Dimension theoretical properties of generalized Baker's Transformations

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Abstract

We show that for generalized Baker's transformations there is a parameter domain where we have an absolutely continuous ergodic measure and in direct neighborhood there is a parameter domain where not even the variational principle for Hausdorff dimension holds.
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1 Introduction

In the modern theory of dynamical systems geometrical invariants like Hausdorff and box-counting dimension of invariant sets and measures seems to have its place beside classical invariants like entropy and Lyapunov exponents. In the last decades a dimension theory of dynamical systems was developed and we have general results for conformal systems, see [15], [5] and references therein. On the other hand the existence of different rates of contraction or expansion in different directions force mathematical problems that are not completely solved. We have general results on hyperbolic measures ([9], [3]) but the question if there exists an ergodic measure with full dimension (the dimension of a given invariant set) is only solved in special cases ([8], [6], [12], [17]). In this work we consider a generalization of the Baker's transformation, a simple example of a 'chaotic' dynamical systems that may be found in many standard text books [18]. In the case that the transformations are invertible dimensional theoretical properties are fairly easy to understand and the results seem to be folklore in the dimension theory of dynamical systems (see Theorem 2.1). We will be interested here in the case that the transformations are not invertible. Our main results describes a praenomen which was in this form not observed before. In fact there is parameter domain were there generically exists an absolutely continuous ergodic measure which obviously has full dimension on the attractor (see Theorem 2.2). On the other hand in the neighborhood there ist a parameter domain were the variational principle for Hausdorff dimension does not hold, the dimension of the attractor can not even be approximated by the dimension of ergodic measure (see Theorem 2.2). A kind of bifurcation occurs. Also we illustrate this phaenomenon only in a very simple case we think it may usually occur for endomorphism.
The rest of the paper is organized as follows. In section two we introduce the systems we study and present our main results. In section three we find a symbolic coding for the dynamics of generalized Baker's transformations through a factor of full shift on two

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symbols and represent all ergodic measures using this coding. In section four we construct absolutely continues ergodic measure for generalized Baker's transformation using our results on overlapping self-similar measure [13] and thus proof Theorem 2.2 . In section we find upper estimates on the dimension of all ergodic measure and proof Theorem 2.3 .

2 Notations and results

We define a *generalized Baker's Transformation* on the square by

$$f_{\beta_1,\beta_2} : [-1,1]^2 \longmapsto [-1,1]^2$$

$$f_{\beta_1,\beta_2}(x,y) = \{ \begin{array}{ll} (\beta_1 x + (1-\beta_1), 2y-1) & \text{if} \quad y \geq 0 \\ (\beta_2 x - (1-\beta_2), 2y+1) & \text{if} \quad y < 0 \end{array}$$

for parameter values $\beta_1, \beta_2 \in (0,1)$. We call this family of maps generalized Baker's transformations because if we set $\beta = \beta_1 = \beta_2$ we get the class of Baker's transformation studied by Alexander and York. and for $\beta = 0.5$ we get the well known classical Baker's transformation [1].

Let us first consider the case $\beta_1 + \beta_2 < 1$. In this case the attractor of the map f_{β_1,β_2}

$$\Lambda_{\beta_1,\beta_2} = \bigcap_{n=0}^{\infty} f^n_{\beta_1,\beta_2}([-1,1]^2)$$

is a product of a cantor set with the interval $[-1,1]$ and the dimensional theoretical properties of the system are easy to deduce.

Theorem 2.1 *Let $\beta_1 + \beta_2 < 1$ and d be the unique positive number satisfying*

$$\beta_1^d + \beta_2^d = 1$$

then

$$\dim_B \Lambda_{\beta_1,\beta_2} = \dim_H \Lambda_{\beta_1,\beta_2} = d + 1$$

and there is an f_{β_1,β_2}-ergodic measure μ of full dimension i.e. $\dim_B \mu = \dim_H \mu = d + 1$.

This result seems to be folklore in the dimension theory of dynamical systems. The box-counting dimension of $\Lambda_{\beta_1,\beta_2}$ is easy to calculate and the ergodic measure of full dimension is constructed as a product of a Cantor measure with weights (β_1^d, β_2^d) on the real line with the normalized Lebesgue measure on $[-1,1]$. See section 23 of [15]) for this martial.

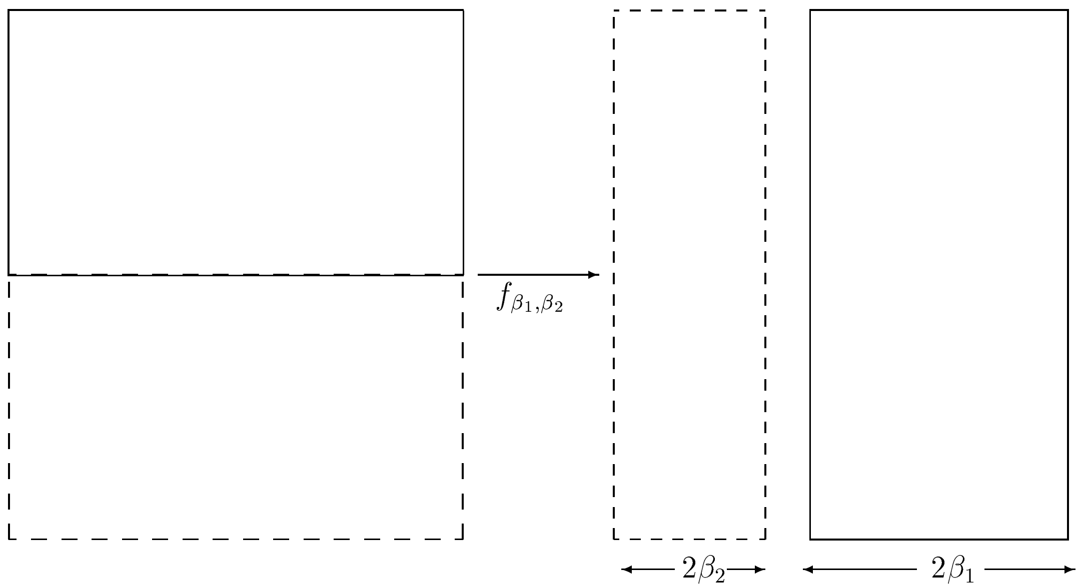

Figure 1: The action of f_{β_1,β_2} on the square $[-1,1]^2$ in the case $\beta_1+\beta_2<1$

Now consider the case $\beta_1+\beta_2\geq 1$ the attractor is obviously the whole square $[-1,1]^2$ which has Hausdorff and box-counting dimension two.

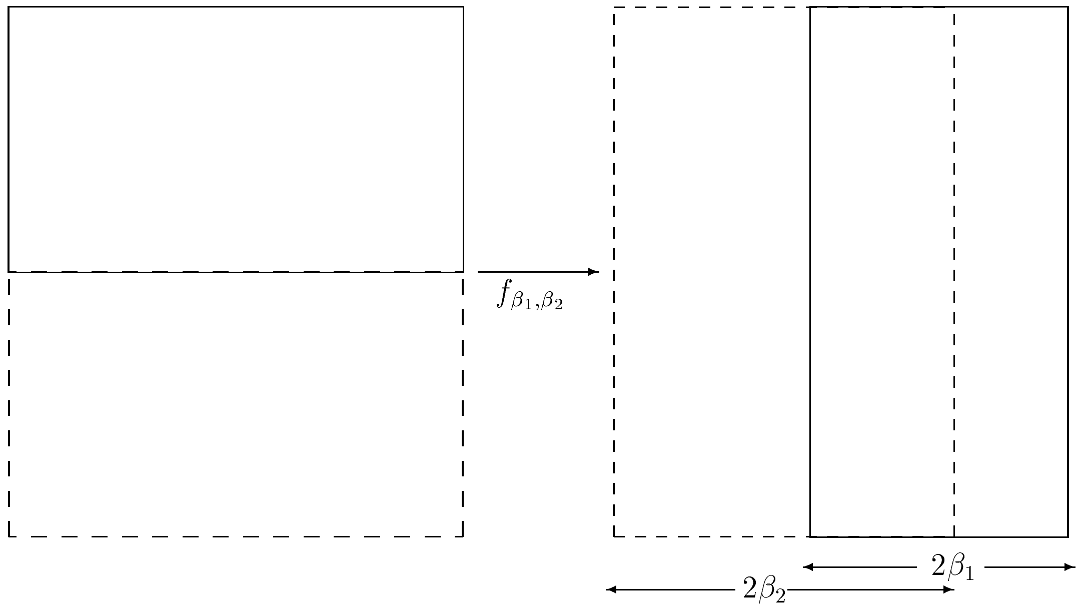

Figure 2: The action of f_{β_1,β_2} on the square $[-1,1]^2$ in the case $\beta_1+\beta_2>1$

The interesting problem in this situation is if there exists an ergodic measure of full dimension. In a restricted domain of parameter values we found generically an absolutely continuous ergodic measure which obviously has dimension two.

Theorem 2.2 *For almost all* $(\beta_1,\beta_2)\in(0,0.649)$ *with* $\beta_1+\beta_2\geq 1$ *and* $\beta_1\beta_2\geq 1/4$ *there is an absolutely continuous ergodic measure for* $([-1,1]^2, f_{\beta_1,\beta_2})$.

This theorem mainly is a consequence of our results about overlapping self-similar measures one the real liner [13]. We will construct the measure of full dimension as a product of an overlapping self-similar with normalized Lebesgue measure. From [] we then deduce absolute continuity of this measure. We do not know if the condition $\beta_1, \beta_2 < 0.649$ in theorem 2.1 is necessary, in fact it is due to the techniques we used in []. On the other hand from our second theorem we see that the condition $\beta_1\beta_2 \geq 1/4$ in theorem 2.1 is necessary.

Theorem 2.3 *For* $(\beta_1, \beta_2) \in (0,1)$ *with* $\beta_1 + \beta_2 \geq 1$ *and* $\beta_1\beta_2 < 1/4$ *we have*

$$\sup\{\dim_H \mu | \mu \quad f_{\beta_1,\beta_2}\text{-ergodic}\} < 2.$$

This example shows that it is not always possible to find the Hausdorff dimension of an invariant set by constructing an ergodic measure of full Hausdorff dimension. Roughly speaking the reason why there is not always an ergodic measure of full Hausdorff dimension here is that one can not maximize the stable and the unstable dimension (the dimension of conditional measures on partitions in stable resp. unstable directions) at the same time. In another context this praenomen was observed before by Manning and McClusky [10].

Now consider the for a moment the Fat Baker's transformation $f_\beta := f_{\beta,\beta}$ with $\beta \in (0.5, 1)$. It follows from the work of Alexander and Yorke [1] together with Solymak's theorem on Bernoulli convolutions [19] that for almost all $\beta \in (0.5, 1)$ we have $\dim_H \mu_{SRB} = 2$ where μ_{SRB} is the Sinai-Ruelle-Bowen measure for the system $([-1,1]^2, f_\beta)$, see [14]. This means that in the symmetric situation, in contrast to the asymmetric case, we generically have an ergodic measure of full dimension in the whole parameter domain.

3 Symbolic coding and representation of ergodic measures

Let $\Sigma = \{-1,1\}^{\mathbb{Z}}$, $\Sigma^+ = \{-1,1\}^{\mathbb{N}_0}$ and $\Sigma^{-1} = \{-1,1\}^{\mathbb{Z}^-}$. The forward shift map σ on Σ (resp. Σ^+) is given by $\sigma((s_k)) = (s_{k+1})$ and the system (Σ, σ) (resp. (Σ^+, σ)) is know as full shift on two symbols [7]. Given $\underline{s} \in \Sigma^+$ we denote by $\sharp_k(\underline{s})$ the cardinality of $\{i | s_i = -1 \ , \ i = 0 \ldots k\}$.

For $\beta_1, \beta_2 \in (0,1)$ with $\beta_1 + \beta_2 \geq 1/2$ we now define a map π_{β_1,β_2} from Σ^+ onto $[-1,1]$ in the following way. Let

$$\pi^*_{\beta_1,\beta_2}(\underline{s}) = \sum_{k=0}^{\infty} s_k \beta_2^{\sharp_k(\underline{s})} \beta_1^{k-\sharp_k(\underline{s})+1}.$$

We scale this map so that it is onto $[-1,1]$. by be the affine transformation L_{β_1,β_2} on the line that maps $\frac{-\beta_2}{1-\beta_2}$ to -1 and $\frac{\beta_1}{1-\beta_1}$ to 1; $\pi_{\beta_1,\beta_2} = L_{\beta_1,\beta_2} \circ \pi^*_{\beta_1,\beta_2}$.
Now define the maps ς from Σ^- onto [-1,1] corresponding to the signed dyadic expansion of a number by

$$\varsigma(\underline{s}) = \sum_{k=1}^{\infty} s_{-k} 2^{-k} \quad \text{where} \quad \underline{s} = (s_k)_{k \in \mathbb{Z}^-} \in \Sigma^-.$$

We are now able to define the coding map for the systems $([-1,1]^2, f_{\beta_1,\beta_2})$ by

$$\bar{\pi}_{\beta_1,\beta_2} : \Sigma \longmapsto [-1,1]^2 \qquad \text{with} \qquad \pi_{\beta_1,\beta_2}((s_k)) = (\pi_{\beta_1,\beta_2}((s_k)_{k\in\mathbb{N}_0}), \varsigma((s_k)_{k\in\mathbb{Z}^-})).$$

Obviously $\bar{\pi}_{\beta_1,\beta_2}$ is onto and continuous if we endow Σ with the natural product topology. Moreover we have

Proposition 3.1 *$\bar{\pi}_{\beta_1,\beta_2}$ conjugates the backward shift σ^{-1} and f_{β_1,β_2} i.e.*

$$f_{\beta_1,\beta_2} \circ \pi_{\beta_1,\beta_2} = \pi_{\beta_1,\beta_2} \circ \sigma^{-1}$$

on

$$\bar{\Sigma} = (\Sigma \backslash \{(s_k) | \exists k_0 \forall k \le k_0 : s_k = 1\}) \cup \{(1)\}.$$

Proof. Let $\underline{s} = (s_k) \in \bar{\Sigma}$. We have $(s_{k+1})_{k\in\mathbf{Z}^-} \neq (\ldots, 1, 1, -1)$ and hence

$$\varsigma((s_{k+1})_{k\in\mathbb{Z}^-}) = \sum_{k=1}^{\infty} s_{-k+1} 2^{-k} \ge 0 \Leftrightarrow s_0 = 1.$$

Thus

$$f_{\beta_1,\beta_2} \circ \bar{\pi}_{\beta_1,\beta_2}((s_{k+1})) =$$

$$\left\{ \begin{array}{l} (\beta_1 \pi_{\beta_1,\beta_2}((s_{k+1})_{k\in\mathbb{N}_0}) + (1-\beta_1), 2\varsigma((s_{k+1})_{k\in\mathbb{Z}^-}) - 1) \text{ if } s_0 = 1 \\ (\beta_2 \pi_{\beta_1,\beta_2}((s_{k+1})_{k\in\mathbb{N}_0}) - (1-\beta_2), 2\varsigma((s_{k+1})_{k\in\mathbb{Z}^-}) + 1) \text{ if } s_0 = -1 \end{array} \right. .$$

On the other hand we have

$$\pi_{\beta_1,\beta_2}(\sigma(\underline{s})) = L_{\beta_1,\beta_2}(\pi^*_{\beta_1,\beta_2}((s_{k+1}))) = \left\{ \begin{array}{lll} L_{\beta_1,\beta_2}(\beta_1^{-1}\pi^*_{\beta_1,\beta_2}((s_k)) - 1) & \text{if} & s_0 = 1 \\ L_{\beta_1,\beta_2}(\beta_2^{-1}\pi^*_{\beta_1,\beta_2}((s_k)) + 1) & \text{if} & s_0 = -1 \end{array} \right.$$

$$= \left\{ \begin{array}{lll} \beta_1^{-1}\pi_{\beta_1,\beta_2}(\underline{s}) + (1-\beta_1^{-1}) & \text{if} & s_0 = 1 \\ \beta_2^{-1}\pi_{\beta_1,\beta_2}(\underline{s}) - (1-\beta_2^{-1}) & \text{if} & s_0 = -1 \end{array} \right. .$$

By these equations and the definition of ς we now see that $f_{\beta_1,\beta_2} \circ \bar{\pi}_{\beta_1,\beta_2}((s_{k+1})) = \bar{\pi}_{\beta_1,\beta_2}((s_k))$. σ as a map of Σ is invertible and we get $f_{\beta_1,\beta_2} \circ \bar{\pi}_{\beta_1,\beta_2}(\underline{s}) = \bar{\pi}_{\beta_1,\beta_2}(\sigma^{-1}(\underline{s}))$ for all $\underline{s} \in \bar{\Sigma}$.

$\square$

Using our symbolic coding we can describe all ergodic measures for f_{β_1,β_2}. To this end we introduce the following notation: $M(X,f)$ denotes the space of all f-ergodic Borel probability measures on X. It is well known in ergodic theory that if X is compact $M(X,f)$ is a nonempty convex *weak** compact metricable space, [20] or [4].

Proposition 3.2 *$\mu \longmapsto \bar{\mu}_{\beta_1,\beta_2} := \mu \circ \bar{\pi}^{-1}_{\beta_1,\beta_2}$ is a continuous affine map from $M(\Sigma,\sigma)$ onto $M([-1,1]^2, f_{\beta_1,\beta_2})$.*

Proof. It is obvious by standard arguments in measure theory [11] that the map in question is continuous and affine since $\bar{\pi}_{\beta_1,\beta_2}$ is continuous. If μ is shift ergodic we have $\mu(\bar{\Sigma}) = 1$. We know from Proposition 3.1 that $\bar{\pi}_{\beta_1,\beta_2}$ conjugates the backward shift and f_{β_1,β_2} on $\bar{\Sigma}$ hence we get that $\bar{\mu}_{\beta_1,\beta_2}$ is f_{β_1,β_2}-ergodic. It remains to show that the map is onto $M([-1,1]^2, f_{\beta_1,\beta_2})$. This is a not completely trivial exercise in ergodic theory. Let us choose an arbitrary measure ξ in $M([-1,1]^2, f_{\beta_1,\beta_2})$.

We first want to show that $\xi(\pi_{\beta_1,\beta_2}(\Sigma\backslash\bar{\Sigma}))=0$. Let D be set of all numbers of the form $k/2^n$ with $n\in I\!N$ and $|k|\leq n-1$. A direct calculation shows that

$$\pi_{\beta_1,\beta_2}(\Sigma\backslash\bar{\Sigma})=(D\times[-1,1])\cup(\{1\}\times[-1,1))=(\bigcup_{k=0}^{\infty}f_{\beta_1,\beta_2}^{-k}(\{0\}\times[1,-1]))\cup(\{1\}\times[-1,1)).$$

Recall that the measure ξ is in particular shift invariant. Hence the measure of the first set in union is zero because it is given by a disjunct infinite union of sets with the same measure. The measure of the second set is zero since $\{1\}\times[-1,1)\subseteq f_{\beta_1,\beta_2}^{-k}(\{1\}\times[1-2\beta_1^k,1))\ \ \forall\ k\geq 0$.
Now take a Borel probability measure μ_{pre} such that $\mu_{pre}\circ\pi_{\beta_1,\beta_2}^{-1}=\xi$. μ_{pre} is not necessary shift invariant so we define a measure μ as a weak* accumulation point of the sequence

$$\mu_n:=\frac{1}{n+1}\sum_{i=0}^{n}\mu_{pre}\circ\sigma^{-n}.$$

From the considerations above we have $\mu_{pre}(\bar{\Sigma})=1$ and hence:

$$\mu_n\circ\pi_{\beta_1,\beta_2}^{-1}=\frac{1}{n+1}\sum_{i=0}^{n}\mu_{pre}\circ\sigma^{-n}\circ\pi_{\beta_1,\beta_2}^{-1}$$

$$=\frac{1}{n+1}\sum_{i=0}^{n}\mu_{pre}\circ\pi_{\beta_1,\beta_2}^{-1}\circ f_{\beta_1,\beta_2}^{-i}=\frac{1}{n+1}\sum_{i=0}^{n}\xi\circ f_{\beta_1,\beta_2}^{-i}=\xi.$$

Thus $\bar{\mu}_{\beta_1,\beta_2}$ is just the measure ξ and μ is shift invariant by definition. We have thus shown that the set $M(\xi):=\{\mu|\mu\ \sigma\text{-invariant and } \mu_{\beta_1,\beta_2}=\xi\}$ of Borel measures on Σ is not empty. Since the map $\mu\longmapsto\bar{\mu}_{\beta_1,\beta_2}$ is continuous and affine on the set of σ-invariant measures we know that $M(\xi)$ is compact and convex. It is a consequence of Krein-Milman theorem that there exists an extremal point μ of $M(\xi)$.
We claim that μ is an extremal point of the set of all σ-invariant Borel measures on Σ and hence ergodic.
If this is not the case then we have $\mu=t\mu_1+(1-t)\mu_2$ where $t\in(0,1)$ and μ_1,μ_2 are two distinct σ-invariant measures. This implies $\xi=t(\mu_1)_{\beta_1,\beta_2}+(1-t)(\mu_2)_{\beta_1,\beta_2}$. Since ξ is ergodic we have $(\mu_1)_{\beta_1,\beta_2}=(\mu_2)_{\beta_1,\beta_2}=\xi$ and hence $\mu_1,\mu_2\in M(\xi)$. This is a contradiction to μ being extremal in $M(\xi)$.

□

4 Construction of absolutely continuous ergodic measures

We now construct absolutely continuous ergodic measures for the systems $([-1,1]^2,f_{\beta_1,\beta_2})$. Let b denote the Bernoulli measure on the shift Σ (resp. Σ^+ or Σ^-, which is the product of the discrete measure giving 1 and -1 the probability 1/2. The Bernoulli measure is ergodic with respect to forward and backward shifts, see [7].
Given b on Σ^- we set $\ell^p=b\circ\varsigma^{-1}$. ℓ is the normalized Lebesgue measure on [-1,1].
Given b on Σ^+ we define two Borel probability measures on the real line by

$$b_{\beta_1,\beta_2}^{*}=b\circ(\pi_{\beta_1,\beta_2}^{*})^{-1}\quad\text{and}\quad b_{\beta_1,\beta_2}=b\circ(\pi_{\beta_1,\beta_2})^{-1}.$$

The measure b_{β_1,β_2} is just $b^*_{\beta_1,\beta_2}$ scaled on the interval $[-1,1]$ by the transformation L_{β_1,β_2}:

$$b_{\beta_1,\beta_2} = b^*_{\beta_1,\beta_2} \circ L^{-1}_{\beta_1,\beta_2}.$$

In the following proposition we describe an ergodic measure fore the generalized Bakers transformations using the Bernoulli measure b.

Proposition 4.1 *For all $\beta_1,\beta_2 \in (0,1)$ with $\beta_1+\beta_2 \geq 1$ we have*

$$\bar{b}_{\beta_1,\beta_2} := b \circ \bar{\pi}^{-1}_{\beta_1,\beta_2} = b_{\beta_1,\beta_2} \times \ell \in M([-1,1]^2, f_{\beta_1,\beta_2}).$$

Proof. By Proposition 3.2 we get that $b \circ \bar{\pi}^{-1}_{\beta_1,\beta_2}$ is f_{β_1,β_2}-ergodic since b is $\sigma-$ergodic. Moreover by the product structure of $\bar{\pi}_{\beta_1,\beta}$ we have

$$b \circ \bar{\pi}^{-1}_{\beta_1,\beta_2} = b \circ \pi_{\beta_1,\beta_2} \times b \circ \iota^{-1} = b_{\beta_1,\beta_2} \times \ell$$

where we use the fact that the Bernoulli measure b on Σ is a product of b on Σ^+ with b on Σ^{-1}

□

The measures $b^*_{\beta_1,\beta_2}$ are by definition a special class of overlapping self similar measures studied in [13]. The following proposition is just a consequence of Theorem I of [13]

Proposition 4.2 *For almost all $(\beta_1,\beta_2) \in (0, 0.649)$ with $\beta_1+\beta_2 \geq 1$ and $\beta_1\beta_2 \geq 1/4$ the measure $b^*_{\beta_1,\beta_2}$ is absolutely continuous.*

Now the proof of Theorem 2.2 is obviously.

Proof of Theorem 2.1. In the relevant parameter domain we generically have absolute continuity of $b^*_{\beta_1,\beta_2}$ by Proposition 4.2. Since $b_{\beta_1,\beta_2} = b^*_{\beta_1,\beta_2} \circ L^{-1}_{\beta_1,\beta_2}$ this clearly implies absolute continuity of b_{β_1,β_2}. Now Proposition 4.1 implies absolute continuity of the measure $\bar{b}_{\beta_1,\beta_2}$ which is f_{β_1,β_2}-ergodic.

□

5 Dimension estimates on all ergodic measures

In this section we proof two upper bounds on the dimension of all f_{β_1,β_2}-ergodic measures using metric entropy of these measures. Theorem 2.2 will be a consequence.
Given $\mu \in M(\Sigma,\sigma)$ we denote by $h(\mu)$ the metric entropy of μ. We refer to [20] or [7] the definition and the properties of this quantity.

Proposition 5.1 *For all $\mu \in M(\Sigma,\sigma)$ and all $\beta_1,\beta_2 \in (0,1)$ with $\beta_1+\beta_2 \geq 1$ we have*

$$\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq \frac{h(\mu)}{\log 2} + 1$$

Proof. Let $\tilde{\mu}_{\beta_1,\beta_2}$ by the projection of the measure $\bar{\mu}_{\beta_1,\beta_2}$ onto the second coordinate axis. Since $\dim_H(B \times [-1,1]) = \dim_H B + 1$ for all sets B we have

$$\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq \dim \tilde{\mu}_{\beta_1,\beta_2} + 1.$$

just by the definition of the Hausdorff dimension of a measure. Now we have to estimate the dimension of the projection. By $\tilde{\mu}_{\beta_1,\beta_2} = \bar{\mu}_{\beta_1,\beta_2} \circ pr_y^{-1} = \mu \circ \bar{\pi}^{-1}_{\beta_1,\beta_2} \circ pr_y^{-1}$ and the product structure of the coding map $\bar{\pi}_{\beta_1,\beta_2}$ we see that $\tilde{\mu}_{\beta_1,\beta_2}$ is ergodic with respect to the map

$$f(y) = \left\{ \begin{array}{ll} 2y - 1 & \text{if} \quad y \geq 0 \\ 2y + 1 & \text{if} \quad y < 0. \end{array} \right.$$

Thus the Hausdorff dimension of $\tilde{\mu}_{\beta_1,\beta_2}$ is well known (see [15])

$$\dim_H \tilde{\mu}_{\beta_1,\beta_2} = \frac{h(\tilde{\mu}_{\beta_1,\beta_2})}{\log 2}.$$

Moreover we know that $([-1,1], f, \tilde{\mu}_{\beta_1,\beta_2})$ is a measure theoretical factor of $([-1,1]^2, f_{\beta_1,\beta_2}, \bar{\mu}_{\beta_1,\beta_2})$. and that this system is a factor of (Σ, σ, μ). Hence we get by well known properties of the entropy (see [4]) $h(\tilde{\mu}_{\beta_1,\beta_2}) \leq h(\bar{\mu}_{\beta_1,\beta_2}) \leq h(\mu)$ which completes the proof.

□

To state the other estimate we need a few notation. Let pr we the projection from Σ to Σ^+. Given μ in $M(\Sigma, \sigma)$ we define $\hat{\mu} \in M(\Sigma^+, \sigma)$ by $\hat{\mu} = \mu \circ pr^{-1}$. Moreover set

$$\Xi_{\beta_1,\beta_2}(\hat{\mu}) = -(\hat{\mu}(\{\underline{s} \in \Sigma^+ | s_0 = 1\}) \log \beta_1 + \hat{\mu}(\{\underline{s} \in \Sigma^+ | s_0 = -1\}) \log \beta_2$$

With these notations we have

Proposition 5.2 *For all $\mu \in M(\Sigma, \sigma)$ and all $\beta_1, \beta_2 \in (0,1)$ with $\beta_1 + \beta_2 \geq 1$ we have*

$$\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq \frac{h(\hat{\mu})}{\Xi_{\beta_1,\beta_2}(\hat{\mu})} + 1$$

Proof. The proof of this proposition has several steps. First we show the following inequality

$$\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq \dim_H \hat{\mu}_{\beta_1,\beta_2} + 1$$

Let B be an arbitrary Borel set with $\hat{\mu}_{\beta_1,\beta_2}(B) = 1$. Since the projection of $\bar{\mu}_{\beta_1,\beta_2}$ onto the first coordinate axis is $\hat{\mu}_{\beta_1,\beta_2}$ we get $\bar{\mu}_{\beta_1,\beta_2}(B \times [-1,1]) = 1$ Thus

$$\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq \dim_H(B \times [-1,1]) = \dim_H(B) + 1$$

Now our claim follows just by the definition of the Hausdorff dimension of a measure.

Now we have to estimate the dimension of the projected measure; we have to show that

$$\dim_H \hat{\mu}_{\beta_1,\beta_2} \leq \frac{h(\hat{\mu})}{\Xi_{\beta_1,\beta_2}(\hat{\mu})}$$

Let us define a metric δ^{β_1,β_2} on Σ^+ by

$$\delta^{\beta_1,\beta_2}(\underline{s},\underline{t}) = \beta_1^{|\underline{s}\wedge\underline{t}|-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}\beta_2^{\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}.$$

where $\sharp_k(\underline{s})$ is the cardinality of $\{i|s_i=-1\ ,i=0\dots k\}$ and $|\underline{s}\wedge\underline{t}|| = \min\{i|s_i\neq t_i\}$. Now we claim that

$$d^{\beta_1,\beta_2}(\underline{s},\hat{\mu}) := \lim_{\epsilon\longrightarrow 0}\frac{\log\hat{\mu}(B_\epsilon^{\beta_1,\beta_2})(\underline{s})}{\log\epsilon} = \frac{h_{\hat{\mu}}(\sigma)}{\Xi_{\beta_1,\beta_2}(\hat{\mu})} \qquad \hat{\mu}\text{-almost everywhere.}$$

Here d^{β_1,β_2} is the local dimension of the measure $\hat{\mu}$ with respect to metric δ^{β_1,β_2} and accordingly $B_\epsilon^{\beta_1,\beta_2}$ is a ball of radius ϵ with respect to this metric. Applying Birkhoff's ergodic theorem (see 4.1.2. of [7]) to $(\Sigma^+,\sigma,\hat{\mu})$ with the function

$$h(\underline{s}) = \left\{ \begin{array}{ll} \log\beta_1 & \text{if} \quad s_0 = 1 \\ \log\beta_2 & \text{if} \quad s_0 = -1 \end{array}\right.$$

we see that

$$\lim_{n\longrightarrow\infty}\frac{1}{n+1}\log\operatorname{diam}_{\beta_1,\beta_2}([s_0,\dots,s_n]) = \lim_{n\longrightarrow\infty}\frac{1}{n+1}\sum_{k=0}^{n+1}h(\sigma^k(\underline{s})) = \int h\ d\hat{\mu}(\underline{s}) = \Xi_{\beta_1,\beta_2}(\hat{\mu})$$

$\hat{\mu}$-almost everywhere. By Shannon-McMillan-Breiman theorem (see [4] 13.4.) we have:

$$\lim_{n\longrightarrow\infty} -\frac{1}{n+1}\log\hat{\mu}([s_0,\dots,s_n]) = h_{\hat{\mu}}(\sigma) \qquad \hat{\mu}\text{-almost everywhere.}$$

Thus we see:

$$\lim_{\epsilon\longrightarrow 0}\frac{\log B_\epsilon^{\beta_1,\beta_2}(\underline{s})}{\log\epsilon} = \lim_{n\longrightarrow\infty}\frac{\log\hat{\mu}([s_0,\dots,s_n])}{\operatorname{diam}_{\beta_1,\beta_2}([s_0,\dots,s_n]_0)} = \frac{h(\hat{\mu})}{\Xi_{\beta_1,\beta_2}(\hat{\mu})}$$

By Billigsly's Lemma about the relation of local and global dimension [15] we get

$$\dim_H^{\beta_1,\beta_2}\hat{\mu} = \frac{h(\hat{\mu})}{-\Xi_{\beta_1,\beta_2}(\hat{\mu})}$$

where the Hausdorff dimension $\dim_H^{\beta_1,\beta_2}$ has to be calculated with respect to the metric δ^{β_1,β_2}.

Now we claim that the map $\pi^*_{\beta_1,\beta_2}$ is Lipschitz with respect to the metric δ^{β_1,β_2}

$$|\pi^*_{\beta_1,\beta_2}(\underline{s}) - \pi^*_{\beta_1,\beta_2}(\underline{t})| \leq \sum_{k=|\underline{s}\wedge\underline{t}|}^{\infty}|s_k\beta_1^{k-\sharp_k(\underline{s})+1}\beta_2^{\sharp_k(\underline{s})} - t_k\beta_1^{k-\sharp_k(\underline{t})+1}\beta_2^{\sharp_k(\underline{t})}|$$

$$= \beta_1^{|\underline{s}\wedge\underline{t}|-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}\beta_2^{\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}$$

$$\sum_{k=0}^{\infty}|s_{k+|\underline{s}\wedge\underline{t}|}\beta_1^{k-\sharp_k(\sigma^{|\underline{s}\wedge-\underline{t}|}(\underline{s}))+1}\beta_2^{\sharp_k(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s}))} - t_{k+|\underline{s}\wedge\underline{t}|}\beta_1^{k-\sharp_k(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t}))+1}\beta_2^{\sharp_k(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t}))}|$$

$$\leq \delta^{\beta_1,\beta_2}(\underline{s},\underline{t})\frac{2}{1-\max\{\beta_1,\beta_2\}}.$$

The the map π_{β_1,β_2} is just $\pi^*_{\beta_1,\beta_2}$ scaled on $[-1,1]$ and hence as Lipschitz with respect δ^{β_1,β_2}. Recall that $\hat{\mu}_{\beta_1,\beta_2} = \hat{\mu}\circ\pi_{\beta_1,\beta_2}$. Since applying a Lipschitz map to the measures $\hat{\mu}$ does obvious not increase its Hausdorff dimension, the proof is complete.

$\square$

Combining proposition 5.1 and 5.2 we now proof Theorem 2.3

Proof of Theorem 2.3. If $\beta_1\beta_2 < 0.25$ then we have $h(b) < \Xi_{\beta_1,\beta_2}(b)$. By upper semi continuity of the metric entropy there is a weak* neighborhood U of b in $M(\Sigma^+,\sigma)$ such that $h(\mu)/\Xi_{\beta_1,\beta_2}(\mu) \leq c_1 < 1$ holds for all $\mu \in U$. By Proposition 5.1 we get $\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq c+1 < 2$ for all $\mu \in \tilde{U} = pr^{-1}(U)$. Obviously $\tilde{U}$ is a neighborhood of b in $M(\Sigma,\sigma)$. Furthermore by Proposition 5.1 and upper-semi continuity of metric entropy we get that $\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq c_2+1 < 2$ for all $\mu \in M(\Sigma,\sigma)\backslash\bar{U}$. Putting these facts together we get

$$\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq \max\{c_1,c_2\}+1 < 2 = \dim[-1,1]^2 \qquad \forall \mu \in M(\Sigma,\sigma).$$

$\square$

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