A Douady-Oesterlé type estimate for the Hausdorff dimension of invariant sets of piecewise smooth maps

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Abstract

We prove a Douady-Oesterlé type estimate for the Hausdorff dimension of invariant sets of piecewise smooth maps under some assumption on the Hausdorff measure of the singularity set and its pre images. Furthermore we apply this result to the Belykh systems, the Lozi systems and a class of piecewise affine solenoid like systems.

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1 Introduction

In the theory of dynamical systems it is well known that invariants sets like attractors or hyperbolic sets often have a non-smooth ("fractal") geometry. The most important quantity that characterise such sets is the Hausdorff dimension (see [12] and chapter 13 of [4]).
For smooth systems we have a general approach that allows us to find upper estimates for the Hausdorff dimension of invariant sets in terms of the singular values of the Jacobi matrix of the smooth map. This approach goes back to an article of Douady-Oesterlé [3] and was developed further in [6], [10] and [9].
But in the theory of dynamical systems we often deal not only with smooth but with piecewise smooth maps. For instance the Poincaré first return map for a smooth flow is as rule only piecewise smooth. Also in concrete physical situations one often use piece wise smooth maps as models. Thus one would like to have estimates of the Douady-Oesterlé type for invariant sets of such maps.
There is one result in this direction found by Reitmann and Schnabel [13]. Roughly speaking Reitmann and Schnabel assumed that the sets of pre images of the singularity are in a special sense controllable to get a dimension estimate. For some well known simple classes of systems it was shown by the authors that their assumptions hold. But in fact in some more complicated examples these assumptions does not hold at all.
In this paper we present a Douady-Oesterlé type estimate for another class of piecewise smooth dynamical systems. We need a special assumption on the Hausdorff measure of the singularity set and its pre images (see Theorem 2.1 below). In fact this as assumption is easy to check for some interesting classes of systems and especially our theorem is applicable in situation where the results in [13] can not be applied.

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We will apply our general result to three classes of systems illustrating these fact. We get estimates of the Hausdorff dimension of invariant sets for the Belykh systems, for the Lozi systems and for a class of piecewise affine solenoid like systems.

2 Definitions and results

Let M be a C^∞-Riemmanian manifold and $U \subseteq M$. We consider a special class of piecewise smooth maps on U.

Definition 2.1 We say that a map $f : U \longmapsto M$ full fills condition (**PC**) if for all $m \in I\!N$ there is a partition $\{U_1^m, \dots U_{i(m)}^m\}$ of U with connected Borel sets that have compact closure such that $f_k^m := f_{|U_k^m}^m$ is a C^1-map and is C^1-extendable to some open neighbourhood of $\overline{U_m^k}$ for all $k = 1 \dots i(m)$.

Remark 2.1 If we would not suppose that the sets U_k^m have compact closure then condition (**PC**) would be equivalent to condition (**H0**) and (**H1**) of [13].

Now we introduce the notations we need to state our main result. First we recall the definitions of Hausdorff measure and Hausdorff dimension.

Definition 2.2 Let K be a subset of M, $\epsilon > 0$ and $d \geq 0$. The (d, ϵ)**-Hausdorff measure** of K is given by

$$H_\epsilon^d(K) = \inf\{\sum_{i\in I} (\mathrm{diam} P_i)^d | I \text{ countable}, \quad Z \subseteq \bigcup_{i\in I} P_i \text{ and } \mathrm{diam}(P_i) \leq \epsilon\}.$$

We define the d**-dimensional Hausdorff measure** $H^d(K)$ of K by $H^d(K) = \lim_{\epsilon \longrightarrow 0} H_\epsilon^d(K)$. The **Hausdorff dimension** $\dim_H K$ of K is now given by $\dim_H K = \sup\{d | H^d(K) = \infty\} = \inf\{d | H^d(K) = 0\}$.

Remark 2.2 The set functions $H^d(.)$ define outer measures on M and measures on the Borel σ-algebra on M (see appendix I and V of [12]).

Now we introduce singular values and the singular value function.

Definition 2.3 Let E and $E^{'}$ be n-dimensional Euclidian spaces and let $L : E \longmapsto E^{'}$ be a linear operator. We denote the singular values of L, i.e. the eigenvalues of $\sqrt{L^* L}$ where L^* is the adjoint operator of L, by $\alpha_1(L) \geq \dots \alpha_n(L) \geq 0$. For $d \in [0, n]$ we define the **singular value function** of L by

$$W_d(L) = \{ \begin{array}{cc} 1 & \text{if } d = 0 \\ \alpha_1(L) \cdot \dots \cdot \alpha_{\lfloor d \rfloor}(L) \alpha_{\lfloor d \rfloor + 1}(L)^{d - \lfloor d \rfloor} & \text{if } d > 0 \end{array}$$

Given $f : U \longmapsto M$ satisfying (**PC**) and $x \in U$ we define the singular value function of f by $W_d(f, x) := W_d(D_x f)$ where $D_x f : T_x M \longmapsto T_{f(x)} M$ is the tangent map of the C^1-extension of f.

Let us state our main result.

Theorem 2.1 *Suppose $f : U \longmapsto M$ satisfying condition* (**PC**). *Let K be a compact f-invariant, i.e. f(K)=K, subset of U. If we have a number $d \in (0, n]$ such that*

$$\sup_{x \in K} W_d(f, x) < 1 \quad \textit{and} \quad \forall m \in I\!N \; \forall k \in \{1, \ldots, i(m)\} \; : \; H^d(K \cap (\overline{U_k^m} \backslash U_k^m)) = 0$$

then $\dim_H K \leq d$ *holds.*

Remark 2.3 We compare our result with Theorem 4.1. of [13] about the Hausdorff dimension estimates for invariant sets piecewise smooth maps. We replaced condition (**H2**) of [13] by the assumption that the intersection of K with the singularity set and its pre images has zero d-dimensional Hausdorff measure. We will see in section four that our theorem is applicable in situations were theorem 4.1. of [13] is not applicable (see remark 3 after Estimate 4.1).

Remark 2.4 In some situations it is useful to replace the map f by a power of f in order to get better dimension estimates by Theorem 2.1 . This fact is due to Lemma 2.2 .

In the following corollary of our main theorem we introduce a Lyapunov function p into the dimension estimate.

Corollary 2.1 *Suppose $f : U \longmapsto M$ satisfying condition* (**PC**). *Let K be a compact f-invariant, i.e. f(K)=K, subset of U. If we have a number $d \in (0, n]$ and a continuous function $p : K \longmapsto I\!R^+ := \{x \in I\!R | x > 0\}$ such that*

$$\sup_{x \in K} \left(\frac{p(f(x))}{p(x)} W_d(f, x)\right) < 1 \quad \textit{and} \quad \forall m \in I\!N \; \forall k \in \{1, \ldots, i(m)\} \; : \; H^d(K \cap (\overline{U_k^m} \backslash U_k^m)) = 0$$

then $\dim_H K \leq d$ *holds.*

Proof We get this Corollary 2.1 from Theorem 2.1 by the same arguments that were used in the proof of Corollary 4.2. in [13]. □

Remark 2.5 It is known in the context smooth maps that the use of a Lyapunov function often allows us to find better dimension estimates (see [6]). It seems obvious that this will be as well true for piecewise smooth maps.

3 Proof of the main result

The following lemma is essential for the proof of our theorem:

Lemma 3.1 *Suppose that $f : U \longmapsto M$ satisfies condition* (**PC**)*. Furthermore suppose that for $d \in (0, n]$, $m \in I\!N$ and for a compact f-invariant set K the inequality $\sup_{x \in K} W_d(f^m, x) < \delta$ holds. Then we have*

$$H^d(f^m(K)) \leq C(d)\delta(H^d(K) + \sum_{k=1}^{i(m)} H^d(K \cap (\overline{U_k^m} \backslash U_k^m)))$$

where $C(d) = 2^{\lfloor d \rfloor}(\lfloor d \rfloor + 1)^d$.

Proof Let $\tilde{f}_k^m$ be the C^1-extension of f_k^m to some open neighbourhood of $\overline{U_k^m}$ and let $c(d) = 2\sqrt{\lfloor d \rfloor + 1}$. We have:

$$f^m(K) = \bigcup_{k=1}^{i(m)} f_k^m(K \cap U_k^m) \subseteq \bigcup_{k=1}^{i(m)} \tilde{f}_k^m(K \cap \overline{U_k^m}).$$

Hence

$$H^d_{c(d)\delta^{1/d}\epsilon}(f^m(K)) \leq H^d_{c(d)\delta^{1/d}\epsilon}(\bigcup_{k=1}^{i(m)} \tilde{f}_k^m(K \cap \overline{U_k^m}))$$
$$\leq \sum_{k=1}^{i(m)} H^d_{c(d)\delta^{1/d}\epsilon}(\tilde{f}_k^m(K \cap \overline{U_k^m})).$$

Fix $k \in \{1, \ldots, i(m)\}$. Since $\tilde{f}_k^m$ is C^1 on some open neighbourhood of the compact set $K \cap \overline{U_k^m}$ and

$$\sup_{x \in K \cap \overline{U_k^m}} W_d(\tilde{f}_k^m) < \delta$$

we get from lemma 3.3. of [10] that there exits $\epsilon_0(k)$ such that for all $\epsilon \in (0, \epsilon_0(k)]$

$$H^d_{c(d)\delta^{1/d}\epsilon}(\tilde{f}_k^m(K \cap \overline{U_k^m})) \leq C(d)\delta H^d_\epsilon(K \cap \overline{U_m^k})$$

Now let $\epsilon_0 := \min\{\epsilon_0(k) | k = 1, \ldots, i(m)\}$ an $\epsilon \in (0, \epsilon_0]$. We get

$$H^d_{c(d)\delta^{1/d}\epsilon}(f^m(K)) \leq C(d)\delta \sum_{k=1}^{i(m)} H^d_\epsilon(K \cap \overline{U_m^k})$$

and using the fact that $K \cap \overline{U_k^m} \subseteq (K \cap U_k^m) \cup (\overline{U_k^m} \backslash U_k^m)$

$$H^d_{c(d)\delta^{1/d}\epsilon}(f^m(K)) \leq C(d)\delta \sum_{k=1}^{i(m)} (H^d_\epsilon(K \cap U_k^m) + H^d_\epsilon(K \cap (\overline{U_k^m} \backslash U_k^m))).$$

Now taking the limit $\epsilon \longrightarrow 0$ we get

$$H^d(f^m(K)) \leq C(d)\delta \sum_{k=1}^{i(m)} (H^d(K \cap U_k^m) + H^d(K \cap (\overline{U_k^m} \backslash U_k^m)))$$

$$= C(d)\delta(H^d(K) + \sum_{k=1}^{i(m)} H^d(K \cap (\overline{U_k^m} \backslash U_k^m)))$$

To get the last equitation we used the fact that $U_k^m \cap K$ are disjoint Borel sets and $H^d(.)$ is a Borel measure (see appendix I and V of [12]) and thus especially additive. □

We need one another simple lemma that is about the singular value function.

Lemma 3.2 *Suppose $f : U \longmapsto M$ satisfying condition* (**PC**). *Let K be a compact f-invariant, i.e. f(K)=K, subset of U and let $d \in (0, n]$. We have*

$$\sup_{x\in K} W^d(f^m, x) < (\sup_{x\in K} W^d(f, x))^m$$

Proof This lemma follows immediately from Lemma 3.1. of [13]. The proof can be done in exactly the same way as in the usual situation of C^1 maps (see the corresponding statements in [10] and [3]). □

Proof of Theorem 2.1

We know from Lemma 2.2 that if $\sup_{x\in K} W^d(f, x) < 1$ holds we have

$$\lim_{m\longrightarrow\infty} \sup_{x\in K} W^d(f^m, x) = 0.$$

Hence under first assumptions of our theorem there exist $m \in N$ and $\delta \geq 0$ such that $C(d)\delta < 1$ and $\sup_{x\in K} W^d(f^m, x) < \delta$. Using the invariance of K we have $H^d(K) = H^d(f^m(K))$. Thus we get from lemma 3.1.

$$H^d(K) \leq C(d)\delta(H^d(K) + \sum_{k=1}^{i(m)} H^d(K \cap (\overline{U_k^m} \backslash U_k^m)))$$

and using Using the second assumption of our theorem we see that $H^d(K) \leq C(d)\delta H^d(K)$ with $C(d)\delta < 1$. Hence we have $H^d(K) = 0$ and consequently $\dim_H K \leq d$. □

4 Applications

The Belykh systems

We consider the class of Belykh systems given by piecewise affine transformations

$$f_{\beta,\tau,k}(x, y) : [-1, 1]^2 \longrightarrow [-1, 1]^2$$

$$f_{\beta,\tau,k}(x, y) = \begin{cases} (\beta x + (1-\beta), \tau y + (1-\tau)) & \text{if} \quad y \geq kx \\ (\beta x - (1-\beta), \tau y - (1-\tau)) & \text{if} \quad y < kx \end{cases}$$

where $\beta \in (0,1)$, $k \in (-1,1)$ and $\tau \in (1, 2/(|k|+1)])$. The original Belykh map was introduced by Belykh in [1]. This version of the Belykh map is due by to Pesin [11] who studied ergodic properties of this map. Dimensional theoretical properties of the Belykh attractor were studied by Schmeling [14].

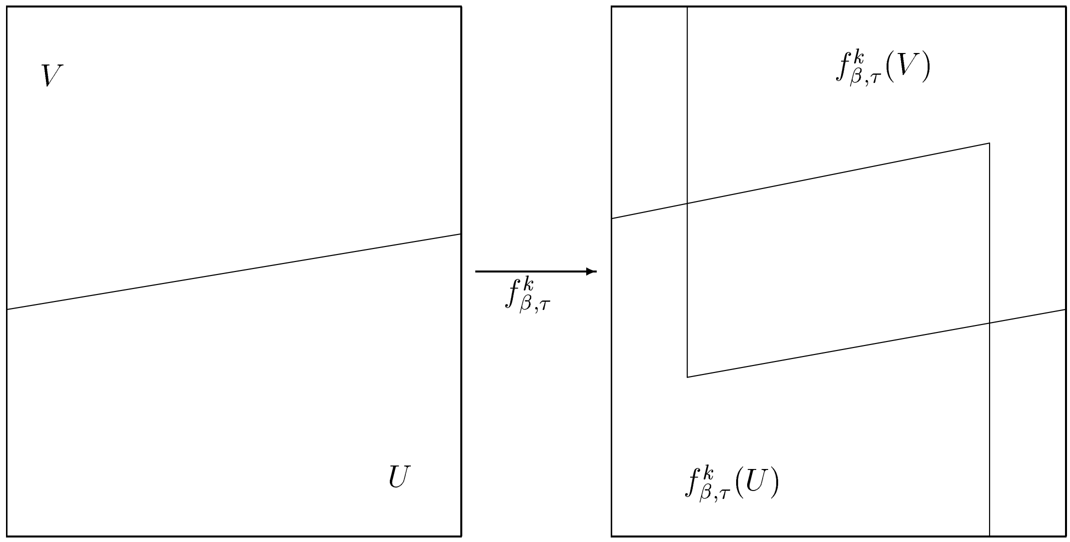

Figure 1: The Belykh maps

We want to apply theorem 2.1. to these systems.

Estimate 4.1 *Let $\beta \in (0,1)$, $k \in (-1,1)$ and $\tau \in (1, 2/(|k|+1)])$. and let K be a compact set which is invariant under $f_{\beta,\tau,k}$ then*

$$\dim_H K \leq 1 - \log\tau/\log\beta$$

Proof
Let $d = (1 - \log\tau/\log\beta) + \epsilon$ where $\epsilon > 0$. Note that $d > 1$. The singular values of $D_x f_{\beta,\tau,k}$ are constant and given by $\tau > \beta > 0$. Hence $W_d(f_{\beta,\tau,k}, (x,y)) = \tau\beta^{d-1} = \beta^\epsilon < 1$. Obviously we can choose the partitions $(\{U_1^m, \ldots, U_{i(m)}^m\}$ in a way such that the partition elements have one dimensional boundary and hence $H^d(\overline{U_k^m} \backslash U_k^m) = 0$. By Theorem 2.1 we now get $\dim_H K \leq d$. Since $\epsilon > 0$ was arbitrary our claim is proved. $\square$

Remark 4.1 Our estimate only gives some information if $\tau < \beta^{-1}$. If this is not the case we have the trivial estimate by dimension two.

Remark 4.2 Schmeling [14] showed that for almost all parameter values (with some technical restrictions) the Hausdorff dimension of the Belykh attractor is given by $\min\{2, 1 - \log\tau/\log\beta\}$. Thus if $\tau < \beta^{-1}$ the estimate obtained by theorem 2.1. is at least generically sharp.

Remark 4.3 Reitmann and Schnabel [13] remarked that they could not apply their dimension estimates to the Belykh systems in general. Thus we see that there are situations where our result is more appropriate.

The Lozi systems

Now for $b \in (0,1)$ and $a \in (0, 2(1-b))$ we consider the class of Lozi systems (see [11]) given by the transformations

$$g_{a,b}(x,y) : [-1/(1-b), 1/(1-b)]^2 \longrightarrow [-1/(1-b), 1/(1+b)]^2$$

$$g_{a,b}(x,y) = (1 + by - a|x|, x).$$

The Lozi map was introduced by Lozi in [7]. Ergodic properties of the map were studied in[2] and estimates of the Hausdorff dimension were given by Ishii [5].

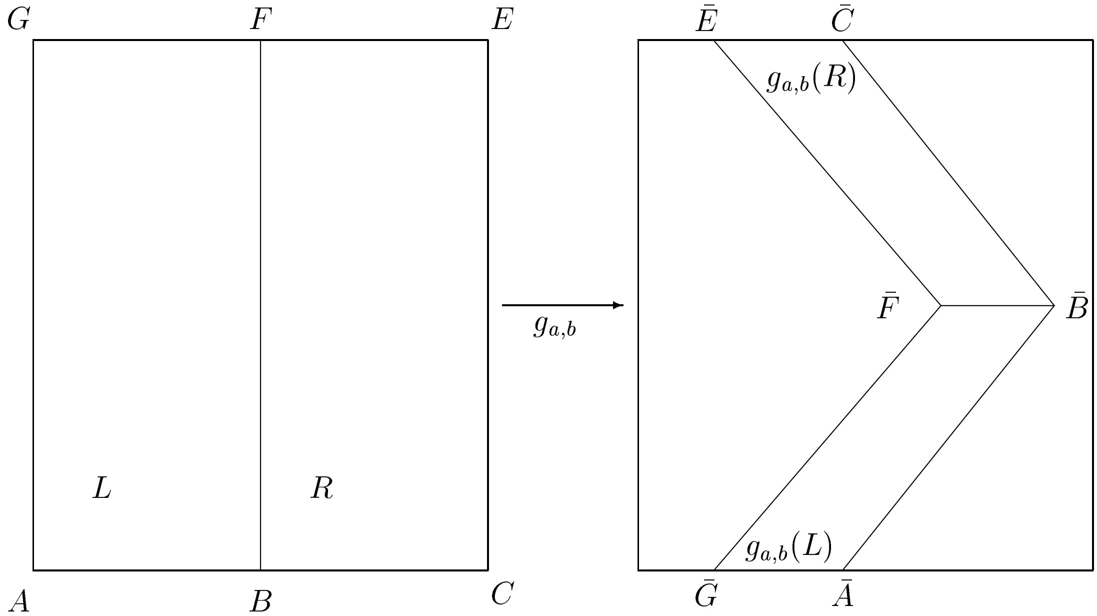

Figure 2: The Lozi maps

By Theorem 2.1 we get the following dimension estimate.

Estimate 4.2 *Let $b \in (0,1)$ and $a \in (0, 2(1-b))$ and let K be compact subset of $[-1/(1-b), 1/(1-b)]^2$ which is invariant under a Lozi map $g_{a,b}$. Set*

$$\beta_1 = 1/2(\sqrt{(a^2+b^2+1)^2 - 4b^2} + a^2 + b^2 + 1)$$

$$\beta_2 = 1/2(-\sqrt{(a^2+b^2+1)^2 - 4b^2} + a^2 + b^2 + 1$$

Furthermore assume that $\beta_2 < 1$. Then we have

$$\dim_H K \leq 1 - \log \beta_1 / \log \beta_2.$$

Proof Let $d = 1 - \log \beta_1 / \log \beta_2 + \epsilon$ where $\epsilon > 0$. Note that for all $(x,y) \in [-1/(1-b), 1/(1-b)]^2$ we have

$$(D_{(x,y)} g_{a,b})^T D_{(x,y)} g_{a,b} = \begin{pmatrix} a^2+1 & -ab \\ -ab & b^2 \end{pmatrix} \text{ if } x \geq 0$$

$$D_{(x,y)} g_{a,b})^T D_{(x,y)} g_{a,b} = \begin{pmatrix} a^2+1 & ab \\ ab & b^2 \end{pmatrix} \text{ if } x < 0$$

A simple calculation shows that the singular values are constant and given by $\alpha_1 = \sqrt{\beta_1}$ and $\alpha_2 = \sqrt{\beta_2}$. We have $W_d(g_{a,b}, (x,y)) = \alpha_1 \alpha_2^{d-1} = \alpha_2^{\epsilon} < 1$. Furthermore the singularity of the system $([-1/(1-b), 1/(1-b)]^2, g_{a,b})$ is given by $S = [-1,1] \times \{0\}$ and since $g_{a,b}$ is just a affine map on $[-1/(1-b), 0]^2$ and $[0, 1/(1-b)]^2$ we see that $\bigcup_{i=0}^{m} g_{a,b}^{-i}(S)$ consists of a finite number of line segments. Hence we can choose the U_k^m as domains bounded by a polygon. The boundary of these sets is thus one dimensional and since $d > 1$ we get $H^d(\overline{U_k^m} \backslash U_k^m) = 0$. We are thus in a situation where theorem 2.1. applies and since $\epsilon > 0$ was arbitrary our claim is proved. $\square$

Remark 4.4 If we set $b = 1.7$ and $a = 0.1$ we get $\dim_H K \leq 1.18761294\ldots$ which is better than the upper estimate by $1.247848\ldots$ found in [5]. By [5] a lower bound on the Hausdorff dimension is given by $1.176669\ldots$. In fact we do not know if our upper bound is sharp.

Piecewise affine solenoid like systems

Consider the following class of three dimensional piecewise affine maps

$$f_\vartheta : [-1,1]^3 \longmapsto [-1,1]^3$$

$$f_\vartheta(x,y,z) = \{ \begin{array}{lll} (\beta_1 x + (1-\beta_1), 2y-1, \tau_1 z + (1-\tau_1)) & \text{if} & y \geq 0 \\ (\beta_2 x - (1-\beta_2), 2y+1, \tau_2 z - (1-\tau_2)) & \text{if} & y < 0 \end{array}$$

where $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in P := \{(u,v,s,t) | u,v,s,t \in (0,1), s+t \geq 1, u+v < 1, u > v, s > t\}$.

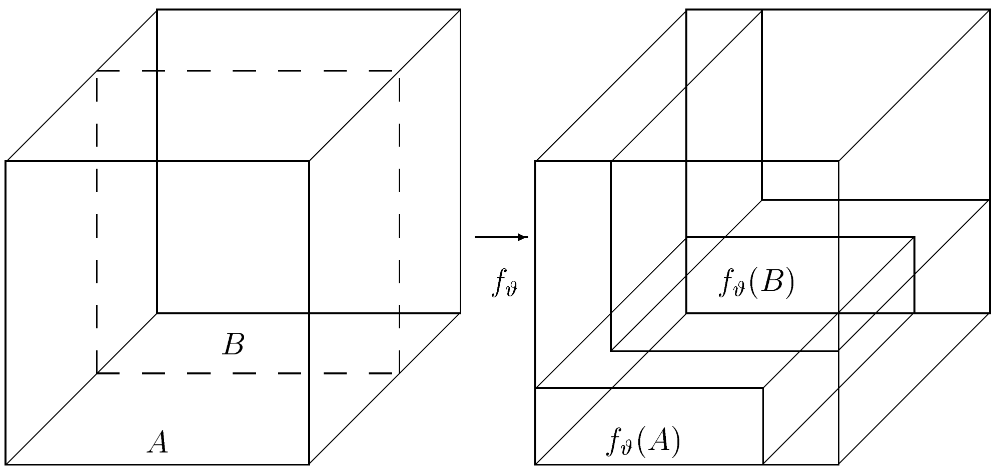

Figure 3: The action of f_ϑ on $[-1,1]^3$.

Again by Theorem 2.1 we get can get a dimension estimates.

Estimate 4.3 *Let $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in P$ and K be a compact f_ϑ invariant set then*

$$\dim_H K \leq 2 - \frac{\log(2\beta_1)}{\log(\tau_1)}.$$

Proof Let $d = 2 - \frac{\log(2\beta_1)}{\log(\tau_1)} + \epsilon$ where $\epsilon > 0$. The singular values in a point (x,y,z) are given by $2 \geq \beta_1 \geq \tau_1$ if $y \geq 0$ and $2 \geq \beta_1 \geq \tau_1$ if $y < 0$. Thus $W^d(f_\vartheta, (x,y,z)) = 2\beta_1\tau_1^{d-2}$ if $y \geq 0$ and $W^d(f_\vartheta, (x,y,z)) = 2\beta_2\tau_2^{d-2}$ if $y < 0$ but anyway $W^d(f_\vartheta, (x,y,z)) < 1$. It is easy to see that one can choose the sets U_i^m with two dimensional boundary and since $d > 2$ we get $H^d(\overline{U_k^m} \backslash U_k^m) = 0$. Thus we are done. $\square$

Remark 4.5 It follows from [8] that the Hausdorff dimension of the attractor of the map f_ϑ is always bounded from above by the solution of $\beta_1\tau_1^{x-2} + \beta_2\tau_2^{x-2} = 1$ and that in some parts of the parameter space this upper bound is at least generically sharp (in the sense of Lebesgue measure). This example shows that the estimates obtained by 2.1 can not be expected to be sharp in general. The problem is that the singular value function is not constant here and Douady-Osterlé type estimates take the worst contraction rates of the system everywhere into consideration. If we consider the symmetric case $\beta_1 = \beta_2 = \beta$ and $\tau_1 = \tau_2 = \tau$ the singular value function is constant and the estimate in 4.3 is generically sharp (see again [8])

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