,Y)$ are stable under changes of Y. That is, if $C(R,Y_1) \approx C(R,Y_2)$ (for Y_1 and $Y_2 < Y_0$, for some suitable constant Y_0) we take it as an indication that the R is close to an eigenvalue and that the components of C are close to the corresponding Fourier coefficients. For more details and justifications see [65, sect. 3.2].

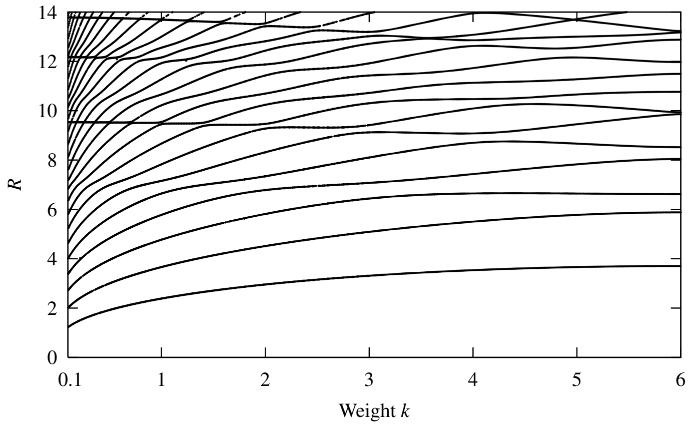
We note that the Phase 2 algorithm (i.e. computation of a larger set of Fourier coefficients) also generalizes to non-zero weight in a similar manner (cf. [65, sect. 3.3]).

8. Numerical results

The experimental excursions have been directed towards three essentially different subjects, but, in each, we have worked in an *exploratory* spirit.

- First we tried to get an over-all picture of the distribution of small to middle-range sized eigenvalues on $PSL(2,\mathbb{Z})$ (and the eta multiplier) for "large" weights, e.g. $R \in [0,14]$ and $k \in [0.1,6]$.
- Second, we continuously "turned off" the multiplier system v_η^{2k} on $PSL(2,\mathbb{Z})$ by letting the weight $k \to 0$ and studied the varying distribution of eigenvalues, $N_k(T)$, as well as the formation of the continuous spectrum.

Figure 1: Section of eigenvalues with $0 < R \leq 14$, and weight $0.1 \leq k \leq 6$.



- There are some cases where arithmeticity plays a role even in nonzero weight. We studied the Shimura correspondence between weight zero forms on $\Gamma_0(2)$ and weight one half forms on $\Gamma_0(4)$. And we also studied weight one forms on $PSL(2,\mathbb{Z})$ which correspond to Hecke eigenforms on $\Gamma_0(144)$ with a Dirichlet character.

8.1. **Varying weight**

The first experiment considered was to tabulate the first few eigenvalues (up to $R = 14$) for $PSL(2,\mathbb{Z})$ and the multiplier system v_η^{2k}, of weight $k \in (0,6]$ (cf. Section 4.4.1). We made the computations for $k \in [0.1, 6]$ with a grid size of 0.01, and the results are presented in Figure 1. We stress here, that the arcs in Figure 1 terminate at $k = 0.1$; it is not excluded (and actually expected) that they might go lower.

For some examples of eigenvalues for "large" weights see Table 1. This data should be compared with data obtained by Mühlenbruch, [46, p. 160], who used a completely different method (with much less accuracy). We note here that as R increases, the negative Fourier coefficients seem to grow rapidly in magnitude (as compared to the positive ones, with the normalization $c(1) = 1$) for "large" weights.

We believe that we have found all eigenvalues with $(R,k) \in [0,14] \times [0.1, 6]$. This belief is founded on the "continuity" of the resulting graphs $R_j(k)$ (cf. Figure 1), where $R_j(k)$ is the j-th eigenvalue at weight k, considered as a function of k. By standard results (e.g. [10, p. 149]) $R_j(k)$ should be a real analytic function in this interval.

Remember that, for $k \geq 0$, the smallest eigenvalue, λ_{min}, corresponds to the function $F(z) = y^{\frac{k}{2}}\eta(z)^{2k}$. A lower bound for the *second smallest* eigenvalue, $\lambda_0(k)$, is discussed in [10, p. 183]. Bruggeman finds two such bounds, both positive, which he calls $\mu_0(k)$ and $\mu_1(k)$ (μ_1 is better than μ_0 in a certain interval $I \subset [0,2]$.) Figure 2 shows a comparison between the R-values corresponding to these bounds and the smallest experimentally found eigenvalues in the interval $k \in [0.1, 6]$; we see that Bruggeman's bounds can be improved quite a bit.

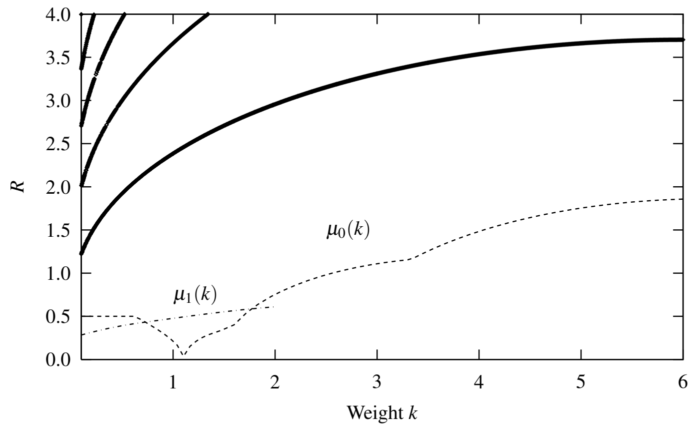
8.2. **Small weights**

The investigation of eigenvalues for small weights k has been done in the interval $R \in [0,20]$, and $k \in \left\{10^{-j} | 1 \leq j \leq 12\right\} \subset \left[10^{-12}, 0.1\right]$. We believe that most cusp forms were found. Let us use the notation

$$\lambda_j(k)$$

for the j-th discrete eigenvalue at weight k, and $\phi_j(k)$ for the corresponding cusp form. It is then a basic fact that $\lambda_j(k)$ depends continuously on k, but it can also be shown that for $k \in (0,12)$ λ_j is even real analytic in k. That is, $\phi_j(k)$ belong to an "analytic family" in the terminology of Bruggeman (see

Figure 2: Comparison with the theoretical lower bounds in $k \in [0.1, 6]$.



[10, 11]). In connection with this, it should also be noted that our experiments support the statement in Observation 71 in [46] that the first few cusp forms at weight 0 do not belong to an analytic family of cusp forms defined in the interval $(-12, 12)$. Indeed we find that in the range considered we actually seem to have $\lambda_j(k) \to 0$ as $k \to 0$ which is consistent with Bruggeman's result (cf. case ii) b) in Prop. 2.17 in [10, p. 149]).

Our experiments indicate that for fixed small k, the "generic" cusp forms $\phi_j(k)$ can be divided into two classes:

- $C(k)$, and
- $E(k)$.

Here $C(k)$ consists of functions $\phi_j(k)$ such that $\lambda_j(k)$ is close to an eigenvalue of a cusp form at weight 0 and the Fourier coefficients are close to the corresponding coefficients of the weight-zero cusp form.

$E(k)$ on the other hand consists of functions $\phi_j(k)$ such that the Fourier coefficients are close to the Fourier coefficients of the Eisenstein series $E(z,s)$ where $\lambda_j(k) = s(1-s)$.

The typical difference between the Fourier coefficients at weight k and weight 0 are in both cases basically of order k; for the forms in $C(k)$, the distance between $\lambda_j(0)$ and the corresponding discrete eigenvalue at weight 0 is much smaller than k.

The "generic" in the statement above means that we exclude certain places where the families $\phi_j(k)$ changes character between $C(k)$ and $E(k)$. In these cases we have a situation of *almost* multiplicity 2, and in too coarse resolution it actually looks like the analytic families intersect.

Weyl's law For non-trivial η-multiplier and a *fixed* non-zero weight $k \in (0, 12)$ on $PSL(2, \mathbb{Z})$ is

$$N_k(T) = \sharp\{R \leq T, \text{weight } k\} = \frac{T^2}{12} - \frac{T}{\pi} \ln \left| 1 - e^{\frac{k\pi}{6} i} \right| + S(T) + O(1), \tag{8.1}$$

for $T \geq 1$ (cf. [22, p. 466] with $\kappa_0 = 0$). Our experiments seem to suggest that as $k \to 0$, the main contribution is proportional to the factor $\ln \left| 1 - e^{\frac{k\pi}{6} i} \right|$, and indeed it is easy show that the $O(1)$ term is even uniformly bounded in k.

To obtain asymptotics for $S(T)$ when $k \to 0$ (and T is bounded) is a bit more involved. We want to generalize [22, thm. 2.29, p. 468] by following the pointers provided in [22, proof of thm. 2.29, p. 468]. Basically, our goal is to single out any terms that grow as $k \to 0$ in the estimates for $S(T)$.

By careful bookkeeping we see that these terms are all of the form $\ln \left| 1 - e^{\frac{k\pi}{6} i} \right|$ and this kind of terms only show up when we apply the functional equation for the logarithmic derivative of the Selberg zeta function, $Z(s)$ ([22, thm. 2.18, p. 441]). To obtain the necessary estimate of $\frac{Z'}{Z}(s)$ we study the integral in the left hand side of [21, prop. 8.6, p. 121]. Let the assumptions of [21, ass. 8.5 p. 120] and [21, p. 125 row 5] apply, e.g. $s=\sigma+iT$, $U=T+10$ and $\frac{1}{2}\leq\frac{1}{2}+\varepsilon=\sigma_1<\sigma$. Through the integral estimates [21, prop. 8.7-8, 8.10-11] we consider either $\frac{Z'}{Z}(\xi)-2\ln\left|1-e^{\frac{k\pi}{6}i}\right|$ or $\frac{Z'}{Z}(\xi)$, depending on whether the functional equation is applied along this part of the path or not and then collect the extra terms at the end. It turns out that we consider the modified integrand in all parts to the left of $\Re(\xi)=\frac{1}{2}-\varepsilon$. With this modification, all estimates in [21, prop. 8.7-8, 8.10-14, thm. 8.15 and 8.17] go through except for the addition of a term $R_m(s)=\frac{2}{\ln x}\ln\left|1-e^{\frac{k\pi}{6}i}\right|I_m(s)$ where $I_m(s)$ is the integral

$$I_m(s) \quad = \quad \frac{1}{2\pi i}\int_\gamma \frac{x^{\xi-s}-x^{2(\xi-s)}}{(\xi-s)^2}d\xi,$$

where the path is $\gamma=\gamma_U=\left[\frac{1}{2}-\varepsilon-iU,-A-iU\right]\cup[-A-iU,-A+iU]\cup\left[-A+iU,\frac{1}{2}-\varepsilon+iU\right]$. The integrand has no poles to the right of $\Re(\xi)=\frac{1}{2}-\varepsilon$ and is hence equal to the integral from $\frac{1}{2}-\varepsilon-iU$ to $\frac{1}{2}-\varepsilon+iU$. Elementary estimates now show that

$$|I_m(s)|\leq\frac{2x^{\left(\frac{1}{2}-\varepsilon-\sigma\right)}}{2\pi}\int_{-U}^{U}\frac{1}{\varepsilon^2+(t-T)^2}dt\leq\frac{x^{-2\varepsilon}}{\pi\varepsilon^2}\int_{-U}^{U}\frac{1}{1+\left(\frac{t-T}{\varepsilon}\right)^2}dt\leq\frac{x^{-2\varepsilon}}{\varepsilon}.$$

Using $x=T^{\frac{1}{4}}$ (cf. [21, p. 138, row -6]) together with $\varepsilon=\frac{1}{\ln x}$ (cf. [21, ass. 8.16(b), p. 130]) we see that

$$|R_m(s)|\leq x^{-\frac{2}{\ln x}}\ln\left|1-e^{\frac{k\pi}{6}i}\right|\ln x\frac{1}{\ln x}=e^{-2}\ln\left|1-e^{\frac{k\pi}{6}i}\right|.$$

This error term is now to be added to $\frac{Z'}{Z}$ in [21, thm. 8.15] and subtracted from the right hand side of [21, thm. 8.17] and finally in [21, p. 135 (8.18)] we need to add $-\int_{\frac{1}{2}}^{2}\Im\left[R_m(s)\right]$ to the expression of $\pi S(T)$. Thus the analogue of [21, thm. 8.1, p. 119] and [22, thm. 2.29, p. 468] is $S(T)=O\left(\frac{T}{\ln T}\right)+O(1)\ln\left|1-e^{\frac{k\pi}{6}i}\right|$ (with implied constants independent of k) and inserted into (8.1) we conclude the following Weyl's law:

$$N_k(T)=\frac{T^2}{12}-\ln\left|1-e^{\frac{k\pi i}{6}}\right|\left[\frac{T}{\pi}+O(1)\right]+O\left(\frac{T}{\ln T}\right), \tag{8.2}$$

uniformly for $0<k\leq 0.1$ (say) and $T\geq 2$ and the $O(1)$ term is bounded in magnitude by $\frac{3}{2\pi}e^{-2}\approx 0.065$.

We have computed $N_k(T)$ experimentally for $k=10^{-j}, j=4,\ldots,12$, and $0\leq T\leq 20$. Figure 3 shows a picture of the experimental values compared to the theoretical values obtained by (8.2) (with the O-terms neglected) and the difference indeed seems to be constant over this range.

From the form of the Weyl's law above, we also see that the successive spacings $\Delta_n(k)=R_{n+1}(k)-R_n(k)$ should look about like $\frac{1}{\frac{dN_k}{dT}}$, i.e. $\Delta_n(k)\approx\frac{1}{\frac{T}{6}+\frac{1}{\pi}\left|\ln\frac{k\pi}{6}\right|}\sim\frac{\pi}{\left|\ln\frac{k\pi}{6}\right|}$ as $k\to 0$. Figure 4 provides a nice illustration of this fact, where it is clearly seen that the average spacings are almost constant for small k and this constant is roughly proportional to $\frac{1}{|\ln k|}$.

The mean gap over 100 typical cases (R_n, $5\leq n\leq 155$) turned out to be 0.117 at $k=10^{-9}$, 0.107 at $k=10^{-10}$, 0.101 at $k=10^{-11}$ and 0.095 at $k=10^{-12}$. The relative quotients, $\Delta_n(k)\left[\frac{R_n}{6}+\frac{1}{\pi}\left|\ln\frac{k\pi}{6}\right|\right]$, are 1.019, 0.990, 1.001 and 0.9934, respectively and it is not inconceivable that one obtains 1 in the (logarithmic) limit.

8.2.1. *Level repulsion.* From figures like 4 one may be tempted to think that there are horizontal lines corresponding to cusp forms at weight 0 which crosses the lines that are going down (i.e. corresponding to the Eisenstein series at weight 0). This is not the case! If we look closer we will see that there is actually "level repulsion" here, i.e. the horizontal "cusp-form-line" turns down before the "near crossing" and

Figure 3: Plot of Weyl's law with constant $T = 20$ and weight $k \to 0$

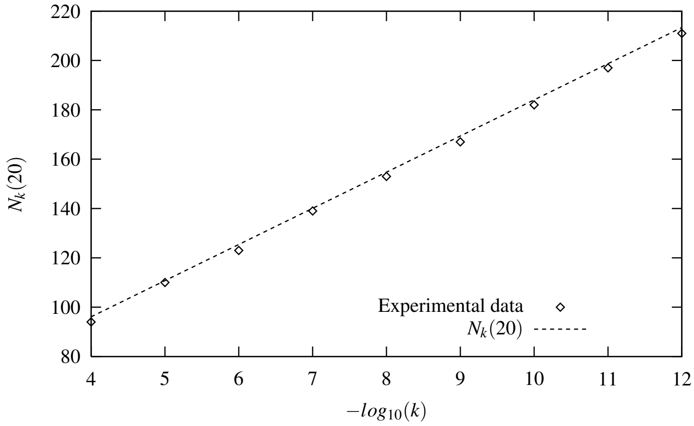
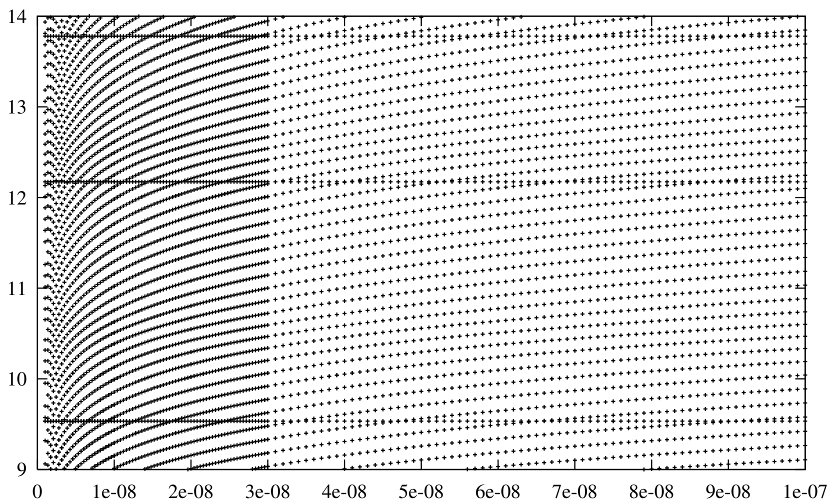


Figure 4: Section of eigenvalues with $9 \le R \le 14$, and $1E-9 \le k \le 1E-7$.



becomes an "Eisenstein-series-line" and the previous "Eisenstein-series-line" turns into a "cusp-form-line". See also Figures 5 and 6. More precisely formulated: if there is a "near crossing" at the weight k_0 close to the eigenvalue $R_0 \approx R_j(k_0) \approx R_{j+1}(k_0)$, then there are two analytic families $\phi_j(k)$ and $\phi_{j+1}(k)$ such that for some $\delta > \varepsilon > 0$:

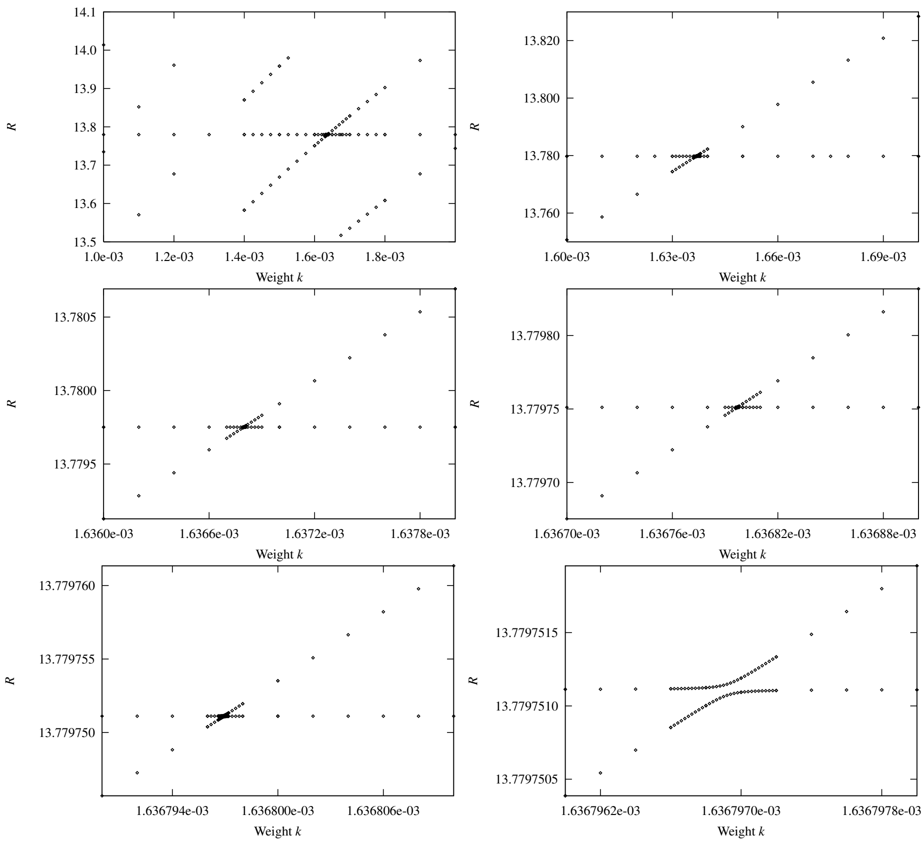
$$\phi_j(k) \in \begin{cases} C(k), & k \in [k_0+\varepsilon, k_0+\delta], \\ E(k), & k \in [k_0-\delta, k_0-\varepsilon], \end{cases}$$

and

$$\phi_{j+1}(k) \in \begin{cases} E(k), & k \in [k_0+\varepsilon, k_0+\delta], \\ C(k), & k \in [k_0-\delta, k_0-\varepsilon], \end{cases}$$

and in the interval $(k_0-\varepsilon, k_0+\varepsilon)$ both families display a mixing between the two types $E(k)$ and $C(k)$. In fact, the Fourier coefficients of ϕ_{j+1} converge (as $k \to k_0-\varepsilon$) to values close to the Fourier coefficients

Figure 5: Level repulsion at $R = 13.779\ldots$



of ϕ_j for $k > k_0 + \varepsilon$ and vice versa. Since the two functions also need to be orthogonal it is clear that the Fourier coefficients exhibit "wild" behavior in the small interval surrounding the "near crossing". Note also that, as $k \to 0$, *all* $\phi_j(k) \in E(k)$ and $R_j(k) \to 0$.

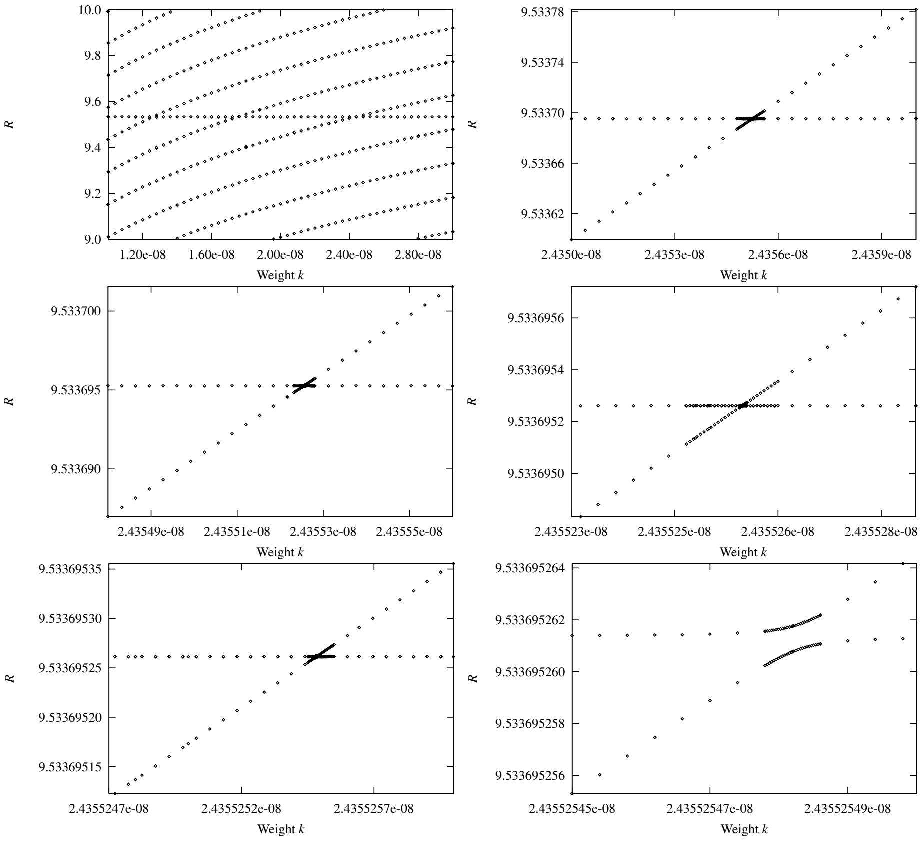
See Table 9 for examples of Fourier coefficients corresponding to eigenfunctions of types $E(k)$ and $C(k)$, close to an avoided crossing at weight $k = 9.044605824E-8$. Table 10 illustrates the agreement between the Fourier coefficients of a more generic cusp form in $E(k)$ and the corresponding coefficients of the Eisenstein series (appropriately normalized) at weight 0. The level of agreement is striking to put it mildly; likewise in Table 9 for the $C(k)$ eigenfunction. The "1 for 1" nature of this convergence *in the presence of a limiting continuous spectrum* seems not to have been suspected earlier. Cf. [24, thm. 6.6 and cor. 6.9] and [23].

The fact that the system seems to avoid accidental degeneracies by means of level repulsion and avoided crossings is in agreement with the Wigner-von Neumann theorem, cf. [72].

8.3. **Lifts at weight 1**

As we saw in Section 4.5.2 we could prove the existence of certain Hecke relations at weight 1 (e.g. (4.19), (4.20) and 4.21)). Tables 2 and 3 contain numerical verifications of these relations. Table 4 contains a list of computed eigenvalues on $\mathcal{M}\left(\Gamma_0(1), v_\eta^2, 1, \lambda\right)$, and the eigenvalues corresponding to cosine CM-forms are indicated. In these cases, we have computed the actual error since we know the exact

Figure 6: Level repulsion at $R = 9.533\ldots$



eigenvalues:

$$R_k = \frac{2\pi k}{\ln\left(7+2\sqrt{12}\right)},\, k \in \mathbb{Z}^+.$$

Note that the actual error is in general much smaller than the error-parameter which is basically $H(Y_1,Y_2) = |c(2)-c'(2)|+|c(3)-c'(3)|+|c(4)-c'(4)|$, where $c(n)$ is computed with Y_1 and $c'(n)$ with Y_2.

8.4. **Half integer weight**

We now consider the case of $\Gamma_0(4)$ and the θ-multiplier system for weight $k=\frac{1}{2}$. The aim of our investigation in this case was to study the Shimura lift, and in particular to investigate the dimensions of the spaces of half integer weight forms. As remarked at the end of section 6 several properties of the Shimura correspondence were observed numerically, and in the original version of these notes, [64, ch. 2], we formulated a number of experimentally inspired conjectures. Except for the question of injectivity these conjectures have now been resolved with Propositions 6.1 and 6.3. (With the obvious correction in the first conjecture: the dimension of $\mathcal{M}(\Gamma_0(4),v_\theta,\frac{1}{2},R)$ corresponding to an oldspace is *at least* two dimensional.)

Tables 5 and 6 contain examples of Fourier coefficients at weight $\frac{1}{2}$. Note especially in table 6 the agreement with Proposition 6.3 c) displayed by the Fourier coefficients $c(n)$ for $n \equiv 5, 1 \bmod 8$ respectively. Here, it is known that 8.92287648699174 and 12.09299487507860 on $\Gamma_0(2)$ correspond to the eigenvalues 1 and -1, respectively, with respect to the involution ω_2.

Table 8 contains a comparison of Fourier coefficients computed both from forms on $\Gamma_0(2)$ via (6.1) and computed directly. Additional Fourier coefficients for the weight 0 forms are available in Table 7.

Acknowledgement. This paper is based on Chapter 2 of my Ph.D. thesis [64] but with additional theoretical discussions in e.g. Section 6 and 4.5. I am grateful to my advisor Dennis Hejhal and my second advisor Andreas Strömbergsson for many giving discussions and lots of valuable comments and suggestions.

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Table 1: Eigenvalues for $N = 1$, $v = v_\eta$

$k = 5.0$			$k = 5.25$		
R	$\lvert c(-1)\rvert$[a]	$H(y1, y2)$	R	$\lvert c(-1)\rvert$[a]	$H(y1, y2)$
3.66240686691	2E+3	8E-09	3.68037312372	3E+3	3E-08
5.77698688079	6E+3	1E-09	5.82067054942	9E+3	6E-09
6.64285171609	1E+4	2E-09	6.63460520751	2E+4	3E-09
7.82634704661	7E+4	8E-07	7.90867228426	2E+5	9E-09
8.66620831839	8E+4	1E-08	8.61646891946	1E+5	8E-09
9.45156176778	4E+4	4E-09	9.56930344151	7E+4	5E-09
10.21802876612	9E+4	2E-08	10.15656706121	2E+5	2E-09
10.65897262925	2E+5	5E-09	10.70911890024	2E+5	4E-10
11.27526358329	2E+5	2E-08	11.34046324165	4E+5	2E-08
12.15792337439	5E+6	2E-06	12.11839521329	4E+6	2E-06
12.55403510011	3E+5	3E-09	12.65021958486	4E+5	9E-09
13.00123950671	4E+4	2E-09	13.02622821839	8E+4	2E-09
13.67542640619	8E+5	8E-09	13.56022943627	1E+6	1E-07
13.71353384347	4E+6	4E-07	13.87057635696	7E+5	4E-07
14.47039277248	6E+5	1E-09	14.48204838116	1E+6	7E-05
15.03845367363	1E+6	6E-09	15.09966704087	4E+7	1E-07
15.39856858318	1E+6	1E-09	15.38981845044	1E+6	4E-09
15.85705128856	7E+4	7E-09	15.94059443942	1E+4	3E-09
16.14536205683	5E+6	2E-07	16.09999759486	8E+6	7E-06
16.45061260141	2E+6	1E-08	16.52671557073	5E+6	1E-09
16.93043847901	2E+7	2E-08	16.90856097808	2E+7	5E-08
17.51562192888	2E+5	4E-09	17.53730159778	1E+6	3E-10
17.59022138300	1E+6	6E-10	17.74142373355	7E+6	4E-09
18.13826107361	7E+6	2E-08	18.02022951826	6E+6	9E-10
18.32637702289	5E+5	5E-09	18.37970066644	8E+5	2E-09
18.76341585136	2E+6	5E-09	18.90587158951	1E+7	8E-09
19.16629116326	4E+6	8E-10	19.09726131554	1E+7	5E-10
19.67214438521	3E+7	2E-07	19.66894569593	5E+6	3E-10
19.68520099819	1E+6	3E-10	19.73195996101	7E+6	4E-10
20.00524829746	2E+6	4E-09	20.12609436572	2E+6	3E-10
20.38266630653	3E+6	3E-10	20.35571778301	1E+7	7E-10
20.67297062056	6E+6	4E-08	20.71020380483	6E+6	8E-09
20.97339376061	6E+6	8E-10	20.88321504381	1E+7	2E-09

[a] The normalization we have used here is the usual $c(1) = 1$.

Table 2: Fourier coefficients for a CM-form $f \in \mathcal{M}(\Gamma_0(1), v_\eta^2, 1, 4.770984191561)$

n		$c(n)/c(0)$[a]	Error
0	1.755930576575		
1	1.000000000000		
2	−1.755930576574	$c(27)$	0.4E-08
3	1.571810322167	$c(40)$	0.1E-08
4	1.755930576575	$c(53)$	0.7E-08
5	−1.770268323978	$c(66)$	0.3E-09
6	−2.474798320759	$c(79)$	0.1E-08
7	0.000000000000	$c(92)$	0.4E-08
8	0.346240855507	$c(105)$	0.5E-08
9	3.510179255561	$c(118)$	0.3E-08
10	1.116593241680	$c(131)$	0.4E-08
11	−0.000000000001	$c(144)$	0.3E-08
12	0.000000000001	$c(157)$	0.1E-07
13	−3.019229958496	$c(170)$	0.8E-08
14	−1.186431979458	$c(183)+c(1)$	0.3E-08
15	−3.079783541463	$c(196)$	0.5E-08
		$-c(n)c(-1)/R^2/c(0)$	
−1	5.055064268188	[b]	0.1E-11
−2	−11.180067729976	$c(21)$	0.4E-07
−3	0.000000000001	$c(32)$	0.2E-07
−4	16.472675660354	$c(43)$	0.2E-08
−5	16.729030199659	$c(54)$	0.5E-07
−6	13.098490835617	$c(65)$	0.5E-08
−7	16.046414105740	$c(76)$	0.1E-07
−8	−0.000000000023	$c(87)$	0.1E-07
−9	13.340740291248	$c(98)$	0.3E-07
−10	0.000000000006	$c(109)$	0.4E-07
−11	2.151215034503	$c(120)$	0.7E-08
−12	2.878852009050	$c(131)-c(1)$	0.2E-07
−13	0.000000000039	$c(142)$	0.6E-08
−14	7.496795049955	$c(153)$	0.4E-07
−15	−12.830881007618	$c(164)$	0.4E-07

[a] This quotient is deduced from formula (4.16) or (4.17) on p. 14.

[b] $c(-1)^2 = -R^2 c(0)(c(10) - c(0))$

Table 3: Fourier coefficients for a non-CM-form $f \in \mathcal{M}(\Gamma_0(1), v_\eta^2, 1, 3.66240686698667)$

n		$c(n)/c(0)$[a]	Error
0	-1.352193685534		
1	1.000000000000		
2	-1.697113317091	$c(27)$	0.5E-08
3	-0.057989599353	$c(40)$	0.4E-07
4	2.461764397786	$c(53)$	0.1E-07
5	0.510856433057	$c(66)$	0.6E-08
6	-0.952325903762	$c(79)$	0.8E-08
7	-1.660630683908	$c(92)$	0.2E-07
8	-2.343382246022	$c(105)$	0.9E-08
9	1.271097206907	$c(118)$	0.1E-07
10	-0.203512820511	$c(131)$	0.6E-08
11	2.110622602834	$c(144)$	0.5E-08
12	2.170616908700	$c(157)$	0.4E-08
13	0.449799127363	$c(170)$	0.4E-07
14	0.612654661780	$c(183)+c(1)$	0.4E-07
15	-1.684453740441	$c(196)$	0.1E-07
16	0.400312170289	$c(209)$	0.2E-07
17	-2.868395110060	$c(222)$	0.1E-07
18	-1.931595991172	$c(235)$	0.2E-07
19	-0.591212766919	$c(248)$	0.1E-07
20	0.792151138999	$c(261)$	0.9E-08
21	-1.717242193922	$c(274)$	0.3E-07
22	1.369169138277	$c(287)$	0.2E-07
23	2.007854712832	$c(300)$	0.8E-09
24	0.447826147902	$c(313)$	0.1E-07
25	3.051006373828	$c(326)$	0.4E-08
26	-0.032419986064	$c(339)$	0.6E-08
27	1.255081531820	$c(352)+a(2)$	0.2E-07
28	-0.707047087424	$c(365)$	0.3E-07
29	1.272283355260	$c(378)$	0.1E-07
30	-0.184187214400	$c(391)$	0.2E-07

[a] This quotient is deduced from formula (4.16) or (4.17) on p. 14.

Table 4: Eigenvalues for $\mathcal{M}(PSL(2,\mathbb{Z}),1,v_\eta^2)$

R	$H(y_1,y_2)$	True error [a]
2.38549209578045	1E-12	1E-15 [b]
3.66240686698667	1E-11	
4.77098419156091	1E-13	1E-14 [b]
5.77698688078694	6E-12	
6.64285171613711	1E-11	
7.15647628734173	1E-11	4E-13 [b]
7.82634704540775	1E-13	
8.66620831896793	7E-14	
9.45156176783224	7E-12	
9.54196838312186	2E-13	6E-14 [b]
10.21802876776059	3E-13	
10.65897262920241	2E-12	
11.27526358349387	2E-13	
11.92746047890219	7E-11	5E-14 [b]
12.15792337422149	7E-13	
12.55403509998720	1E-13	
13.00123950642372	9E-13	
13.67542640643589	2E-13	
13.71353384358095	1E-12	
14.31295257468268	1E-12	1E-14 [b]
14.47039277253940	1E-12	
15.03845367358721	2E-13	
15.39856858348441	1E-12	
15.85705128717333	3E-10	
16.14536205734475	3E-11	
16.45061260131967	4E-11	
16.69844467046304	4E-12	1E-13 [a]
16.93043847896222	4E-15	
17.51562192885174	2E-10	
17.59022138305996	1E-10	
18.13826107340244	3E-11	
18.32637702205910	4E-11	
18.76341585146817	1E-11	

[a] For CM-forms, the true error is computed with respect to the eigenvalue $R_k = \frac{2\pi k}{\ln(\eta_0)}$, where $\eta_0 = 7+2\sqrt{12}$.

[b] These forms correspond to CM-forms.

Table 5: Fourier coefficients for
$f_{1,2} \in \mathcal{M}(\Gamma_0(4), \frac{1}{2}, 6.889875675945)$

Fourier coefficients for $f_1 \in V^+$, observe that $a(n) = 0$ for $n \equiv 2, 3 \bmod 4$

n	$a(n)$	n	$a(-n)$
4	0.84219769675471	3	1.01825299171456
5	0.18355821406443	4	2.18968040385979
8	0.56907998524429	7	2.06305218095270
9	−0.33045049673565	8	−1.17157116610978
12	−0.41296169213831	11	−1.02718719694121
13	0.60153537988057	12	2.29759751514531
16	0.30482066289397	15	−4.08935474990375
17	−0.88689598690620	16	3.39248165496032
20	0.41418282092059	19	−1.31804633824673
21	0.50212917023175	20	1.87725570455517
23	−0.00000000000110	23	−2.22265197818114
24	−1.04429548341249	24	0.58816620330140
25	0.28984678984391		

$$|c(4)c(9) - c(36)| = 0.2E - 08$$

Fourier coefficients for $f_2 \notin V^+$, observe that $a(n) = 0$ for $n \equiv 1 \bmod 8$

n	$a(n)$	n	$a(n)$
0	0.000000000000	19	−0.936283350934
1	0.000000000000	20	0.192087703124
2	1.000000000000	21	1.247834928335
3	−0.725665465042	22	0.411443268212
4	−0.710754741008	23	−0.018564690418
5	0.456158224759	24	−1.297582830232
6	−1.835059236817	25	0.000000000000
7	0.481289183972	26	−0.610656711780
8	0.707106781187	27	−0.179166638197
9	0.000000000000	28	0.340322845698
10	−0.651049038069	29	0.001563849679
11	−0.470914040036	30	1.462745418892
12	−0.513122971204	31	0.400763051265
13	1.494868058150	32	0.095523702475
14	0.968484734380	33	0.000000000000
15	−0.945563574521	34	−0.879782299796
16	−1.101175502961	35	−1.833603623066
17	0.000000000000	36	0.234869257223
18	0.824250041644		

Table 6: Fourier coefficients for $f \in \mathcal{M}\left(\Gamma_0(4), \frac{1}{2}, R\right)$

$R = 4.461438243496$

n	$a(n)$
0	0.00000000000000
1	1.00000000000000
2	0.63334968449036
3	0.63517832947402
4	−0.70710678118667
5	−0.00000000000003
6	1.28035706400142
7	−0.90756258916698
8	−0.44784585676555
9	0.52643872643776
10	−0.57763498966972
11	1.12485377915641
12	−0.44913890403389
13	0.00000000000001
14	0.48078071833327
15	−1.58539012005784
16	0.50000000000015
17	−0.14882069214483
18	1.06474902295605
19	0.21268916863632
20	0.00000000000003
21	0.00000000000002
22	−1.56248056455209
23	0.85478501960318
24	−0.90534916229569
25	0.45696099733973
36	−0.37224839333969

$|c(4)c(9) - c(36)| = 4E-12$
$|c(21)| = 2E-14$

$R = 6.046497437542$

n	$a(n)$
0	0.000000000000
1	0.000000000000
2	1.000000000000
3	1.770795863371
4	0.000000000029
5	1.331470494003
6	−0.749395507674
7	0.074369313625
8	0.707106781209
9	−0.000000000016
10	−1.214451367832
11	0.620756243833
12	1.252141763204
13	−1.123686021586
14	−0.711286031630
15	0.964706032413
16	−0.000000000112
17	−0.000000000108
18	−0.128644484609
19	−0.499664871594
20	0.941491814843
21	−0.447011950327
22	−0.805501416928
23	1.236716356721
24	−0.529902643583
25	0.000000008756
36	−0.000000000168
288	−0.064322242377

$|c(2 \cdot 3^2)c(2 \cdot 4^2) - c(2 \cdot 12^2)| = 8E-11$
$|c(17)| = 1E-10$

Table 7: Supplemental table of Fourier coefficients for $\mathcal{M}(\Gamma_0(2), 0, R)$

R	8.92287648699174	R	12.0929948750786
n	$A(n)$	n	$A(n)$
2	−0.70710678118654	2	0.70710678118655
3	1.10378899562734	3	−0.70599475399569
4	0.49999999999993	4	0.49999999999999
5	0.90417459283958	5	−0.79974825694039
6	−0.78049668380711	6	−0.49921367803249
7	0.82934246755499	7	−1.71337067862845
8	−0.35355339059330	8	0.35355339059328
9	0.21835014686776	9	−0.50157140733054
10	−0.63934798597339	10	−0.56550741572467

R	13.77975135189073 (even wrt $z \mapsto -\frac{1}{2z}$)	R	13.77975135189073 (odd wrt $z \mapsto -\frac{1}{2z}$)
n	$A(n)$	n	$A(n)$
2	2.96351804031448	2	0.13509091556820
3	0.24689977245401	3	0.24689977245398
4	3.59139177031902	4	−0.79070303958101
5	0.73706038534834	5	0.73706038534830
6	0.73169192981688	6	0.03335391631439
7	−0.26142007576500	7	−0.26142007576538
8	2.60064131148226	8	−1.36013067551284
9	−0.93904050235826	9	−0.93904050236089
10	2.18429174878080	10	0.09957016228575

Table 8: Comparison of Fourier coefficients $A(n)$ computed directly for $\mathcal{M}(\Gamma_0(N),0,R)$ $(N=1,2)$, vs $\hat{A}(n)$ computed on $\mathcal{M}\left(\Gamma_0(4),\frac{1}{2},\frac{1}{2}R\right)$ and using (6.1).

n	$\hat{A}(n)$	$A(n)$	$\lvert A(n)-\hat{A}(n)\rvert$
	$f\in\mathcal{M}(\Gamma_0(2),0,8.922876486992)$ $(t=1)$		
2	-0.70710678118665	-0.707106781186	$0.6E-12$
3	1.10378899562739	1.103788995627	$0.4E-12$
5	0.90417459283969	0.904174592840	$0.3E-12$
	$f\subseteq\mathcal{M}(\Gamma_0(1),0,13.779751351891)$ $(t=1)$		
2	1.54930447794126	1.549304477941	$0.7E-12$
3	0.24689977245398	0.246899772454	$0.3E-12$
4	1.40034436536892	1.400344365369	$0.2E-12$
5	0.73706038534387	0.737060385348	$0.4E-11$
6	0.38252292109716	0.382522923066	$0.2E-08$
	$f\in\mathcal{M}(\Gamma_0(1),0,13.779751351891)$ $(t=2)$		
3	0.24689977245437	0.246899772454	$0.8E-13$
5	0.73706038535004	0.737060385348	$0.2E-11$
7	-0.26142007624377	-0.261420075765	$0.5E-09$
9	-0.93904050238904	-0.939040502362	$0.3E-10$
	$f\in\mathcal{M}(\Gamma_0(2),0,12.092994875079)$ $(t=2)$		
3	-0.70599475379863	-0.705994753996	$0.2E-09$
5	-0.79974825694696	-0.799748256940	$0.7E-11$
7	-1.71337067860377	-1.713370678628	$0.2E-10$
9	-0.50157140750090	-0.501571407330	$0.2E-09$

In segment 3, the calculation is based on (6.1) and the second portion of Table 5.

Table 9: Comparison of Fourier coefficients for weights $k = 9.044605824E-08$ and $k = 0$ near an "avoided crossing".

Corresponds to the cusp form

k	$9.0446058240E-08$	0	Difference
R	13.77975135189074	13.77975135189074	
$c(2)$	1.54930480559976	1.54930447794069	$0.3E-06$
$c(3)$	0.24689988546553	0.24689977245411	$0.1E-06$
$c(4)$	1.40034433555250	1.40034436536841	$0.1E-06$
$c(5)$	0.73706067260516	0.73706038534787	$0.2E-06$
$c(6)$	0.38252272069428	0.38252292306557	$0.2E-06$

Corresponds to the Eisenstein series

k	$9.0446058240E-08$	0	Difference
R	13.77975135138225	13.77975135138225	
$c(2)$	-2.06525760334129	-1.98398933080188	$0.8E-01$
$c(3)$	-1.72891679536648	-1.68449330640991	$0.4E-01$
$c(4)$	2.97153986917404	2.93621366473571	$0.4E-01$
$c(5)$	-2.02747287754385	-1.96531634618530	$0.6E-01$
$c(6)$	3.41008729668221	3.34201674772446	$0.7E-01$

The coefficients for the Eisenstein series at $k = 0$ were computed using (5.1), i.e.:

$c(2) = 2\cos(R\ln 2)$

$c(3) = 2\cos(R\ln 3)$

$c(4) = 1 + 2\cos(R\ln 4)$

$c(5) = 2\cos(R\ln 5)$

$c(6) = 2\cos(R\ln 6) + 2\cos(R(\ln 3 - \ln 2))$

Table 10: Comparison of Fourier coefficients for weights $k = 9.044605824E-08$ and $k = 0$ "far" from an "avoided crossing". The weight 0 coefficients were computed using the formulas in Table 9.

k	$9.0446058240E-08$	0	Difference
R	13.62696884857618	13.62696884857618	
$c(2)$	-1.99957085683552	-1.99957081810438	$0.4E-07$
$c(3)$	-1.48069687587703	-1.48069680342062	$0.7E-07$
$c(4)$	2.99828354611637	2.99828345661464	$0.7E-07$
$c(5)$	-1.99647405201235	-1.99647406885962	$0.2E-07$
$c(6)$	2.96075820067617	2.96075811858031	$0.8E-07$