COMPUTATION OF SELBERG ZETA FUNCTIONS ON HECKE TRIANGLE GROUPS

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ABSTRACT. In this paper, a heuristic method to compute the Selberg zeta function for Hecke triangle groups, G_q is described. The algorithm is based on the transfer operator method and an overview of the relevant background is given. We also present some numerical results obtained by implementing the algorithm.

CONTENTS

1. INTRODUCTION

The Selberg zeta function, $Z_\Gamma(s)$, for a co-finite Fuchsian group Γ plays an important role in the spectral theory or harmonic analysis on the corresponding orbifold $\mathscr{M} = \Gamma\backslash\mathscr{H}$, a surface with constant negative curvature. Selberg's [25] motivation to introduce $Z_\Gamma(s)$ was the similarity between a trace formula he developed (cf. in particular [25, p. 74]), now called the *Selberg trace formula* and Weil's explicit formula [29]. The role of the Riemann zeta function $\zeta(s)$ in the latter is analogous to the role of $Z_\Gamma(s)$ in the former. For a more detailed account of this motivation see Hejhal [9] (in particular sections 4-6). Since then the Selberg trace formula has been worked out in detail for $PSL_2(\mathbb{R})$ (by e.g. Hejhal [10, 11]) and the properties of $Z_\Gamma(s)$ has been extensively studied in many other contexts.

Despite the importance of $Z_\Gamma(s)$ and the fact that one can obtain an abundance of its properties through the Selberg trace formula, numerical studies of its behavior inside the critical strip, $|\Re s| \leq \frac{1}{2}$ have been surprisingly scarce in the literature. The main reason is of course the fact that the defining formula does not represent an analytic function in this domain so one is forced to, one way or the other, analytically continue this expression.

To the authors knowledge, even for the simple case of the modular surface, the only successful numerical evaluation of Z_Γ on the critical line was made by Matthies and Steiner

[19]. They overcame the difficulty by desymmetrizing the modular surface $\mathscr{H}/PSL_2(\mathbb{Z})$ with respect to reflection in the imaginary axis and then restricting their analysis to the odd part, which conveniently avoids any interference by the continuous part of the spectra. For this system, corresponding to a billiard with Dirichlet boundary conditions, they consider a modified Selberg Zeta function, $Z_-(s)$, which has a Dirichlet series representation which seems to be conditionally convergent up to $\Re s = \frac{1}{2}$. For convex, co-compact Schottky groups Guillopé, Lin and Zworski [8] presented numerical results for $Z(s)$ in a large range of $\Im s$. They use a method based on transfer operators and due to the co-compactness they are able to evaluate the related Fredholm determinants in a more or less straight-forward manner in terms of fixed points of the corresponding maps (cf. e.g. also Jenkinson and Pollicott [15]).

In this paper, we also consider an approach to the Selberg zeta function using a transfer operators. This method is applied to the family of Fuchsian groups known as Hecke triangle groups, generalizing the modular group. These groups have finite area but are not co-compact, so the evaluation of the corresponding Fredholm determinants is more involved.

It can not be stressed too much that at least one of the steps in our proposed method is not entirely rigorous but rather supported by heuristic arguments and the entire method is supported by numerical evidence.

2. Hyperbolic geometry and Hecke surfaces

Let $\mathscr{H} = \{z \in \mathbb{C} \,|\, \Im z > 0\}$ be the hyperbolic upper half-plane together with the metric given by $ds = \frac{|dz|}{y}$, the group of isometries of $\mathscr{H}$ is $PSL_2(\mathbb{R}) \cong SL_2(\mathbb{R})/\{\pm I_2\}$ where $SL_2(\mathbb{R})$ is the group of 2×2 real matrices with determinant 1 and I_2 is the 2×2 identity matrix. Elements of $PSL_2(\mathbb{R})$ acts on $\mathscr{H}$ via Möbius transformations. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL_2(\mathbb{R})$ then $z \mapsto \frac{az+b}{cz+d}$ and we say that we say that g is *elliptic*, *hyperbolic* or *parabolic* depending on whether $|\operatorname{Tr} g| = |a+d| < 2$, > 2 or $= 2$. The same notation applies for fixed points of g. A parabolic fixed point is a degenerate fixed point, belongs to $\partial\mathscr{H}$ and is usually called a cusp. Elliptic points z appear in pairs, one belongs to $\mathscr{H}$ and the other one is in the lower half-plane $\overline{\mathscr{H}}$ and its stabilizer subgroup Γ_z in Γ is cyclic of finite order m. Hyperbolic fixed points appear also in pairs with $x, x^* \in \partial\mathscr{H}$, where x^* is said to be the conjugate point of x. A geodesics γ on $\mathscr{H}$ is either a half-circle orthogonal to $\mathbb{R}$ or a line parallel to the imaginary axis and the endpoints of γ are denoted by $\gamma_\pm \in \partial\mathscr{H}$.

Let $\pi : \mathscr{H} \to \mathscr{M} = \Gamma\backslash\mathscr{H}$ be the natural projection map, i.e. $\pi(z) = \Gamma z$ then $\gamma^* = \pi(\gamma)$ is a closed geodesic on $\mathscr{M}$ if and only if each $\gamma \in \pi^{-1}(\gamma^*)$ has endpoints which are conjugate hyperbolic fixed points. This gives a one-to-one correspondence between hyperbolic conjugacy classes in Γ, i.e. the set $\{[P] \,|\, P \in \Gamma, |\operatorname{Tr} P| > 2\}$ where $[P] = \{APA^{-1} \,|\, A \in \Gamma\}$. It is known that any hyperbolic element P can be written as a power of a primitive hyperbolic element, P_0, i.e. $P = P_0^m$ for some $m \geq 1$. We denote this integer by $m(P)$. In terms of closed geodesics on $\mathscr{M}$ this means that every closed geodesic has a minimal length obtained by traversing it once only. We can now define the Selberg zeta function for Γ as

$$Z_\Gamma(s) = \prod_{[P_0]\in\mathscr{H}_q^0} \prod_{k\geq 0} \left(1 - \mathscr{N}(P_0)^{-k-s}\right) \tag{1}$$

where $\mathscr{H}_q^0$ is the set of primitive hyperbolic conjugacy classes in Γ, P_0 is a representative in this class with $\operatorname{Tr} P_0 > 2$ and the norm of P, $\mathscr{N}(P)$ is the solution of $|\operatorname{Tr} P| = \mathscr{N}^{\frac{1}{2}} + \mathscr{N}^{-\frac{1}{2}}$ with $1 < \mathscr{N} < \infty$. We observe that $\mathscr{N}(P_0^n) = \mathscr{N}(P_0)^n$ and since the trace is invariant

under conjugation $\mathscr{N}$ is constant over conjugacy classes. If γ is the geodesic corresponding to P then the length of γ is $l(\gamma) = \ln \mathscr{N}(P)$. For $\Re s > 1$ the logarithm of $Z_\Gamma(s)$ can be written

$$
\begin{aligned}
(2) \qquad -\ln Z_\Gamma(s) &= -\sum_{k\geq 0}\sum_{[P_0]\in\mathscr{H}_q^0} \ln\left(1-\mathscr{N}(P_0)^{-k-s}\right) \\
&= \sum_{k\geq 0}\sum_{[P_0]\in\mathscr{H}_q^0}\sum_{n\geq 1}\frac{1}{n}\mathscr{N}(P_0)^{-kn-sn} \\
&= \sum_{[P_0]\in\mathscr{H}_q^0}\sum_{n\geq 1}\frac{1}{n}\mathscr{N}(P_0)^{-sn}\frac{1}{1-\mathscr{N}(P_0)^{-n}} \\
&= \sum_{[P_0]\in\mathscr{H}_q^0}\sum_{n\geq 1}\frac{1}{n}\frac{\mathscr{N}\left(P_0^n\right)^{-s}}{1-\mathscr{N}\left(P_0^n\right)^{-1}} = \sum_{[P]\in\mathscr{H}_q}\frac{1}{m(P)}\frac{\mathscr{N}(P)^{-s}}{1-\mathscr{N}(P)^{-1}}.
\end{aligned}
$$

For an integer $q \geq 3$ the *Hecke triangle group* G_q is generated by the maps $S : z \mapsto -\frac{1}{z}$ and $T : z \mapsto z+\lambda_q$ where $\lambda_q = 2\cos\left(\frac{\pi}{q}\right) \in [1,2)$. Let $I_q = \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]$. One can show (cf. e.g. [17, VII]) that G_q is a Fuchsian group (discrete subgroup of $PSL_2(\mathbb{R})$) with the only relations $S^2 = (ST)^q = Id$ and which has $\mathscr{F}_q = \left\{z \in \mathscr{H} \mid \Re z \in I_q, |z| \geq 1\right\}$ as a closed fundamental domain (with sides properly pair-wise identified). It follows that G_q is co-finite, meaning that the *Hecke triangle surface,* $\mathscr{M}_q = G_q \backslash \mathscr{H}$, has finite hyperbolic area. In the following we usually write λ for λ_q, $\mathscr{H}_q$ and $\mathscr{H}_q^0$ denotes the set of hyperbolic respectively primitive hyperbolic conjugacy classes in G_q.

3. Symbolic Coding

In [26] we showed that the geodesic flow on the unit tangent bundle of $\mathscr{M}_q$, $\mathscr{T}^1\mathscr{M}_q \cong \mathscr{M}_q \times S^1$ can be coded in terms of regular λ-fractions (nearest λ-multiple continued fractions). If $\{x\}_\lambda = \left\lfloor \frac{x}{\lambda} + \frac{1}{2} \right\rfloor$ is a nearest λ-multiple function we define $F_q : I_q \to I_q$ by $F_q(0) = 0$ and $F_q(x) = -\frac{1}{x} - \{x\}_\lambda \lambda$ for $x \neq 0$. For any number $x \in \mathbb{R}$ we obtain the regular λ-fraction of x, $\mathfrak{c}_q(x) = \left[a_0; a_1, a_2, \ldots\right]$ by first setting $a_0 = \{x\}_\lambda$, $x_1 = x - a_0\lambda$, and then recursively set $a_n = \{Sx_n\}_\lambda$ and $x_{n+1} = F_q(x_n)$ for $n \geq 1$. Note that $x = \lim_{n\to\infty} T^{a_0}ST^{a_1}\cdots ST^{a_n}(0)$. If x is a cusp of G_q this algorithm terminates and we get a finite λ-fraction and if x is a hyperbolic fixed point of G_q then it has an eventually periodic λ-fraction. It follows that F_q acts as a left shift map on $\mathscr{A}_q$, the set of regular λ-fractions viewed as a subset of $\mathbb{Z}^{\mathbb{N}}$. If $a_0 = 0$ we usually omit the leading "a_0;", repetitions in the λ-fraction are denoted by powers and infinite repetitions by overlines.

In [20] it was shown that F_q is almost orbit equivalent to G_q, that is, two points $x, y \in \mathbb{R}$ are equivalent under the action of G_q if and only if, either they have regular λ-fractions with the same tail or x has the same tail as r and y the same tail as $-r$ (or vice versa). Here $r \in I_q$ is a special hyperbolic point which can be given either in terms of its regular λ-fraction or explicitly. For even q one has $r = 1-\lambda$ and $\mathfrak{c}_q(r) = \left[\overline{1^{h-1}, 2}\right]$ with $h = \frac{q-2}{2}$ and for odd q one has $r = R - \lambda$ where R is the positive solution of $R^2 + (2-\lambda)R - 1 = 1$ and $\mathfrak{c}_q(r) = \left[\overline{1^h, 2, 1^{h-1}, 2}\right]$ with $h = \frac{q-3}{2}$.

Let $\mathscr{P}$ be the set of all purely periodic regular G_q-inequivalent λ-fractions and set $\mathscr{P}^r = \mathscr{P} \backslash \{-r\}$, i.e. the set of purely periodic regular λ-fractions with tail not equivalent to $-r$. Let $\mathscr{P}_k^r$ denote the subset with minimal period $k \geq 1$ and set $\mathscr{P}_0^r = \cup_{k\geq 1}\mathscr{P}_k^r$.

It was also shown in [26] that for the part of the geodesic flow not disappearing into the cusp there exists a cross section Σ and a first return map $\mathscr{T}:\Sigma\to\Sigma$ which has as a factor map in the expanding direction the map $\mathscr{T}_x : I_q \to I_q$ given by powers of the generating map of the nearest λ-multiple fractions, F_q. Closed geodesics on $\mathscr{M}_q$ correspond to the orbits of fixed points of $\mathscr{T}$ and it is easy to verify that these correspond in fact to fixed points of F_q, i.e. points with purely periodic regular λ-fractions. It follows that there is a one-to-one correspondence between $\mathscr{P}_0^r$ and $\mathscr{H}_q^0$.

In practice, if $\mathfrak{c}_q(x) = \left[\overline{a_1,\ldots,a_n}\right]$ then $A_{\vec{a}} = ST^{a_1}\cdots ST^{a_n} \in G_q$ is hyperbolic, has attractive fixed-point x, repelling fixed-point x^* and the geodesic $\gamma(x,x^*)$ is closed. Furthermore $y=\frac{1}{x^*}$ has dual regular λ-fraction (cf. [26]) $\mathfrak{c}_q^*(y) = \left[\overline{a_n,a_{n-1},\ldots,a_1}\right]^*$ and $y\in[-R,-r]\operatorname{sgn}(x)$.

This connection (coding) between primitive hyperbolic conjugacy classes and periodic orbits of F_q is precisely what allows us to relate the Fredholm determinant of the transfer operator for F_q to the Selberg zeta function.

4. The transfer operator

In this section we will construct the so-called transfer operator for the map F_q defined in the previous section.

4.1. Markov partitions.

There is a particular Markov partition of I_q with respect to F_q which is important here, namely the one determined by the orbit $\left\{F_q^j\left(\pm\frac{\lambda}{2}\right), j=1\ldots,\kappa\right\}$ of the endpoints $\pm\frac{\lambda}{2}$ under F_q. Let $\left\{\overline{\mathscr{I}}_j\right\}_{j\in\mathscr{J}_\kappa}$ be the decomposition of $\left[-\frac{\lambda}{2},\frac{\lambda}{2}\right]$ determined by this orbit with $\mathscr{J}_\kappa=\{1,2,\ldots,\kappa,-\kappa,\ldots,-2,-1\}$, $\mathscr{I}_j=\left[\phi_{j-1},\phi_j\right)=-\mathscr{I}_{-j}$ where the order of $\mathscr{O}\left(-\frac{\lambda}{2}\right)=\left\{F_q^j\phi_0\right\}=\left\{\phi_j\right\}_{j=0}^{\kappa}$ given as $-\frac{\lambda}{2}=\phi_0<\phi_1<\cdots<\phi_\kappa=0$ and $\phi_{-j}=-\phi_j$. If q is even $\kappa=\frac{q-2}{2}=h$ and $-\frac{\lambda}{2}=\left[1^h\right]$. If q is odd $\kappa=\frac{q-3}{2}=2h+1$ and $-\frac{\lambda}{2}=\left[1^h,2,1^h\right]$. It is easy to verify that the closure of the intervals, $\left\{\overline{\mathscr{I}}_j\right\}$ is indeed a Markov partition of I_q for F_q. Let $\varphi_n(y)=ST^ny=\frac{-1}{n\lambda+y}$ then the most important property of the partition $\left\{\mathscr{I}_j\right\}$ is the fact that if $y\in\mathscr{I}_j$ then ${F_q}^{-1}(y)=\left\{\varphi_n(y)\,|\,n\in\mathscr{N}_j\right\}$ where $\mathscr{N}_j$ is a fixed set of integers depending only on j. It is now easy to show that if $l\geq 1$, $i\in\mathscr{J}_\kappa$ and $y\in\mathscr{I}_i$ then

$$\left({F_q}^{-1}\right)^l(y)=\bigcup_{j\in\mathscr{J}_\kappa}\left\{\varphi_{n_l}\circ\ldots\circ\varphi_{n_2}\circ\varphi_{n_1}(y)\,\middle|\,(n_1,\ldots,n_l)\in\mathscr{N}_{ij}^l\right\}$$

where we define $\mathscr{N}_{ij}^l:=\left\{(n_1,n_2,\ldots,n_l)\in\mathbb{Z}^l\,\middle|\,ST^{n_l}ST^{n_{l-1}}\cdots ST^{n_1}\mathscr{I}_i\subset\mathscr{I}_j\right\}$. Let $\mathbb{Z}_{\geq m}=\{j\in\mathbb{Z}\,|\,j\geq m\}$ and for $A\subseteq\mathbb{Z}$ let $-A=\{j\in\mathbb{Z}\,|\,-j\in A\}$. It is shown in [21] that $\mathscr{N}_{ij}\in\{\{1\},\{2\},\mathbb{Z}_{\geq 2},\mathbb{Z}_{\geq 3}\}$ for $i,j\in\mathscr{J}_\kappa$, $j\geq 0$ and that $\mathscr{N}_{i-j}=-\mathscr{N}_{-ij}$. It is also shown that the non-empty elements of $\mathscr{N}_{ij}$ (for $j\geq 0$) are given by the following expressions:

$$\begin{aligned}
&\mathscr{N}_{1,2h}=\{2\},\mathscr{N}_{1,2h+1}=\mathbb{Z}_{\geq 3},\mathscr{N}_{-1,2h}=\mathscr{N}_{-2,2h}=\{1\},\\
&\mathscr{N}_{i,i-2}=\mathscr{N}_{-i,2h}=\{1\},\mathscr{N}_{i,2h+1}=\mathscr{N}_{-i,2h+1}=\mathbb{Z}_{\geq 2},\,3\leq i\leq 2h+1,\\
&\mathscr{N}_{2,2h+1}=\mathscr{N}_{-1,2h+1}=\mathscr{N}_{-2,2h+1}=\mathbb{Z}_{\geq 2}
\end{aligned}$$

if q is odd and

$$\mathscr{N}_{1,h}=\mathbb{Z}_{\geq 2},\ \mathscr{N}_{-1,h}=\mathbb{Z}_{\geq 1},$$
$$\mathscr{N}_{i,i-1}=\{1\},\mathscr{N}_{i,h}=\mathbb{Z}_{\geq 2},\mathscr{N}_{-i,h}=\mathbb{Z}_{\geq 1},\,2\leq i\leq h$$

if q is even. For example

$$(\mathscr{N}_{ij})=\begin{pmatrix}\mathbb{Z}_{\geq 3} & -\mathbb{Z}_{\geq 2}\\ \mathbb{Z}_{\geq 2} & -\mathbb{Z}_{\geq 3}\end{pmatrix},\text{ for } q=3,\text{ and }(\mathscr{N}_{ij})=\begin{pmatrix}\mathbb{Z}_{\geq 2} & -\mathbb{Z}_{\geq 1}\\ \mathbb{Z}_{\geq 1} & -\mathbb{Z}_{\geq 2}\end{pmatrix}\text{ for } q=4.$$

4.1.1. *Transfer Operator corresponding to F_q.* For any interval $I\subset\mathbb{R}$ let $C(I)$ denote the space of continuous real-valued functions on I. If $f\in C(I_q)$ the *transfer*, or *generalized Perron-Frobenius operator* $\mathscr{L}_\beta$ corresponding to F_q, acts for real $\beta>\frac{1}{2}$ on f by

$$\begin{aligned}\mathscr{L}_\beta f(x) &= \sum_{y\in F_q^{-1}(x)}\left|\frac{d}{dx}F_q^{-1}(x)\right|^\beta f(y(x))\\ &= \sum_{i\in\mathscr{J}_\kappa}\chi_{\mathscr{I}_i}(x)\sum_{n\in\mathscr{N}_i}\left|\varphi_n'(x)\right|^\beta f(\varphi_n(x))\end{aligned}$$

where $\chi_{\mathscr{I}_j}$ is the characteristic function of $\mathscr{I}_j$. It is important to note here, that $\mathscr{L}_\beta f(x)$ is in general not continuous, but only piece-wise continuous. For this reason consider the action of $\mathscr{L}_\beta$ on vector-valued functions in $\mathscr{C}=\bigoplus_{i\in\mathscr{J}_\kappa}C(\mathscr{I}_i)$. For $\vec{f}\in\mathscr{C}$ we set $\vec{f}(x)=f_i(x)$ if $x\in\mathscr{I}_i$. We can now write $\mathscr{L}_\beta:\mathscr{C}\to\mathscr{C}$ as

$$\left(\mathscr{L}_\beta\vec{f}\right)_i(x) = \sum_j\sum_{n\in\mathscr{N}_{ij}^1}\left|\varphi_n'(x)\right|^\beta f_j(\varphi_n(x)),\ i\in\mathscr{J}_\kappa$$

respectively, for any $l\geq 1$

$$\left(\mathscr{L}_\beta^l\vec{f}\right)_i(x) = \sum_j\mathscr{L}_{\beta,ij}^l f_j(x),\ i\in\mathscr{J}_\kappa,$$

where

$$\mathscr{L}_{\beta,ij}^l f_j(x)=\sum_{(n_1,\ldots,n_l)\in\mathscr{N}_{ij}^l}\left|(ST^{n_1}\cdots ST^{n_l})'x\right|^\beta f_j(ST^{n_1}\cdots ST^{n_l}x).$$

It is convenient to use a composition operator π_β related to the principal series representation of $PSL_2(\mathbb{R})$: Define $\pi_\beta(A)f(x):=|A'(x)|^\beta f(Ax)=|cx+d|^{-2\beta}f\left(\frac{ax+b}{cx+d}\right)$ for $A\in PSL_2(\mathbb{R})$. Note that $\pi_\beta(AB)=\pi_\beta(B)\pi_\beta(A)$. With this notation one gets

$$\mathscr{L}_{\beta,ij}^l f_j(x)=\sum_{(n_1,\ldots,n_l)\in\mathscr{N}_{ij}^l}\pi_\beta(ST^{n_1}\ldots ST^{n_l})f_j(x).$$

To obtain better spectral properties for the operator $\mathscr{L}_\beta$, we have to restrict its domain of definition even more. For any open disk D in $\mathbb{C}$ we let $\mathscr{B}(D)$ be the Banach space of functions holomorphic in D and continuous on the closure $\overline{D}$ together with the supremum norm. Let $\{D_i\}_{i\in\mathscr{J}_\kappa}$ be a set of open disks with diameter which contains an ε-neighborhood of $\mathscr{I}_i$, constructed in such a way that for $(n_1,\ldots,n_l)\in\mathscr{N}_{ij}^l$ one has $\varphi_{n_1}\circ\varphi_{n_2}\circ\cdots\circ\varphi_{n_l}\left(\overline{D}_i\right)\subset D_j$. That such a choice is possible is shown in [21]. Let $\mathscr{B}_i=\mathscr{B}(D_i)$ and define the Banach space $\mathscr{B}=\bigoplus_{i\in\mathscr{J}_\kappa}\mathscr{B}_i$ with norm given by $\left\|\vec{f}\right\|=\max_j\left\|f_j\right\|_{\mathscr{B}_i}$. Then we want to consider $\mathscr{L}_\beta$ as acting $\mathscr{L}_\beta:\mathscr{B}\to\mathscr{B}$. For this purpose we also need an analytic extension of π_β. If $A=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in PSL_2(\mathbb{R})$ and $A(D)\subseteq D$ for some disk D then $\pi_\beta(A):\mathscr{B}(D)\to\mathscr{B}(D)$ is defined for any $\beta\in\mathbb{C}$ by $\pi_\beta(A)f(z)=\left((cz+d)^{-2}\right)^\beta f\left(\frac{az+b}{cz+d}\right)$. Usually we simply write

the first factor as $(cz+d)^{-2\beta}$, but remember that there is a choice of sign involved, i.e. $(-cz-d)^{-2\beta} = (cz+d)^{-2\beta}$. For $l \geq 1$ and $\vec{f} \in \mathscr{B}$:

$$\left(\mathscr{L}_\beta^l \vec{f}\right)_i (z) = \sum_{j \in \mathscr{J}_\kappa} \mathscr{L}_{\beta,ij}^l f_j(z), \; i \in \mathscr{J}_\kappa \text{ with} \tag{3}$$

$$\mathscr{L}_{\beta,ij}^l f_j(z) = \sum_{(n_1,\ldots,n_l) \in \mathscr{N}_{ij}^l} \pi_\beta \left(ST^{n_1} \cdots ST^{n_l}\right) f_j(z).$$

We now have a representation of the operator $\mathscr{L}_\beta$ as a $(\kappa+1)\times(\kappa+1)$ matrix of operators $\left(\mathscr{L}_{\beta,ij}\right)_{i,j \in \mathscr{J}_\kappa}$ with $\mathscr{L}_{\beta,ij} : \mathscr{B}_j \to \mathscr{B}_i$.

Next we need some facts from Grothendieck's theory of Fredholm determinants and nuclear operators on Banach spaces [7] (Ruelle [24] provides more detailed references). The following Lemmas follow from this theory.

Lemma 1. *Let D be any open disk in $\mathbb{C}$ and let $\mathscr{B}(D)$ be as above. If $\Psi : \mathscr{B}(D) \to \mathscr{B}(D)$ is a simple composition operator $\Psi f(z) = \psi(z) f(\varphi(z))$ with ψ, φ continuous in D and $\overline{\varphi(D)} \subset D$. Then φ has an attractive fixed-point $z_* \in D$, Ψ is nuclear of order zero and has trace $Tr_{\mathscr{B}(D)} \Psi = \frac{\psi(z_*)}{1-\varphi'(z_*)}$.*

The formula for the trace, sometimes referred to as a special case of the Atiyah-Bott trace formula is easy to verify directly since the eigenvalues of Ψ are all of the form $\mu_n = \psi(z_*)(\varphi'(z_*))^n$, $n \geq 0$ and $|\varphi'(z_*)| < 1$.

Lemma 2. *If $\mathscr{L}$ is a nuclear operator of order zero on a Banach space we can express the Fredholm determinant* $\det(1-\mathscr{L})$ *in two different ways:*

$$-\log\det(1-\mathscr{L}) = \sum_{l=1}^{\infty} \frac{1}{l} Tr\, \mathscr{L}^l = -\log \prod_{j=1}^{\infty} (1-\lambda_j)$$

where $\left\{\lambda_j\right\}_{j=1}^{\infty}$ are the eigenvalues of $\mathscr{L}$ (counted with multiplicity). Furthermore, if $\mathscr{L} = \mathscr{L}(s)$ is a meromorphic function of s then $\det(1-\mathscr{L}(s))$ *is also meromorphic in s.*

Proof. Cf. e.g. [7, prop. 1, pp. 346-347]. □

Lemma 3. *Let $A \in SL_2(\mathbb{R})$ be hyperbolic with attractive and repelling fixed points x_+ and x_- respectively. If D is a disk with diameter on $\mathbb{R}$ containing only the attractive fixed point of A then $\overline{A(D)} \subset D$.*

Proof. This Lemma is easy to verify by conjugating with the map in $SL_2(\mathbb{R})$ which takes x_+ to 0, x_- to $i\infty$ and A to $z \mapsto l^2 z$ with $0 < l < 1$. □

If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is hyperbolic with attractive fixed point x_+ it is easy to verify that $\mathscr{N}(A) = j_A(x_+)^2$ where $j_A(x) = cx+d$. Since $\pi_\beta(A) f(x) = j_A(x)^{-2\beta} f(Ax)$ it is easy to see that if $x_+ \in D$ and $x_- \notin D$ then by Lemma 1 $\pi_\beta(A)$ is nuclear of order zero and

$$\mathrm{Tr}_{\mathscr{B}(D)} \pi_\beta(A) = \frac{\mathscr{N}(A)^{-\beta}}{1-\mathscr{N}(A)^{-1}}.$$

Let $\vec{n} = (n_1,\ldots,n_l) \in \mathscr{N}_{jj}^l$ and set $A_{\vec{n}} = ST^{n_1}ST^{n_2}\cdots ST^{n_l}$. Then $A_{\vec{n}}\overline{\mathscr{I}}_j \subsetneq \mathscr{I}_j$ so the attractive fixed point of $A_{\vec{n}}$, $x_{\vec{n}} = \left[\overline{n_1,\ldots,n_l}\right] \in \mathscr{I}_j$ and by Lemma 3 it is clear that $A_{\vec{n}}\left(\overline{D}_j\right) \subset D_j$. This demonstrates that all composition operators showing up in the operators $\mathscr{L}_{\beta,jj}^l$ appearing in the trace of $\mathscr{L}_\beta^l$ are nuclear of order zero for $\Re\beta > \frac{1}{2}$. The arguments in [22] or

[14] can be generalized to show that $\mathscr{L}_\beta^l$ is also of trace class and nuclear of order zero for $\Re\beta > \frac{1}{2}$. It is clear that $\operatorname{Tr}_{\mathscr{B}}\mathscr{L}_\beta^l = \sum_{i\in\mathscr{J}_\kappa} \operatorname{Tr}_{\mathscr{B}_i}\mathscr{L}_{\beta,ii}^l$ for any $l \geq 1$ and it is also not hard to see that any off-diagonal term, $\mathscr{L}_{\beta,ij}^l : \mathscr{B}_j \to \mathscr{B}_i$ is a bounded operator. One can now use similar arguments as those in [5] to show the following lemma.

Lemma 4. If $\Re\beta > \frac{1}{2}$ then $\mathscr{L}_\beta$ is nuclear of order zero and hence of trace class.

From the identification of hyperbolic conjugacy classes with purely periodic λ-fractions we may now calculate the trace of $\mathscr{L}_\beta^l$

$$\begin{aligned}\operatorname{Tr}_{\mathscr{B}}\mathscr{L}_\beta^l &= \sum_i \operatorname{Tr}_{\mathscr{B}_i}\mathscr{L}_{ii,\beta}^l = \sum_j \sum_{\vec{n}\in\mathscr{N}_{ii}^l} \operatorname{Tr}_{\mathscr{B}_i}\pi_\beta\left(A_{\vec{n}}\right) \\ &= \sum_{\left[\overline{n_1,n_2,\ldots,n_l}\right]\in\mathscr{P}_l} \frac{\mathscr{N}\left(ST^{n_1}\cdots ST^{n_l}\right)^{-s}}{1-\mathscr{N}\left(ST^{n_1}\cdots ST^{n_l}\right)^{-1}}\end{aligned}$$

By the standard Grothendieck theory the Fredholm determinant of $1-\mathscr{L}_\beta$ is well-defined and can be calculated by

$$\det\left(1-\mathscr{L}_\beta\right) = \sum_{l=1}^\infty \frac{1}{l}\operatorname{Tr}\mathscr{L}_\beta^l = \sum_{l=1}^\infty \frac{1}{l}\sum_{\left[\overline{n_1,n_2,\ldots,n_l}\right]\in\mathscr{P}_l} \frac{\mathscr{N}\left(A_{\vec{n}}\right)^{-\beta}}{1-\mathscr{N}\left(A_{\vec{n}}\right)^{-1}}$$

We now need to study the relation between λ-fractions and hyperbolic conjugacy classes in more detail. Let $x = \left[\overline{n_1,\ldots,n_l}\right]$ correspond to the hyperbolic $A = A_{\vec{n}} = ST^{n_1}\cdots ST^{n_l} = P_0^m$ where P_0 is a primitive hyperbolic and $m = m(A)$ then $x = \left[\overline{n_1,\ldots,n_{l_0}}\right]$ where l_0 is the minimal period and $l_0 m = l$. Furthermore, all shifts, $\left[\overline{n_i,n_{i+1},\ldots,n_{i-1}}\right]$ for $1 \leq i \leq n_{l_0}$ belong to the same conjugacy class $[A]$ and the norm is also constant over conjugacy classes. Let $\mathscr{B}(x) = \mathscr{B}(D_j)$ where $x \in D_j$, set $A_r r = r$ and $\mathscr{K}_\beta = \pi_\beta(A_r)$. Then

$$\begin{aligned}&-\log\det\left(1-\mathscr{L}_\beta\right) + \log\det\left(1-\mathscr{K}_\beta\right) \\ &= \sum_{l=1}^\infty \frac{1}{l}\sum_{\vec{n}\in\mathbb{Z}^l,\, x=\left[\overline{n_1,n_2,\ldots,n_l}\right]\in\mathscr{P}} \operatorname{Tr}_{\mathscr{B}_{j(x)}}\pi_\beta\left(A_{\vec{n}}\right) - \sum_{l=1}^\infty \frac{1}{l}\operatorname{Tr}_{\mathscr{B}(r)}\pi_\beta\left(A_r^l\right) \\ &= \sum_{l=1}^\infty \frac{1}{l}\sum_{\vec{n}\in\mathbb{Z}^l,\, x=\left[\overline{n_1,n_2,\ldots,n_l}\right]\in\mathscr{P}^r} \operatorname{Tr}_{\mathscr{B}_{j(x)}}\pi_\beta\left(A_{\vec{n}}\right) \\ &= \sum_{l_0=1}^\infty \frac{1}{l_0}\sum_{\vec{n}_0\in\mathbb{Z}^{l_0},\, x=\left[\overline{n_1,\ldots,n_{l_0}}\right]\in\mathscr{P}^r} \frac{1}{m(P)}\operatorname{Tr}_{\mathscr{B}_{j(x)}}\pi_\beta\left(A_{\vec{n}_0}^{m(P)}\right) \\ &= \sum_{l_0=1}^\infty \frac{1}{l_0}\sum_{[P_0]\in\mathscr{H}_q^0} \frac{l_0}{m(P)}\operatorname{Tr}_{\mathscr{B}_{j(\vec{n}_0)}}\pi_\beta\left(P_0^m\right) \\ &= \sum_{[P]\in\mathscr{H}_q} \frac{1}{m(P)}\frac{\mathscr{N}(P)^{-s}}{1-\mathscr{N}(P)^{-1}}\end{aligned}$$

and by comparing with (2) we see that for $\Re\beta > 1$ we have $\ln Z_q(\beta) = \log\det\left(1-\mathscr{L}_\beta\right) - \log\det\left(1-\mathscr{K}_\beta\right)$ and thus

$$Z_q(\beta) = \frac{\det\left(1-\mathscr{L}_\beta\right)}{\det\left(1-\mathscr{K}_\beta\right)}. \tag{4}$$

But since the right hand side is in fact meromorphic for $\beta \in \mathbb{C}$ this equation provides an analytic continuation of $Z(\beta)$ for $\beta \in \mathbb{C}$. Note that the operator $\mathscr{K}_\beta$ is a simple composition operator and we can evaluate $\det\left(1-\mathscr{K}_\beta\right)$ explicitly. It is easy to show that all eigenvalues of $\mathscr{K}_\beta$ are of the form $\mu_n = (2+\lambda R)^{-2(n+\beta)}$, $n \geq 0$ and hence

$$\det\left(1-\mathscr{K}_\beta\right) = \prod_{n\geq 0}\left(1-\mu_n\right). \tag{5}$$

To evaluate the factor $\det\left(1-\mathscr{L}_\beta\right)$ we use the same identity $\det\left(1-\mathscr{L}_\beta\right) = \prod_{n\geq 1}\left(1-\lambda_n\right)$ where $\{\lambda_n\}_{n\geq 1}$ are the eigenvalues of $\mathscr{L}_\beta$ counted with multiplicity. In the next chapter we will discuss how to calculate eigenvalues of $\mathscr{L}_\beta$.

Remark 1. The method to relate $Z_\Gamma(s)$ to Fredholm determinants of nuclear operators can be extended to any finite dimensional representation χ of Γ. The identity (4) will hold with Z_Γ, $\mathscr{L}_\beta$ and $\mathscr{K}_\beta$ replaced by $Z_\Gamma^\chi(s) = \prod_{[P_0]}\prod_k \det\left(1-\chi(P)\mathscr{N}(P)^{-s-k}\right)$, respectively $\mathscr{L}_\beta^\chi$ and $\mathscr{K}_\beta^\chi$. Here $\mathscr{L}_\beta^\chi$ and $\mathscr{K}_\beta^\chi$ are obtained by replacing $\pi_\beta(A)$ with $\pi_\beta^\chi(A) = \chi(A)\pi_\beta(A)$ in all formulas. The only problem is to obtain explicit expressions for the truncated operator $\mathscr{A}_\beta^{(N)}$ which will be introduced in the next section. The algorithm has been implemented and tested for representations induced by the trivial representation of the Hecke congruence subgroups $\Gamma_0(p)$ with prime p. This allowed us to compute e.g. $Z_{\Gamma_0(p)}(s)$ for $p = 2,5$.

5. Analytic Continuation of $\mathscr{L}_\beta$ and Computation

It turns out that the same analysis which enables us to deduce an analytic continuation of $\mathscr{L}_\beta$ to $\beta \in \mathbb{C}$ is also vital to compute the eigenvalues of $\mathscr{L}_\beta$.

We follow the same procedure as in e.g. [23, 6] to demonstrate that $\mathscr{L}_\beta$ admits a meromorphic extension to the whole complex plane. First of all we have to change domains once more. To make some of the calculations easier it is desirable to work with functions which have power series expansions around zero. For this purpose we choose open disks $\tilde{D}_i \supset D_i$ such that $0 \in \tilde{D}_i$ and $ST^n\tilde{D}_i \subset \tilde{D}_j$ for $n \in \mathscr{N}_{ij}$. That this construction is possible is shown in [21]. If $\tilde{\mathscr{B}}_i = \mathscr{B}\left(\tilde{D}_i\right)$ then $f_i \in \tilde{\mathscr{B}}_i$ has a power series expansion centered at zero.

If $i \in \mathscr{J}_\kappa$ it can be shown that either $\mathscr{N}_{ij} = \left\{n_{ij} \in \mathbb{Z}\setminus\{0\}\right\}$ or $\mathscr{N}_{ij} = \emptyset$ for $1 \leq j \leq \kappa-1$, $\mathscr{N}_{i\kappa} = \{n \in \mathbb{Z} \,|\, n \geq n_{i\kappa}\}$ for some $n_{i\kappa} \geq 1$ and $\mathscr{N}_{i,-j} = -\mathscr{N}_{-i,j}$. Let $1 \leq i \leq \kappa$ and consider $\mathscr{L}_{\beta,ij} : \tilde{\mathscr{B}}_j \to \tilde{\mathscr{B}}_i$. Let $f \in \tilde{\mathscr{B}}_j$ and $N \geq 1$. Taylor's theorem with remainder gives $f(z) = \sum_{k=0}^N a_k z^k + R_N(z)$ with $R_N(z) = O\left(|z|^{N+1}\right)$. Then

$$\begin{aligned}
\mathscr{L}_{\beta,i\kappa} f(z) &= \sum_{n\in\mathscr{N}_{i\kappa}} \pi_\beta(ST^n) f(z) = \sum_{n\geq n_{i\kappa}} \pi_\beta(ST^n) f(z) \\
&= \sum_{n\geq n_{i\kappa}} \left(\frac{1}{z+n\lambda}\right)^{2\beta} f\left(\frac{-1}{z+n\lambda}\right) \\
&= \sum_{n\geq n_{i\kappa}} \left(\frac{1}{z+n\lambda}\right)^{2\beta} \left[\sum_{k=0}^N a_k \left(\frac{-1}{z+n\lambda}\right)^k + R_N\left(\frac{1}{z+n\lambda}\right)\right] \\
&= \mathscr{A}_{\beta,i\kappa}^{(N)} f(z) + \mathscr{L}_{\beta,i\kappa}^{(N)} f(z)
\end{aligned}$$

where $\mathscr{L}_{\beta,i\kappa}^{(N)} f(z) = \mathscr{L}_{\beta,i\kappa}\left[f(z) - \sum_{k=1}^N a_k z^k\right]$ is analytic for $\Re\beta > \frac{1-N}{2}$ and in fact nuclear of order 0. We also have $\left\|\mathscr{L}_{\beta,i\kappa}^{(N)} f(z)\right\|_\infty \leq C\sum_{n\geq n_{i\kappa}} \left|\frac{1}{n\lambda}\right|^{2\Re\beta+N+1} \to 0$ as $N \to \infty$. The

operator $\mathscr{A}_{\beta,i\kappa}^{(N)}$ on the other hand can be written as

$$\begin{aligned}\mathscr{A}_{\beta,i\kappa}^{(N)}f(z) &= \sum_{k=0}^{N}(-1)^k a_k \sum_{n\geq n_{i\kappa}}\left(\frac{1}{z+n\lambda}\right)^{2\beta+k}\\ &= \sum_{k=0}^{N}(-1)^k a_k \lambda^{-2\beta-k}\zeta\left(2\beta+k,\frac{z}{\lambda}+n_{i\kappa}\right)\end{aligned}$$

where $\zeta(s,z)$ is the Hurwitz zeta function. It is known that for any $z\in\mathbb{C}$ the function $\zeta(s,z)$ is meromorphic with only one simple pole at $s=1$ with residue 1. Hence $\mathscr{A}_{\beta,i\kappa}^{(N)}$ is of finite rank and meromorphic in β with at most simple poles at the points $\beta_k=\frac{-k+1}{2}$, $0\leq k\leq N$. For the operator corresponding to $\mathscr{N}_{ij}=\{n_{ij}\}$ one has

$$\begin{aligned}\mathscr{L}_{\beta,ij}f(z) &= \pi_\beta(ST^{n_{ij}})f(z)=(z+n_{ij}\lambda)^{-2\beta}f\left(\frac{-1}{z+n_{ij}\lambda}\right)\\ &= (z+n_{ij}\lambda)^{-2\beta}\left[\sum_{k=0}^{N}\left(\frac{-1}{z+n_{ij}\lambda}\right)^k+R_N\left(\frac{-1}{z+n_{ij}\lambda}\right)\right]\\ &= \mathscr{A}_{\beta,ij}^{(N)}f(z)+\mathscr{L}_{\beta,ij}^{(N)}f(z)\end{aligned}$$

where $\mathscr{A}_{\beta,ij}^{(N)}f(z)=\sum_{k=0}^{N}a_k(-1)^k(z+n_{ij})^{-k-2\beta}$ and $\mathscr{L}_{\beta,ij}^{(N)}f(z)=(z+n_{ij}\lambda)^{-2\beta}R_N\left(\frac{-1}{z+n_{ij}\lambda}\right)=O\left(\left|z+n_{ij}\lambda\right|^{-N-1-2\Re\beta}\right)$. It is clear that in this case $\mathscr{A}_{\beta,ij}^{(N)}$ is entire of finite rank and that $\mathscr{L}_{\beta,ij}^{(N)}$ is entire and nuclear of order 0.

Since $N\geq 1$ was arbitrary, we conclude that all components $\mathscr{L}_{\beta,jk}$, have meromorphic continuations to the entire complex plane with at most simple poles at the points $\beta_k=\frac{-k+1}{2}$, $k=1,\ldots$. The same clearly holds true for the operator $\mathscr{L}_\beta$. Note, that in the determinant $\det\left(1-\mathscr{L}_\beta\right)$ poles may well cancel against zeros due to the presence of eigenvalues equal to one.

5.1. **Computation of $\mathscr{A}_\beta^{(N)}$.** Let $\mathscr{L}_\beta=\mathscr{A}_\beta^{(N)}+\mathscr{L}_\beta^{(N)}$ where $\mathscr{A}_\beta^{(N)}$ and $\mathscr{L}_\beta^{(N)}$ have the components $\mathscr{A}_{\beta,jk}^{(N)}$ respectively $\mathscr{L}_{\beta,jk}^{(N)}$ given above. To obtain a numerical approximation of $\det\left(1-\mathscr{L}_\beta\right)$ it is necessary to approximate the spectrum of $\mathscr{L}_\beta$. For this purpose we construct another finite rank approximation of $\mathscr{A}_\beta^{(N)}$ in terms of a matrix which is more suitable for computations.

Let $\mathscr{P}_N$ be the space of polynomials of degree less than or equal to N. Then $\mathscr{P}_N$ is a subspace of all $\mathscr{B}_i$'s and we let Π_N denote the projection $\mathscr{B}_i\to\mathscr{P}_N$ given by truncation of the power series, i.e. $\Pi_N\left(\sum_{k=0}^{\infty}a_kz^k\right)=\sum_{k=0}^{N}a_kz^k$. We will also use Π_N to denote the projection from $\mathscr{B}=\bigoplus_{i\in\mathscr{J}_\kappa}\mathscr{B}_i$ to the space $\bigoplus_{i\in\mathscr{J}_\kappa}\mathscr{P}_N$ obtained by truncating each component.

We saw, that $\mathscr{A}_\beta^{(N)}$ maps $\bigoplus_{i\in\mathscr{J}_\kappa}\mathscr{P}_N=:\mathscr{P}_N^{2\kappa}$ into a space spanned by Hurwitz zeta functions. By applying Π_N to the resulting expression we obtain an operator $\mathscr{A}_\beta^{(N,N)}:\mathbb{C}^{2\kappa(N+1)}\to\mathbb{C}^{2\kappa(N+1)}$ which can be represented by a $\kappa_N\times\kappa_N$ complex matrix where $\kappa_N=2\kappa(N+1)$. This construction will now be explained in detail.

Let $N \geq 1$ be a fixed integer, then with the notation as above

$$\begin{aligned} \mathscr{A}_{\beta,i\kappa}^{(N)} f(z) &= \sum_{k=0}^{N} a_k \frac{(-1)^k}{\lambda^{k+2\beta}} \zeta\left(k+2\beta, \frac{z}{\lambda}+n_{i\kappa}\right) \\ &= \sum_{k=0}^{N} a_k \frac{(-1)^k}{\lambda^{k+2\beta}} \sum_{n=0}^{\infty} \frac{(-1)^n (k+2\beta)_n}{n!\lambda^n} \zeta(2\beta+k+n, n_{i\kappa}) z^n \\ &= \sum_{n=0}^{\infty} z^n \sum_{k=0}^{N} a_k \alpha_{i\kappa,nk} \end{aligned} \tag{6}$$

where $\alpha_{i\kappa,nk} = \alpha_{i\kappa,nk}(\beta) = \frac{(-1)^{k+n}}{n!\lambda^{n+k+2\beta}} (k+2\beta)_n \zeta(2\beta+k+n, n_{i\kappa})$. For $i \in \mathscr{J}_\kappa$ and $1 \leq j \leq \kappa-1$ we get

$$\begin{aligned} \mathscr{A}_{\beta,ij}^{(N)} f(z) &= \sum_{k=0}^{N} a_k (-1)^k (z+n_{ij}\lambda)^{-2\beta-k} \\ &= \sum_{k=0}^{N} a_k (-1)^k \sum_{n=0}^{\infty} \frac{(-1)^n (2\beta+k)_n}{n!(n_{ij}\lambda)^{2\beta+k+n}} z^n \\ &= \sum_{n=0}^{\infty} z^n \sum_{k=0}^{N} a_k \alpha_{ij,nk} \end{aligned}$$

where $\alpha_{ij,nk} = \alpha_{ij,nk}(\beta) = \frac{(-1)^{n+k}}{n!\lambda^{2\beta+k+n}} (2\beta+k)_n n_{ij}^{-2\beta-k-n}$. For $n \leq -1$ we use a slightly modified definition of $\pi_\beta(ST^n)$, namely $\pi_\beta(ST^{-n}) f(z) = \left((-z+n\lambda)^{-2}\right)^\beta f\left(\frac{1}{-z+n\lambda}\right)$. It is then easy to see that $\alpha_{ij,nk} = (-1)^{n+k} \alpha_{i-j,nk}$ for $1 \leq j \leq \kappa$.

By truncating the sum over n at N in the formula for $\mathscr{A}_{\beta,ij}^{(N)}$ we get operators $\mathscr{A}_{\beta,ij}^{(N,N)} : \mathscr{P}_N \to \mathscr{P}_N$ and $\mathscr{A}_\beta^{(N,N)} = \left(\mathscr{A}_{\beta,ij}^{(N,N)}\right)_{i,j\in\mathscr{J}_\kappa}$. Then $\mathscr{A}_\beta^{(N)} = \mathscr{L}_\beta \circ \Pi_N$ and $\mathscr{A}_\beta^{(N,N)} = \Pi_N \circ \mathscr{L}_\beta \circ \Pi_N$ and by the identification $\mathscr{P}_N^{2\kappa} \cong \mathbb{C}^{2\kappa(N+1)}$ it is clear that $\mathscr{A}^{(N,N)} : \mathscr{P}_N \to \mathscr{P}_N$ can be represented by the $\kappa_N \times \kappa_N$ complex matrix

$$A = \left(\alpha_{ij,nk}\right)_{i,j\in\mathscr{J}_\kappa, 0\leq n,k\leq N}.$$

In the next section we will discuss the relation between the eigenvalues of $\mathscr{L}_\beta$ and those of $\mathscr{A}_\beta^{(N,N)}$.

5.2. **Approximation of the spectrum of $\mathscr{L}_\beta$.** Let Σ_β denote the spectrum of the operator $\mathscr{L}_\beta$ and Σ_β^N the spectrum of $\mathscr{A}_\beta^{(N,N)}$. Since $\mathscr{L}_\beta$ is nuclear, setting $\Pi^N = Id - \Pi_N$ then Π^N is bounded and it is easy to verify, that the conditions of Theorem 2 and Proposition 3 in Baladi and Holschneider [2] are satisfied by the approximations $\mathscr{A}_\beta^{(N,N)}$, $N \geq 1$. Hence the following Lemma can be deduced:

Lemma 5. *Let $\vec{f} \in \mathscr{B}$ be an eigenfunction of $\mathscr{L}_\beta$ corresponding to the eigenvalue $\lambda_\beta \in \Sigma_\beta$ with algebraic multiplicity d. Then there exists $N_0 \geq 0$ such that for all $N \geq N_0$ there exist eigenvalues $\lambda_{N,j} \in \Sigma_\beta^N$ with corresponding eigenfunctions $\vec{f}_{N,j}$, $1 \leq j \leq l$ such that the sum of the algebraic multiplicities of $\lambda_{N,j}$ equals d and*

$$\max_{1\leq j\leq l} \left(\left|\lambda - \lambda_{N,j}\right|, \left\|\vec{f} - \vec{f}_{N,j}\right\|\right) \leq c(N),$$

where $c(N) \to 0$ as $N \to \infty$.

Definition 1. If $\lambda_{N,j} \in \Sigma_\beta^N$ is one of the eigenvalues in Lemma 5 approximating a $\lambda_\beta \in \Sigma_\beta$ then $\lambda_{N,j}$ is said to be *regular*, otherwise it is said to be *spurious*.

A problem in computing the spectra of $\mathscr{L}_\beta$ using $\mathscr{A}_\beta^{(N,N)}$ is that we do not know a priori which eigenvalues of $\mathscr{A}_\beta^{(N,N)}$ are regular and which are spurious. A trivial consequence of Lemma 5 is the following Lemma which gives a necessary condition for a sequence of eigenvalues $\lambda_{N_i,j_i} \in \Sigma_\beta^{N_i}$ to be regular.

Lemma 6. *Let $\left\{\lambda_{N_i,j_i}\right\}_{i\geq 1}$ be a sequence of eigenvalues of $\mathscr{A}_\beta^{(N_i,N_i)}$ such that $\lambda_{N_i,j_i} \to \lambda \in \Sigma_\beta$ as $j \to \infty$. Then for any $\varepsilon > 0$ there exists $N_0 \geq 0$ such that $\left|\lambda_{N_i,j_i} - \lambda_{M_i,j_i}\right| < \varepsilon$ for all $N_i, M_i \geq N_0$.*

Remark 2. Let $\Sigma_\beta = \left\{\lambda_{\beta,n}\right\}_{n\geq 1}$ (where eigenvalues are counted with multiplicity). By Bandtlow-Jenkinson [3, 4] there exist positive constants A, c such that $\left|\lambda_{\beta,n}\right| \leq Ae^{-cn}$. Numerically we fond that a similar bound seems to hold for the operators $\mathscr{A}_\beta^{(N,N)}$. It follows that 0 is a limit point of Σ_β and there exist many sequences of spurious eigenvalues $\left\{\lambda_{N_i,j_i}\right\}$ converging to 0.

We will now present an algorithm which uses Lemma 6 to compute an approximation to $Z_q(s)$, but to put Lemma 6 into praxis we first need to make the following heuristic claims.

Claim 1. There is no sequence $\lambda_{N_i,j_i} \in \Sigma_\beta^{N_i}, i \geq 1$ such that $\lambda_{N_i,j_i} \to \lambda$ unless $\lambda \in \Sigma_\beta$ or $\lambda = 0$.

Claim 2. Suppose that $\Sigma_\beta^N = \{\lambda_{N,i}\}_{1\leq i\leq \kappa_N}$, $|\lambda_{N,n_1}| \geq |\lambda_{N,n_2}| \geq \cdots \geq |\lambda_{N,n_K}| > 0$ are regular eigenvalues with an estimated error $< \varepsilon$ and $|\lambda_{N,n_K}| < \varepsilon$. Then there does not exist an eigenvalue $\lambda_\beta \in \Sigma_\beta$ in the region $\{z \in \mathbb{C} \,|\, |z| \geq |\lambda_{N,n_K}|, \max_{i=1,\ldots,K} |z - \lambda_{N,n_i}| > \varepsilon\}$. I.e. the sequence $\{\lambda_{N,n_i}\}_{1\leq i\leq K}$ approximates all eigenvalues of $\mathscr{L}_\beta$ with absolute value greater than or equal to $|\lambda_{N,n_K}|$.

Algorithm. *Let $\delta, \varepsilon > 0$ and consider N and M for some $M \geq N+1$.*

Step 1: Compute the two spectra $\Sigma_\beta^N = \{\lambda_{N,i}\}_{1\leq i\leq \kappa_N}$ and $\Sigma_\beta^M = \left\{\lambda_{M,j}\right\}_{1\leq j\leq \kappa_N}$ (both ordered with non-increasing magnitude and repeated according to multiplicity) and the relative differences $\delta_{i,j} = \frac{\left|\lambda_{N,i} - \lambda_{M,j}\right|}{\left|\lambda_{N,i}\right| + \left|\lambda_{M,j}\right|}$.

Step 2: Let $k = 0$ and consider in sequence each $i = 1, \ldots, \kappa_N$. If there exists a j such that $\delta_{i,j} < \delta$ we assume that λ_{N,i_k} and λ_{M,j_k} are approximating some $\lambda \in \Sigma_\beta$ and accordingly increase k by 1, set $i_k = i$, $j_k = j$, $\delta_k = \delta_{i_k j_k}$ and $\tilde{\lambda}_{\beta,k} = \lambda_{M,j_k}$.

Step 3: Let K denote the last value of k. Then $\{\tilde{\lambda}_{\beta,k}\}_{k=1}^K$ is an ordered sequence of eigenvalues believed to approximate eigenvalues of $\mathscr{L}_\beta$ and we define

$$\tilde{d}_{N,M}(\beta) = \prod_{j=1}^{K} \left(1 - \tilde{\lambda}_{\beta,k}\right).$$

If $|\tilde{\lambda}_{\beta,K}| > \varepsilon$ we increase N and M and start from Step 1. As will be explained in section 6.2 below it might also be necessary to increase the working precision simultaneously with N in this step. If $|\tilde{\lambda}_{\beta,K}| < \varepsilon$ we assume that $\tilde{d}_{N,M}(\beta)$ approximates $\det\left(1 - \mathscr{L}_\beta\right)$ and return

$$\tilde{Z}_q(\beta) = \tilde{Z}_{q,N,M}(\beta) = \tilde{d}_{N,M}(\beta) \det\left(1 - \mathscr{K}_\beta\right)^{-1}$$

as a tentative value of $Z_q(s)$ with an assumed error depending only on δ, ε and the working precision. The factor $\det\left(1-\mathscr{K}_\beta\right)$ *can be computed using relation (5) to any desired accuracy.*

6. Discussion of the Results

The numerical method, Algorithm 5.2, which is proposed as a means to evaluate the Selberg zeta function relies on the heuristic Claims 1 and 2 above. It is thus clear that no amount of internal "consistency tests", e.g. stability under change of order of approximation and variation of the parameters ε and δ, can certify that the result returned by the algorithm is correct. If Claim 1 is wrong we would obtain extra eigenvalues not associated to $\mathscr{L}_\beta$ and on the other hand, if Claim 2 is incorrect we might actually miss eigenvalues of comparatively large magnitude. In both cases we would only be able to approximate $Z_\Gamma(s)$ times some unknown factor.

The need of an independent test to verify the accuracy of our numerical results is thus obvious. We propose to use a test relying on the functional equation of $Z_\Gamma(s)$. The setup will be discussed in Subsection 6.1.

Remark 3. If we were only concerned about zeros of Z_Γ on the real axis, i.e. eigenvalues equal to 1 of $\mathscr{L}_\beta$ for real β much more is known about approximation of eigenvalues and eigenfunctions, cf. e.g. [16, 18].

6.1. **The functional equation for $Z_q(s)$.** Let $Z_q(s)$ be the Selberg Zeta function for G_q. We know [11, p. 499] that

$$\frac{Z_q(1-s)}{Z_q(s)} = \varphi_q(s)\, c\Psi_q(s), \tag{7}$$

where $\varphi_q(s)$ is the scattering matrix (here a 1×1-matrix), $c=\varphi_q\left(\frac{1}{2}\right)=\pm 1$ and

$$\begin{aligned}\Psi_q(s) \quad = \quad & \frac{\Gamma\left(\frac{3}{2}-s\right)}{\Gamma\left(s+\frac{1}{2}\right)}\exp\left(-\frac{q-2}{q}\pi\int_0^{s-\frac{1}{2}} t\tan(\pi t)\,dt+\right.\\ & +\pi\sum_{k=1}^{q-1}\frac{1}{q\sin\frac{k\pi}{q}}\int_0^{s-\frac{1}{2}}\left(\frac{e^{-\frac{2\pi ikt}{m}}}{1+e^{-2\pi it}}+\frac{e^{\frac{2\pi ikt}{m}}}{1+e^{2\pi it}}\right)dt\\ & \left.+(1-2s)\ln 2\right).\end{aligned}$$

The function $\Psi_q(s)$ can be computed to any desired degree of accuracy using standard methods of numerical (e.g. Gauss) quadrature. Evaluation of $\varphi_q(s)$ on the other hand is more tricky. For $q=3$ there is an explicit formula [11, p. 508]

$$\varphi_3(s) = \sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\,\zeta(2s)}. \tag{8}$$

For $q\geq 4$ the only explicit formula is in terms of a Dirichlet series with abscissa of absolute convergence equal to 1, cf. e.g. [11, p. 569] and [28]. Note that for $q=4,6$ it might still be possible to work out explicit formulas for $\varphi_q(s)$ using the relations between G_4, G_6 and $\Gamma_0(2), \Gamma_0(3)$ respectively. We do not pursue this approach and for all $q\geq 4$ we use values of $\varphi_q(s)$ obtained by an algorithm of Helen Avelin [1]. The main idea of Avelin's algorithm is that φ_q occurs in the zeroth Fourier coefficient of the Eisenstein series $E(s,z)$ for the group G_q and one can use a method based on the G_q-invariance of $E(s,z)$ to compute its Fourier coefficients and thus also $\varphi_q(s)$. This method was first introduced to compute cuspidal Maass waveforms on Hecke triangle groups by Hejhal [12, 13]. Later it

was generalized to the setting of general subgroups of $PSL_2(\mathbb{Z})$ [27, Ch. 1] and finally it was generalized to compute Eisenstein series on Fuchsian groups with one cusp by Avelin [1].

Another application of the functional equation is that we may define a real-valued function

$$\mathscr{Z}_q(t) = Z_q\left(\frac{1}{2}+it\right)e^{-i\Theta(t)} \tag{9}$$

where $\Theta(t) = \frac{1}{2}\arg\left(\varphi\left(\frac{1}{2}+it\right)\Psi\left(\frac{1}{2}+it\right)\right)$ and the branch of the argument is chosen so that $\mathscr{Z}_q(t)$ becomes continuous. Note that a single choice of a branch cut is in general not possible because $\varphi\left(\frac{1}{2}+it\right)\Psi\left(\frac{1}{2}+it\right)$ winds around zero as $t \in \mathbb{R}^+$ varies. The advantage of considering $\mathscr{Z}_q(t)$ is in our case purely aesthetic, in that we may plot graphs of $\mathscr{Z}_q(s)$.

It is known [11, p. 498] that $Z_q(s)$ is zero for $s = s_k = \frac{1}{2}+ir_k$ where $\frac{1}{4}+r_k^2$ is an eigenvalue of Δ and at $s = 1-\gamma$ where $\varphi_q(\gamma) = 0$. In Figures 1-3 we plot $\mathscr{Z}_q(t)$ together with blue vertical lines at $t = r_k$ and green vertical lines at $t = \Im\gamma$. The zeros of $Z_q(s)$ on and off the half-line are clearly visible as zeros and "dips" of $\mathscr{Z}_q(t)$ at the corresponding points. The eigenvalues of Δ were computed by the method of Hejhal indicated above, see e.g. [12, 13, 27] and zeros of $\varphi_q(s)$ were located using Avelin's algorithm.

Verification of these zeros as well as the zeros on the real axis of $Z_q(s)$ ([11, p. 498]) does of course also lend credibility to our proposed algorithm but since this verification does not tell us anything about the accuracy for general s we prefer to concentrate on the error estimate using $\varphi_q(s)$.

Remark 4. The actual value of $\varphi_q\left(\frac{1}{2}\right) \in \{\pm 1\}$ can be computed experimentally in two different ways. The straight-forward way is to use Avelin's method but it is also known (cf. [11, p. 498]) that $Z_q(s)$ has a simple pole at $s = \frac{1}{2}$ if and only if $\varphi_q\left(\frac{1}{2}\right) = -1$. Experiments performed using both methods indicate that $\varphi_q\left(\frac{1}{2}\right) = -1$ for all $q \geq 3$.

In certain cases one can use the transfer operator to show that $Z_q(s)$ has a singularity at $s = \frac{1}{2}$ by showing that $\mathscr{L}_\beta$ has an eigenvalue $\mu_\beta \sim \frac{1}{\lambda\left|\beta-\frac{1}{2}\right|}$ in a neighborhood of $\beta = \frac{1}{2}$, but it is not possible to exclude that this singularity in $\det\left(1-\mathscr{L}_\beta\right)$ is canceled by the appearance of an eigenvalue $= 1$ for $\mathscr{L}_{\frac{1}{2}}$.

6.2. **Discussion of data and error analysis.** The procedure for testing and producing error estimates of the proposed algorithm to compute $Z_q(s)$ is now clear. Given tentative values of $Z_q(s)$ and $Z_q(1-s)$, denoted by $\tilde{Z}_q(s)$ and $\tilde{Z}_q(s)$ we compute the quantity

$$\tilde{\varphi}_q(s) = \tilde{Z}_q(1-s)\tilde{Z}_q(s)^{-1}\Psi(s)^{-1}$$

and compare this with the value of $c\,\varphi_q(s)$ obtained as described above (in all cases considered here we have $c = 1$). The difference $\left|\varphi_q(s)-\tilde{\varphi}_q(s)\right|$ or relative difference in the neighborhood of a zero of $\varphi(s)$ gives an estimate of the accuracy of the values $\tilde{Z}_q(s)$ and $\tilde{Z}_q(1-s)$.

To confirm a value $\tilde{Z}_q(s)$ we thus need also to compute $\tilde{Z}_q(1-s)$, but on the critical line with $s = \frac{1}{2}+it$ we have $Z_q(1-s) = \overline{Z_q(s)}$ so we need only compute $\tilde{Z}_q(s)$.

Using this "φ-test" we may verify the correctness of Claims 1 and 2. As it turns out, these two claims seems to be correct in theory. In practice, however, they and the entire algorithm may fail unless the working precision is increased as necessary. This phenomenon is clearly visible in Table 1 where we investigate the case $q = 3$ and $s = \frac{1}{2}+5i$ using different degrees of approximation N (here $M = N+3$ always) and working precision WP. In this table we list the estimated error, $\left|\tilde{\varphi}_3(s)-\varphi_3(s)\right|$, the time it took to compute $Z_3(s)$

in seconds, the number of eigenvalues of $\mathscr{L}_\beta$ which were used in the computation, the size of the smallest of those eigenvalues and the maximum of differences between eigenvalues of $\mathscr{A}^{(N,N)}$ and $\mathscr{A}^{(M,M)}$. With working precision of 50 digits we see that the error decreases as $N = 25$,50 and 75. To further increase N up to 100 does not improve the accuracy and increasing N up to 200 actually results in a worse approximation than at $N = 25$. The reason for this phenomenon is that Claim 1 is violated due to an increasing number of spurious eigenvalues and in particular there appear spurious eigenvalues which do not vary fast with N. This problem can be overcome by increasing the working precision, which is demonstrated in the remainder of the table, where the precision has been increased to $WP = 100$, 150 and 200 digits respectively. To know a priori when the precision has to be increased one must study more closely the relative differences δ_k. For example, in the case $WP = 50$ and $N = 200$, the relative differences for the spurious eigenvalues of a certain magnitude are much larger than the relative differences of regular eigenvalues. If one sees such a break from the otherwise almost monotonously increasing δ_k it is a clear sign to increase the working precision. What is not visible in this table, is that the need for increase in precision is actually dependent on the matrix size $\kappa_N = 2\kappa(N+1)$ and not only on N.

Table 2 contains values of $\tilde{\varphi}_3\left(\frac{1}{2}+ni\right)$, $1 \le n \le 10$, computed using $N = 100$, $M = 103$, $\delta = 10^{-7}$ and 100 digits working precision. The third column contains the true error, i.e. $\tilde{\varphi}_3$ compared to the explicit formula (8) for φ_3. One can see that in this case the true error agrees well with the error estimate in the fourth column given by the absolute value of the smallest eigenvalue used in computing $\tilde{d}(s)$. The fifth column contains the difference between $\varphi_3(s)$ computed by Avelin's method (using double precision) and by the explicit formula.

In Table 6.3 we list values $\varphi_4^A(s)$ of $\varphi_4\left(\frac{1}{2}+ni\right)$, for $1 \le n \le 10$ given by Avelin's algorithm and of $\tilde{\varphi}_4(s)$ by our algorithm using $N = 100, M = 103$, $\delta = 10^{-7}$ and 100 digits precision. The fourth column contains the difference between these values and the fifth column contains the size of the last eigenvalue $\tilde{\lambda}_K$ used in the evaluation of $\tilde{d}_{NM}(s)$.

Comparing the values of $\left|\tilde{\lambda}_K\right|$ in Tables 2 and 6.3 we observe that the eigenvalues of $\mathscr{L}_\beta$ for $q = 3$ seem to decay more rapidly than for $q = 4$. We would also expect the errors in the tabulated approximations of $\varphi_4(s)$ to be greater than those of $\varphi_3(s)$ even though we use the same level of precision and approximation. However, there is no reason to believe that the error in φ_4^A is any worse than in φ_3^A. Moreover it is very unlikely that the values $\varphi_4^A(s)$ should agree with our values $\tilde{\varphi}_4(s)$ to a much larger degree than the true accuracy, cf. e.g. $s = \frac{1}{2}+9i$ where $|\tilde{\lambda}_K| = 2\cdot 10^{-10}$ but $|\varphi_4^A(s) - \tilde{\varphi}_4(s)| = 2\cdot 10^{-15}$. We conclude that the values of $\left|\tilde{\lambda}_K\right|$ do not give an accurate estimate of the true magnitude of the error for $q = 4$ but that they still provide us with an upper bound.

The final conclusion we can draw from Tables 1- 6.3 is that the value of $|\tilde{\lambda}_K|$ alone is not enough to estimate the error in $\tilde{Z}_q(s)$ unless the working precision is high enough. To completely eliminate the external test by using φ_q to confirm values produced by our algorithm one needs a better understanding of when it is necessary to increase the working precision.

It is clear, that high precision eigenvalue computations are very time consuming. To evaluate $Z_q(s)$ to a fixed precision it is necessary to increase the approximation level N as $\Im s$ grows and this forces a simultaneous increase in the working precision. Altogether this makes it very time consuming to compute values of $Z_q\left(\frac{1}{2}+it\right)$ for large t's and to reach even values of $t \approx 1000$ for $q = 3$ seem to be out of reach with current methods and hardware. Remember that the size of the matrix κ_N grows with q, so similar problems arise

when computing $Z_q(s)$ for large q's. To end this discussion I would like to give a feeling of the necessary CPU-times.

To compute $Z_q\left(\frac{1}{2}+it\right)$ with an estimated error of 10^{-5} for $q=3$ takes 12 seconds at $t=15$ and 30 minutes at $t=100$. Decreasing the error to 10^{-9} at $t=15$ only increases the time to 26 seconds. For $q=8$, to compute $Z_8\left(\frac{1}{2}+15i\right)$ with an estimated error of 10^{-4} takes 5 hours and 57 minutes. If these figures seem outrageous, remember that for $q=8$ the size of $|\mathscr{J}_\kappa|=6$ and in this case $N=190$ so $\kappa_N=1146$ compared to $\kappa_N=352$ for $q=3$ at $t=100$. We see that $\frac{1146}{352}\approx 3.25$ and $\frac{5\text{h}57\text{m}}{30\text{m}}=\frac{21412}{1800}\approx 11.9\approx 3.45^2$ so the CPU-time increases roughly like the square of the size of the approximating matrix, which is to be expected from the eigenvalue computations.

6.3. **Implementation.** The algorithm outlined on p. 11 above has been implemented in FORTRAN90 using the ARPREC library for arbitrary precision computations. It was also necessary to write arbitrary precision Riemann and Hurwitz zeta functions as well as an arbitrary precision version of the linear algebra system LAPACK.

For the interested reader who is not comfortable with any version of Fortran there is a version of the algorithm written for MuPAD which is available from the homepage of the author.

Avelin's algorithm is currently only implemented in double precision FORTRAN77 hence the MuPAD version only contains the complete error check using $\varphi_q(s)$ for $q=3$.

TABLE 1. Demonstrating interplay between working precision and level of approximation in computing $Z_3(s)$

WP	N	$Z_3\left(\frac{1}{2}+5\mathrm{i}\right)$	$\lvert\tilde{\varphi}-\varphi\rvert$	Time (s)	K	$\lvert\tilde{\lambda}_K\rvert$	$\max_k \delta_k$
50	25	$1.1954+0.0811i$	$1\cdot 10^{-2}$	12	3	$5\cdot 10^{-2}$	$8\cdot 10^{-9}$
	50	$1.192213397499979+0.074413721696096i$	$5\cdot 10^{-9}$	65	12	$2\cdot 10^{-8}$	$5\cdot 10^{-8}$
	75	$1.192213402687674941183+0.07441372136992775i$	$7\cdot 10^{-13}$	202	17	$3\cdot 10^{-12}$	$2\cdot 10^{-8}$
	100	$1.192213402687674941270+0.074413721369927750i$	$7\cdot 10^{-13}$	839	19	$3\cdot 10^{-12}$	$2\cdot 10^{-8}$
	200	$1.0165+0.0782i$	$3\cdot 10^{-2}$	3764	19	$4\cdot 10^{-9}$	$5\cdot 10^{-8}$
100	100	$1.192213402686855883038325+0.074413721370273731516 8i$	$4\cdot 10^{-19}$	599	26	$4\cdot 10^{-18}$	$2\cdot 10^{-10}$
	200	$1.192213402686855883047193363+$ $0.074413721370273731779018337 3i$	$3\cdot 10^{-25}$	4324	34	$1\cdot 10^{-24}$	$2\cdot 10^{-8}$
	250	$1.1837+0.0575i$	$3\cdot 10^{-2}$	10164	27	$4\cdot 10^{-18}$	$3\cdot 10^{-8}$
150	200	$1.192213402686855883047193551130117623672+$ $0.074413721370273731779018511569381822 65i$	$1\cdot 10^{-36}$	6637	50	$5\cdot 10^{-36}$	$1\cdot 10^{-9}$
	250	$1.192213402686855883047193551130117623672+$ $0.074413721370273731779018511569381822 65i$	$1\cdot 10^{-36}$	12847	50	$5\cdot 10^{-36}$	$2\cdot 10^{-9}$
200	250	$1.192213402686855883047193551130117621955253934465021290+$ $0.074413721370273731779018511569381823164472321153746259i$	$6\cdot 10^{-51}$	18788	68	$7\cdot 10^{-49}$	$8\cdot 10^{-8}$

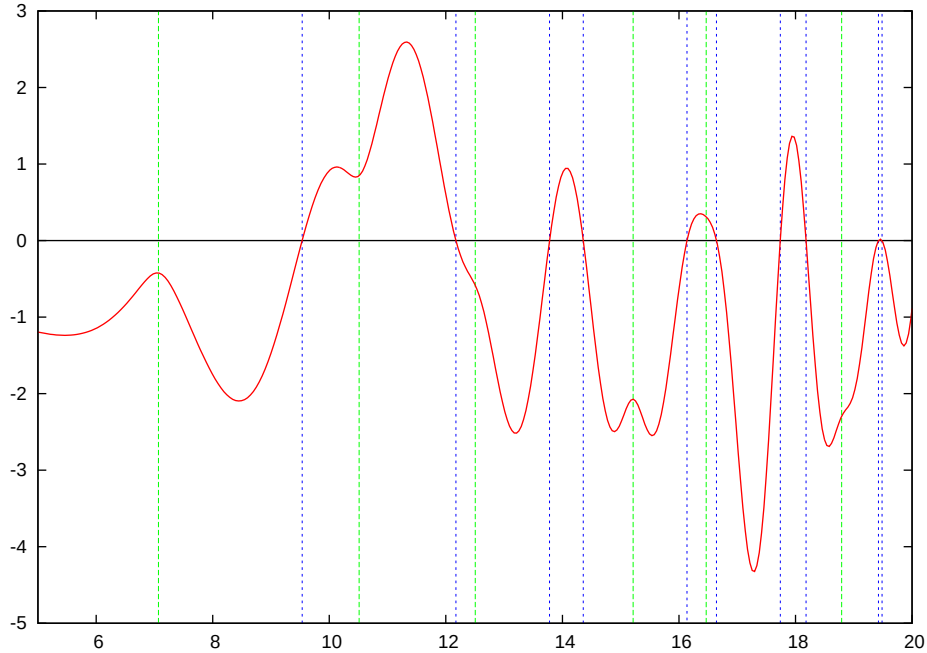
TABLE 2. Comparing values of $\varphi_3(s)$ together with different error estimates ($N = 100$, $WP = 100$, $\delta = 10^{-7}$).

n	$\tilde{\varphi}_3(s) = \tilde{Z}_3(1-s)/\tilde{Z}_3(s)/\Psi_3(s), \quad s = \frac{1}{2}+ni$	$\lvert\tilde{\varphi}_3 - \varphi_3\rvert$	K	$\lvert\tilde{\lambda}_K\rvert$	$\max \delta_k$	$\lvert\varphi_3^A - \varphi_3\rvert$
1	$0.52312715169438121771 8 - 0.85225464689852167578 8i$	$1\cdot 10^{-20}$	27	$6\cdot 10^{-21}$	$8\cdot 10^{-9}$	$8\cdot 10^{-17}$
2	$0.77770987086343080140 2 - 0.62862338228989365730 1i$	$3\cdot 10^{-21}$	24	$1\cdot 10^{-20}$	$3\cdot 10^{-8}$	$1\cdot 10^{-16}$
3	$0.81030753643965055089 5 - 0.58600486038010332753 0i$	$1\cdot 10^{-20}$	24	$9\cdot 10^{-21}$	$2\cdot 10^{-8}$	$4\cdot 10^{-16}$
4	$0.78411602629814366093 7 - 0.62061425805600766807 3i$	$4\cdot 10^{-20}$	27	$1\cdot 10^{-19}$	$9\cdot 10^{-8}$	$6\cdot 10^{-16}$
5	$0.62061425805600766807 3 - 0.70990764919907814131 7i$	$4\cdot 10^{-19}$	26	$4\cdot 10^{-18}$	$2\cdot 10^{-10}$	$7\cdot 10^{-16}$
6	$0.47376947672198515501 2 - 0.88064889878235603590 5i$	$4\cdot 10^{-20}$	26	$2\cdot 10^{-18}$	$2\cdot 10^{-9}$	$1\cdot 10^{-15}$
7	$-0.98266683804842746663 5 + 0.18538038029927985066 9i$	$2\cdot 10^{-19}$	26	$1\cdot 10^{-18}$	$9\cdot 10^{-9}$	$1\cdot 10^{-14}$
8	$0.94728094544443085019 5 - 0.32040413604993494728 1i$	$3\cdot 10^{-19}$	26	$2\cdot 10^{-18}$	$1\cdot 10^{-8}$	$2\cdot 10^{-15}$
9	$0.67870227473724821670 6 - 0.73441352266041836622 0i$	$1\cdot 10^{-18}$	26	$3\cdot 10^{-18}$	$1\cdot 10^{-8}$	$1\cdot 10^{-15}$
10	$-0.06335576668736108160 0 - 0.99799100538404486556 1i$	$2\cdot 10^{-18}$	26	$5\cdot 10^{-18}$	$2\cdot 10^{-8}$	$5\cdot 10^{-15}$

TABLE 3. Comparing values of $\varphi_4(s)$ ($N = 100$, $WP = 100$, $\delta = 10^{-7}$).

n	$\varphi_4^A\left(\frac{1}{2}+ni\right)$	$\tilde{\varphi}_4(s) = \tilde{Z}_4(1-s)/\tilde{Z}_4(s)/\Psi_4(s)$	$\lvert\varphi_4^A - \tilde{\varphi}_4\rvert$	K	$\lvert\tilde{\lambda}_K\rvert$	$\max \delta_k$
1	$-0.2632601861373177 - 0.9647248697918721i$	$-0.2632601861373176 - 0.9647248697918723i$	$2\cdot 10^{-16}$	27	$5\cdot 10^{-14}$	$7\cdot 10^{-8}$
2	$-0.7021440712594831 - 0.7120349030736887i$	$-0.7021440712594827 - 0.7120349030736895i$	$9\cdot 10^{-16}$	25	$3\cdot 10^{-13}$	$5\cdot 10^{-8}$
3	$-0.9912520623526865 + 0.1319823809511951i$	$-0.9912520623526863 + 0.1319823809511939i$	$1\cdot 10^{-15}$	24	$6\cdot 10^{-12}$	$5\cdot 10^{-9}$
4	$0.2148427612152942 + 0.9766486512320531i$	$0.2148427612152920 + 0.9766486512320534i$	$2\cdot 10^{-15}$	26	$3\cdot 10^{-13}$	$8\cdot 10^{-8}$
5	$-0.8749676464424498 - 0.4841813892323421i$	$-0.8749676464424484 - 0.4841813892323440i$	$2\cdot 10^{-15}$	24	$3\cdot 10^{-12}$	$3\cdot 10^{-8}$
6	$-0.0732387210885128 + 0.9973144387470341i$	$-0.0732387210885146 + 0.9973144387470377i$	$2\cdot 10^{-15}$	24	$2\cdot 10^{-11}$	$1\cdot 10^{-8}$
7	$-0.0299908075389591 - 0.9995501745601175i$	$-0.0299908075389498 - 0.9995501745601176i$	$9\cdot 10^{-15}$	24	$4\cdot 10^{-11}$	$3\cdot 10^{-8}$
8	$0.8554598916720125 + 0.5178690700751991i$	$0.8554598916721716 + 0.5178690700748949i$	$3\cdot 10^{-13}$	24	$1\cdot 10^{-10}$	$1\cdot 10^{-8}$
9	$0.7163471899280185 - 0.6977440099938026i$	$0.7163471899280196 - 0.6977440099938012i$	$2\cdot 10^{-15}$	24	$2\cdot 10^{-10}$	$7\cdot 10^{-8}$
10	$-0.7358033312663973 - 0.6771952877104747i$	$-0.7358033312663925 - 0.6771952877104797i$	$7\cdot 10^{-15}$	24	$3\cdot 10^{-10}$	$1\cdot 10^{-7}$

FIGURE 1. $\mathscr{Z}_3(t)$



REFERENCES

1. H. Avelin, *Deformation of* $\Gamma_0(5)$*-cusp forms*, Math. Comp. **76** (2007), no. 257, 361–384 (electronic). MR MR2261026
2. V. Baladi and M. Holschneider, *Approximation of nonessential spectrum of transfer operators*, Nonlinearity **12** (1999), no. 3, 525–538. MR MR1690191 (2000h:47008)
3. O. F. Bandtlow and O. Jenkinson, *Explicit a priori bounds on transfer operator eigenvalues*, Comm. Math. Phys. **276** (2007), no. 3, 901–905. MR MR2350441
4. _____, *Explicit eigenvalue estimates for transfer operators acting on spaces of holomorphic functions*, (2008), arXiv:0802.1638v1 [math.DS].
5. C.-H. Chang and D. Mayer, *Thermodynamic formalism and Selberg's zeta function for modular groups*, Regul. Chaotic Dyn. **5** (2000), no. 3, 281–312. MR MR1789478 (2001k:37045)
6. C.-H. Chang and D. H. Mayer, *The transfer operator approach to Selberg's zeta function and modular and Maass wave forms for* $\mathrm{PSL}(2,\mathbf{Z})$, Emerging applications of number theory (Minneapolis, MN, 1996), IMA Vol. Math. Appl., vol. 109, Springer, New York, 1999, pp. 73–141. MR MR1691529 (2000e:11112)
7. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **1955** (1955), no. 16, 140. MR MR0075539 (17,763c)
8. L. Guillopé, K. K. Lin, and M. Zworski, *The Selberg zeta function for convex co-compact Schottky groups*, Comm. Math. Phys. **245** (2004), no. 1, 149–176. MR MR2036371 (2005f:11193)
9. D. A. Hejhal, *The selberg trace formula and the riemann zeta function*, Duke Math. J. **43** (1976), no. 3.
10. _____, *The Selberg Trace Formula for* PSL(2,$\mathbb{R}$), *Vol.1*, Lecture Notes in Mathematics, vol. 548, Springer-Verlag, 1976.
11. _____, *The Selberg Trace Formula for* PSL(2,$\mathbb{R}$), *Vol.2*, Lecture Notes in Mathematics, vol. 1001, Springer-Verlag, 1983.
12. _____, *On eigenfunctions of the Laplacian for Hecke triangle groups*, Emerging applications of number theory (D. Hejhal, J. Friedman, et al., eds.), IMA Vol. Math. Appl., vol. 109, Springer, New York, 1999, pp. 291–315.
13. _____, *On the Calculation of Maass Cusp Forms*, Proceedings of the "International School on Mathematical Aspects of Quantum Chaos II", Lecture Notes in Physics, Springer, 2004, to appear.

FIGURE 2. $\mathscr{Z}_4(t)$

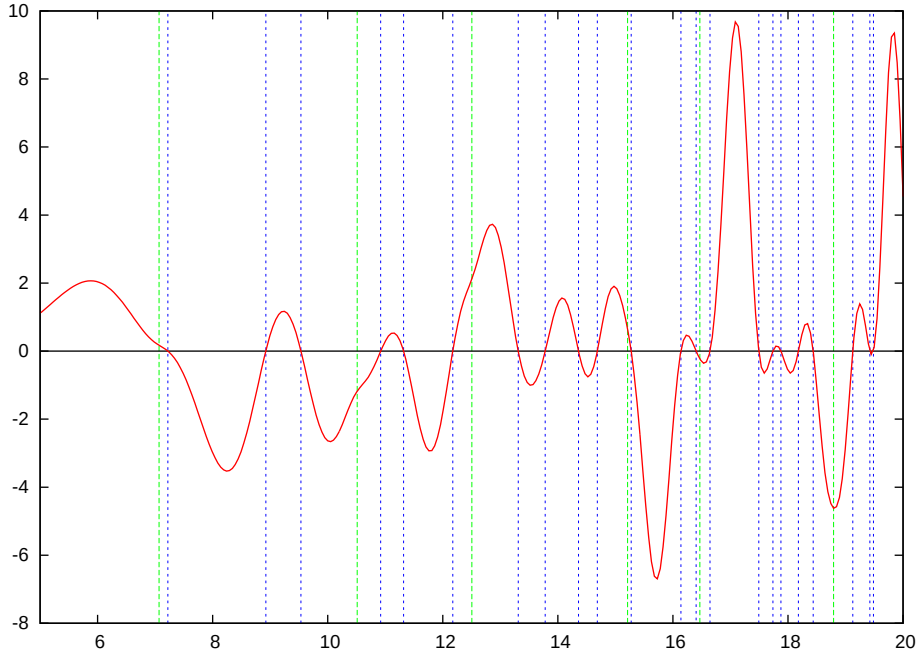
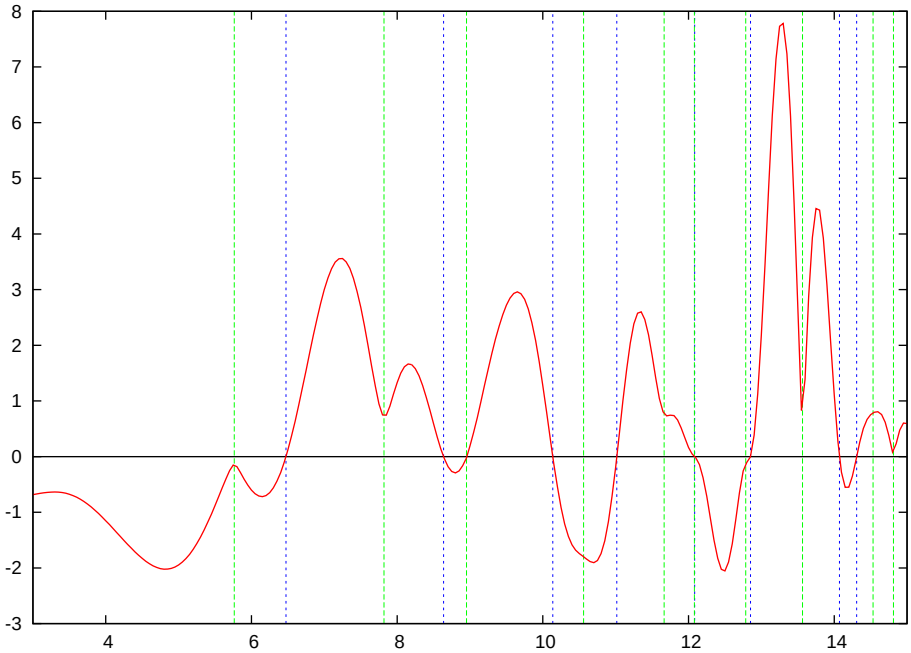


FIGURE 3. $\mathscr{Z}_5(t)$



14. O. Jenkinson, L. F. Gonzalez, and M. Urbański, *On transfer operators for continued fractions with restricted digits*, Proc. London Math. Soc. (3) **86** (2003), no. 3, 755–778. MR MR1974398 (2004d:37032)
15. O. Jenkinson and M. Pollicott, *Calculating Hausdorff dimensions of Julia sets and Kleinian limit sets*, Amer. J. Math. **124** (2002), no. 3, 495–545. MR MR1902887 (2003c:37064)
16. G. Keller and C. Liverani, *Stability of the spectrum for transfer operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), no. 1, 141–152. MR MR1679080 (2000b:47030)
17. J. Lehner, *Discontinuous groups and automorphic functions*, Mathematical Surveys, No. VIII, American Mathematical Society, Providence, R.I., 1964. MR MR0164033 (29 #1332)
18. C. Liverani, *Rigorous numerical investigation of the statistical properties of piecewise expanding maps. A feasibility study*, Nonlinearity **14** (2001), no. 3, 463–490. MR MR1830903 (2002b:37128)
19. C. Matthies and F. Steiner, *Selberg's ζ function and the quantization of chaos*, Phys. Rev. A (3) **44** (1991), no. 12, R7877–R7880. MR MR1141108 (92k:81042)
20. D. Mayer, T. Mühlenbruch, and F. Strömberg, *Nearest λ-multiple fractions*, In preparation.
21. _____, *On transfer operators for the geodesic flow on hecke triangle surfaces*, In preparation.
22. D. H. Mayer, *On a ζ function related to the continued fraction transformation*, Bull. Soc. Math. France **104** (1976), no. 2, 195–203. MR MR0418168 (54 #6210)
23. Dieter H. Mayer, *On the thermodynamic formalism for the Gauss map*, Comm. Math. Phys. **130** (1990), no. 2, 311–333. MR MR1059321 (91g:58216)
24. D. Ruelle, *Zeta-functions for expanding maps and Anosov flows*, Invent. Math. **34** (1976), no. 3, 231–242. MR MR0420720 (54 #8732)
25. A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. (N.S.) **20** (1956), 47–87. MR MR0088511 (19,531g)
26. F. Strömberg, *Symbolic dynamics for the geodesic flow on hecke triangle surfaces*, In preparation.
27. _____, *Computational aspects of maass waveforms*, Ph.D. thesis, Uppsala University, 2004.
28. A. B. Venkov, *The automorphic scattering matrix for the Hecke group* $\Gamma(2\cos(\pi/q))$, Trudy Mat. Inst. Steklov. **163** (1984), 32–36, International conference on analytical methods in number theory and analysis (Moscow, 1981). MR MR769866 (86e:11041)
29. A. Weil, *Sur les "formules explicites" de la théorie des nombres premiers*, Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] **1952** (1952), no. Tome Supplementaire, 252–265. MR MR0053152 (14,727e)