Properties of some overlapping self-similar and some self-affine measures

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Abstract

We generalise the Theorems of Peres and Solomyak about the absolute continuity resp. singularity of Bernoulli convolutions ([19], [16], [17]) to a broader class of self-similar measures on the real line. Using the dimension theory of ergodic measures (see [11] and [2]) we find a formula for the dimension of certain self-affine measures in terms of the dimension of the above mentioned self-similar measures. Combining these results we show the identity of Hausdorff and box-counting dimension of a special class of self-affine sets.

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1. Introduction

Self-similar sets and measures are the simplest objects of fractal geometry. They raised great interest in mathematics and also natural science. If the images of the similarities are separated we know the Hausdorff and box-counting dimension of the corresponding self-similar sets and measures.[2]. But if these images overlap the situation is more difficult. Recently one major progress was achieved by Peres and Solomyak (see [19], [16], [17]). They succeeded in determining generic properties of symmetric overlapping self-similar measures on the real line (Bernoulli convolutions). In this article we will use techniques developed by Peres and Solomyak to find a generalisation of these results to asymmetric overlapping self-similar measure on the real line (see Theorem I below).
Self-affine sets and measures provide a more general class of fractal objects. The dimensional theoretical properties of these sets and measures are very difficult to understand. The existence of different rates of contraction in different directions of the affine maps inducing these sets and measures forces mathematical problems that are not solved in general these days. We will use here a dynamical approach relying on the dimension theory of ergodic measures(see [11] and [2]). We will find an expression of the dimension of certain self-affine measures in terms of the dimension of overlapping self-similar measures (see Theorem II below).
Using these results we are able to find generically self-affine measures of full dimension on a

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[2]We recommend the book of Falconer [6] or the book of Pesin [15] for an introduction to fractal geometry and dimension theory. All notions we presuppose here can be found in this books.

special class of self-affine sets. As a consequence we show that the Hausdorff dimension of these sets equals theirs box-counting dimension, which is easy to calculate (see Theorem III below). The rest of the paper is organised as follows: In section 2 we give some definitions and state our main results. In section 3 we proof Theorem I, in section 4 we proof Theorem II and in the last section we proof Theorem III.

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2. Notations and results

Let $\Sigma^+ = \{-1,1\}^{\mathbb{N}_0}$ and let $\Sigma = \{-1,1\}^{\mathbb{Z}}$. Define a metric $\tilde{d}$ on Σ^+ resp. Σ by

$$\tilde{d}(\underline{s},\underline{t}) = \sum_{k=j}^{\infty} |s_k - t_k| 2^{-|k|}$$

where $j = 0$ resp. $j = -\infty$ and $\underline{s} = (s_k)$, $\underline{t} = (t_k)$. Moreover let b^p for $p \in (0,1)$ be the Bernoulli measure on Σ^+ which is the product of the discrete measure on $\{-1,1\}$ assigning 1 the probability p and -1 the probability $(1-p)$. Given $\underline{s} \in \Sigma^+$ we denote by $\sharp_k(\underline{s})$ the cardinality of the set $\{s_i | s_i = -1 \quad i = 0\dots k\}$. Now for $\beta_1, \beta_2 \in (0,1)$ we define a map $\pi_{\beta_1,\beta_2} : \Sigma^+ \longmapsto \mathbb{R}$ by

$$\pi_{\beta_1,\beta_2}(\underline{s}) = \sum_{k=0}^{\infty} s_k \beta_2^{\sharp_k(\underline{s})} \beta_1^{k-\sharp_k(\underline{s})+1}.$$

Obviously this map is continuous with respect to the metric $\tilde{d}$ and its image is in the interval $[\frac{-\beta_2}{1-\beta_2}, \frac{\beta_1}{1-\beta_1}]$. Hence $b^p_{\beta_1,\beta_2} := \pi_{\beta_1,\beta_2}(b^p) = b^p \circ \pi^{-1}_{\beta_1,\beta_2}$ is a Borel probability measure concentrated on $[\frac{-\beta_2}{1-\beta_2}, \frac{\beta_1}{1-\beta_1}]$. It is easy to see that $b^p_{\beta_1,\beta_2}$ is a **self-similar measure** in the following sense

$$b^p_{\beta_1,\beta_2} = pL_1(b^p_{\beta_1,\beta_2}) + (1-p)L_2(b^p_{\beta_1,\beta_2})$$

where $L_1(x) = \beta_1 x + \beta_1$ and $L_2(x) = \beta_2 x - \beta_2$ are linear maps on the real line.
If $\beta_1 + \beta_2 \geq 1$ we call the measure $b^p_{\beta_1,\beta_2}$ overlapping and if $\beta_1 + \beta_2 < 1$ we call it non-overlapping. Properties of non-overlapping self-similar measures are well known. In the non-overlapping case $b^p_{\beta_1,\beta_2}$ is concentrated on a Cantor set and it's Hausdorff and box-counting dimension are given by $\frac{p\log p + (1-p)\log(1-p)}{p\log\beta_1 + (1-p)\log\beta_2}$ (see [8]). Now let $\beta \in [0.5,1)$. The symmetric overlapping self-similar measures $b^p_\beta := b^p_{\beta,\beta}$ are usually called **Bernoulli convolutions**. The equal weighted Bernoulli convolutions $b_\beta := b^{0.5}_\beta$ raised great interest in mathematics since Erdös ([4], [5]) discovered that b_β is absolutely continuous for almost all β in some neighbourhood of one and singular if β is the reciprocal of a Pisot-Vijayarghavan number.[3] Recently one major progress about Bernoulli convolutions was achieved by Peres and Solomyak:

[3] Recall that a Pisot-Vijayarghavan number is an algebraic integer $\alpha > 1$ which has all it's conjugates inside the unit circle; see [1]. Consequences of number theoretical peculiarities in the dimension theory of dynamical systems are part of my dissertation (see [13], [14] and references in there).

Theorem ([17])

Let $p \in (0,1)$ and $q \in (1,2]$. Let $I = [0.5,1]$ if $p \in [1/3,2/3]$ and let $I = [0.5,0.649]$ if not. For almost all $\beta \in I$ the measures b_β^p are absolutely continuous if $\beta \geq p^p(1-p)^{1-p}$ and have a density in L^q if $\beta \geq (p^q+(1-p)^q)^{\frac{1}{q-1}}$. If $\beta < p^p(1-p)^{1-p}$ the measures b_β^p are singular with

$$\overline{\dim}_B b_\beta^p \leq \frac{p\log p + (1-p)\log(1-p)}{\log\beta}$$

Remark

Solomyak [19] first proved that b_β is absolutely continuous with density in L^2 for almost all $\beta \in [0.5,1]$. Then Peres and Solomyak [16] gave a considerably simplified proof of this statement and then proved the more general result stated here.

As far as we know the overlapping asymmetric self-similar measures b_{β_1,β_2}^p have not been studied jet. This will be done here:

Theorem I

Let $p \in (0,1)$ and $q \in (1,2]$. For almost all $(\beta_1,\beta_2) \in (0,0.649)^2$ the measures b_{β_1,β_2}^p are absolutely continuous if $(\beta_2 p)^p(\beta_1(1-p))^{1-p} \leq \beta_1\beta_2$ and have a density in L^q if $(\beta_2^{q-1}p^q + \beta_1^{q-1}(1-p)^q)^{\frac{1}{q-1}} \leq \beta_1\beta_2$. If $(p\beta_2)^p((1-p)\beta_1)^{1-p} > \beta_1\beta_2$ the measures b_{β_1,β_2}^p are singular with

$$\overline{\dim}_B b_\beta^p \leq \frac{p\log p + (1-p)\log(1-p)}{p\log\beta_1 + (1-p)\log\beta_2}.$$

Remarks

(1) We have to say a few word about the bound 0.649 that appears in theorem I. In step 4 of the proof we will see that it is due to a certain **transversality condition** that we need. In fact the bound is given by the infimum of all double zeros of power series with absolute value of the coefficients less or equal to one and first coefficient equal to one. An lower approximation of this quantity is 0.649 (see [19]). Peres and Solomyak ([19], [16], [17]) used some additional arguments concerning Fourier transformations to improve this bound to 1 in the symmetric situation if $p \in [1/3,2/3]$. These arguments do not work if $p < 1/3$. Moreover we have not been able to improve the bound in the asymmetric situation and do not know if this bound is really essential.

(2) It is an interesting question if there are numbertheoretical exceptions to our generic result in the asymmetric situation. We have thought about this but have been not able to solve the problem.

Now given $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in (0,1)^4$ with $\tau_1 + \tau_2 \leq 1$ we consider two affine contractions in $I\!R^2$:

$$T_1(x,y) = (\beta_1 x + \beta_1, \tau_1 y + \tau_1) \qquad \text{and} \qquad T_{-1}(x,y) = (\beta_2 x - \beta_2, \tau_2 y - \tau_2).$$

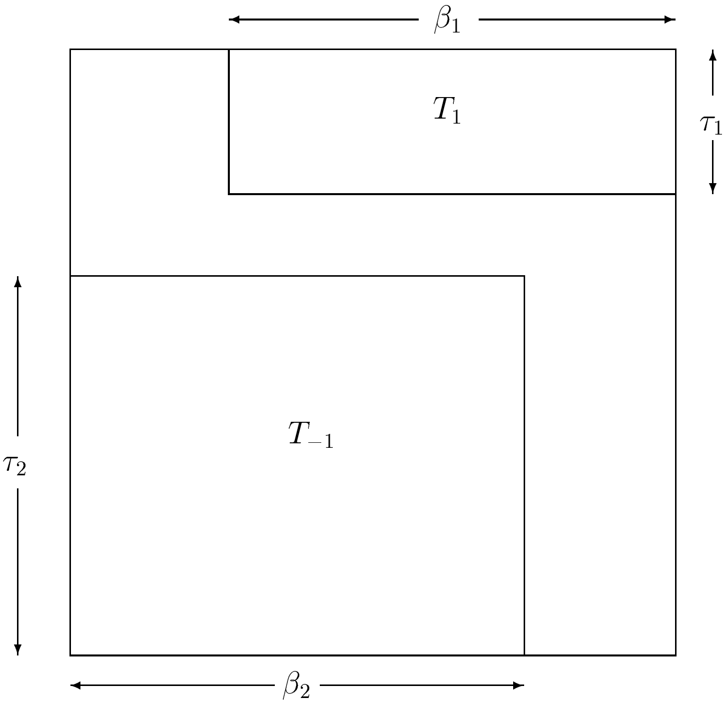

Figure 1: The action of the transformations T_1 and T_{-1} scaled on the unite square

We know that there is a unique **self-affine set** Λ_ϑ with $\Lambda_\vartheta = T_1(\Lambda_\vartheta) \cup T_{-1}(\Lambda_\vartheta)$ (see [9]). Define the natural coding homeomorphism $\hat{\pi}_\vartheta$ from Σ onto Λ_ϑ by

$$\hat{\pi}_\vartheta((s_k)) = \lim_{n \longrightarrow \infty} T_{s_1} \circ T_{s_2 2} \circ \ldots \circ T_{s_n}([\frac{-\beta_2}{1-\beta_2}, \frac{\beta_1}{1-\beta_1}] \times [\frac{-\tau_2}{1-\tau_2}, \frac{\tau_1}{1-\tau_1}]).$$

Given $p \in (0,1)$ we define a Borel probability measure $\hat{b}^p_\vartheta$ on Λ_ϑ by $\hat{b}^p_\vartheta = \hat{\pi}_\vartheta(b^p) = b^p \circ \hat{\pi}_\vartheta^{-1}$. It is obvious that $\hat{b}^p_\vartheta$ is a **self-affine measure** in the following sense:

$$\hat{b}^p_\vartheta = pT_1(\hat{b}^p_\vartheta) + (1-p)T_{-1}(\hat{b}^p_\vartheta).$$

Using the results of Ledrappier and Young [11] we can express the dimension of the self-affine measures $\hat{b}^p_\vartheta$ in terms of the dimension of the self-similar measures $b^p_{\beta_1,\beta_2}$:

Theorem II

For all $p \in (0,1)$ and all $\vartheta = (\tau_1, \tau_2, \beta_1, \beta_2) \in (0,1)^4$ with $\tau_1 + \tau_2 \leq 1$ and $p \log \tau_1 + (1-p) \log \tau_2 \leq p \log \beta_1 + (1-p) \log \beta_2$ we have

$$\dim_B \hat{b}^p_\vartheta = \dim_H \hat{b}^p_\vartheta = \frac{p \log p + (1-p)\log(1-p)}{p \log \tau_1 + (1-p) \log \tau_2} + (1 - \frac{p \log \beta_1 + (1-p) \log \beta_2}{p \log \tau_1 + (1-p) \log \tau_2}) \dim_H b^p_{\beta_1,\beta_2}.$$

Remark

The condition $p \log \tau_1 + (1-p) \log \tau_2 \leq p \log \beta_1 + (1-p) \log \beta_2$ means that the second coordinate axis forms the strong unstable direction with respect to the measure $\hat{b}^p_\vartheta$ (see section 4).

With the help of Theorem I and Theorem II we get the following result about the dimension of the self-affine sets Λ_ϑ:

Theorem III

For almost all $\vartheta \in \{(\beta_1, \beta_2, \tau_1, \tau_2) \in (0, 0.649)^2 \times (0,1)^2 \mid \beta_1 + \beta_2 \geq 1, \quad \tau_1 + \tau_2 < 1\}$ we have

$$\dim_H \hat{b}^p_\vartheta = \dim_H \Lambda_\vartheta = \dim_B \Lambda_\vartheta = d + 1,$$

where d is the solution of $\beta_1 \tau_1^x + \beta_2 \tau_2^x = 1$ and $p = \beta_1 \tau_1^d$.

Remarks

(1) We like to emphasise the difference of our result to the "classical" theorem of Falconer [7] about the identity of box-counting and Hausdorff dimension of self-affine sets. First of all Falconer's result is generic with respect to translations of given linear maps but our result is generic with respect to the contraction rates. Secondly Falconer had to assume the the contraction rates are less than 1/3. Solomyak [20] improved this bound to 1/2 and showed that Falconer's theorem does not hold if we replace 1/2 by $1/2 + \delta$ where $\delta > 0$. In our situation we have larger contraction rates.

(2) Consider the symmetric self-similar sets $\Lambda_{\beta,\tau} := \Lambda_{\beta,\beta,\tau,\tau}$. It follows from Theorem II and [19] that for almost all $\beta \in (0,5,1)$ and all $\tau \in (0, 0.5)$ the identity

$$\dim_H \hat{b}^{0.5}_{\beta,\beta,\tau,\tau} = \dim_H \Lambda_{\beta,\tau} = \dim_B \Lambda_{\beta,\tau} = \frac{\log 2\beta}{\log \tau} + 1$$

holds. This identity has been shown before by Pollicott and Weiss [18] under the assumption that β is a Garsia-Erdös number.[4]

[4]This means $\exists\, C > 0\ \forall x \in I\!R : \mathrm{card}\{(s_0, \ldots, s_{n-1}) \in \{-1,1\}^n \mid \sum_{k=0}^{n-1} s_k \beta^k \in [x, x + \beta^n)\} \leq C(2\beta)^n\ \forall n \geq 1.$

3. Proof of Theorem I

Part I: Absolute continuity

By the theorem of Fubini the following claim is stronger than our assertions about the absolute continuity of the measures $b^p_{\beta_1,\beta_2}$ in theorem I.

Claim: Let $p \in (0,1)$, $q \in (1,2]$ and $c \in (0,1]$. The density of the measures $b^p_{\gamma,c\gamma}$ is in L^q for almost all $\gamma \in [\gamma_0(c,q,p), 0.649]$ where $\gamma_0(c,q,p) = (p^q + c^{1-q}(1-p)^q)^{\frac{1}{q-1}}$.

We proof this claim. Fix p, q and c during the proof.

1. Step: An integral condition for the measures to have density in L^q

We define the (lower) local density of a measure μ on the real line by

$$\underline{D}(\mu, x) = \underline{\lim}_{r\longrightarrow 0}\frac{\mu(B_r(x))}{2r}.$$

If we have

$$\int (\underline{D}(\mu, x))^{q-1} d\mu(x) < \infty$$

then μ is absolute continuous and has density in L^q. This follows from 2.12 of [12]. Thus it is sufficient for us to show that

$$\Im(\gamma_0) := \int_{\gamma_0}^{0.649} \int \ (D(b^{p*}_{\gamma,c\gamma}, x))^{q-1} db^{p*}_{\gamma,c\gamma}(x) \ d\gamma < \infty$$

holds for all $\gamma_0 > \gamma_0(c,q,p)$.

2. Step: Some estimates on the integral $\Im$

By applying Fatou's lemma then changing variables using the definition of the measures $b^p_{\gamma,c\gamma}$ and reversing the order of integration we obtain:

$$\Im(\gamma_0) \leq \underline{\lim}_{r\longrightarrow 0}\frac{1}{(2r)^{q-1}}\int_{\gamma_0}^{0.649}\int \ (b^p_{\gamma,c\gamma}(B_r(x)))^{q-1} \ db^p_{\gamma,c\gamma}(x)d\gamma$$

$$= \underline{\lim}_{r\longrightarrow 0}\frac{1}{(2r)^{q-1}}\int_{\gamma_0}^{0.649}\int_{\Sigma+}(b^p_{\gamma,c\gamma}(B_r(\pi_{\gamma,c\gamma}(\underline{s}))))^{q-1} \ db^p(\underline{s})d\gamma$$

$$= \underline{\lim}_{r\longrightarrow 0}\frac{1}{(2r)^{q-1}}\int_{\Sigma+}\int_{\gamma_0}^{0.649}(b^p_{\gamma,c\gamma}(B_r(\pi_{\gamma,c\gamma}(\underline{s}))))^{q-1} \ d\gamma db^p(\underline{s}).$$

Applying Hölder's inequality, $\int f^\alpha \leq C_1(\int f)^\alpha$ where $\alpha \in (0,1]$ and $f \geq 0$, we get

$$\Im(\gamma_0) \leq C_1\underline{\lim}_{r\longrightarrow 0}\frac{1}{(2r)^{q-1}}\int_{\Sigma+}(\int_{\gamma_0}^{0.649} b^p_{\gamma,c\gamma}(B_r(\pi_{\gamma,c\gamma}(\underline{s})) \ d\gamma)^{q-1}db^p(\underline{s}).$$

Now note that

$$\int_{\gamma_0}^{0.649} b^p_{\gamma,c\gamma}(B_r(\pi_{\gamma,c\gamma}(\underline{s})))\ d\gamma = \int_{\gamma_0}^{0.649} \int \mathbf{1}_{B_r(\pi_{\gamma,c\gamma}(\underline{s}))}(x)\ db^p_{\gamma,c\gamma}(x) d\gamma$$

$$= \int_{\gamma_0}^{0.649} \int_{\Sigma^+} \mathbf{1}_{\{\underline{t}|\ |\pi_{\gamma,c\gamma}(\underline{s}) - \pi_{\gamma,c\gamma}(\underline{t})| \leq r\}} db^p(\underline{t}) d\gamma$$

$$= \int_{\Sigma^+} \ell(\{\gamma \in [\gamma_0, 0.649]|\ |\pi_{\gamma,c\gamma}(\underline{s}) - \pi_{\gamma,c\gamma}(\underline{t})| \leq r\})\ db^p(\underline{t}).$$

where ℓ denotes the Lebesgue measure. Thus $\Im(\gamma_0)$ is bounded from above by

$$C_1\ \underline{\lim}_{r\longrightarrow 0} \frac{1}{(2r)^{q-1}} \int_{\Sigma^+} (\int_{\Sigma^+} \ell(\{\gamma \in [\gamma_0, 0.649]|\ |\pi_{\gamma,c\gamma}(\underline{s}) - \pi_{\gamma,c\gamma}(\underline{t})| \leq r\})\ db^p(\underline{t}))^{q-1} db^p(\underline{s}).$$

3. Step: Using the structure of the map π_{β_1,β_2}

For $\underline{s} = (s_k)$ and $\underline{t} = (t_k)$ in Σ^+ let $|\underline{s} \wedge \underline{t}| = \min\{k|s_k \neq t_k\}$. We have

$$\phi_{\underline{s},\underline{t}}(\gamma) := \pi_{\gamma,c\gamma}(\underline{s}) - \pi_{\gamma,c\gamma}(\underline{t}) = \sum_{k=0}^{\infty} (s_k c^{\sharp_k(\underline{s})} - t_k c^{\sharp_k(\underline{t})})\gamma^{k+1}$$

$$= \gamma^{|\underline{s}\wedge\underline{t}|+1} \sum_{k=0}^{\infty} (s_{k+|\underline{s}\wedge\underline{t}|} c^{\sharp_{k+|\underline{s}\wedge\underline{t}|}(\underline{s})} - t_{k+|\underline{s}\wedge\underline{t}|} c^{\sharp_{k+|\underline{s}\wedge\underline{t}|}(\underline{t})})\gamma^k$$

$$= \gamma^{|\underline{s}\wedge\underline{t}|+1} (s_{|\underline{s}\wedge\underline{t}|} c^{\sharp_{|\underline{s}\wedge\underline{t}|}(\underline{s})} - t_{|\underline{s}\wedge\underline{t}|} c^{\sharp_{|\underline{s}\wedge\underline{t}|}(\underline{t})})(1 + \sum_{k=1}^{\infty} \underbrace{\frac{s_{k+|\underline{s}\wedge\underline{t}|} c^{\sharp_{k+|\underline{s}\wedge\underline{t}|}(\underline{s})} - t_{k+|\underline{s}\wedge\underline{t}|} c^{\sharp_{k+|\underline{s}\wedge\underline{t}|}(\underline{t})}}{s_{|\underline{s}\wedge\underline{t}|} c^{\sharp_{|\underline{s}\wedge\underline{t}|}(\underline{s})} - t_{|\underline{s}\wedge\underline{t}|} c^{\sharp_{|\underline{s}\wedge\underline{t}|}(\underline{t})}}}_{:=a_k(\underline{s},\underline{t})} \gamma^k)$$

$$= 1/2(s_{|\underline{s}\wedge\underline{t}|} - t_{|\underline{s}\wedge\underline{t}|})(1+c)\gamma^{|\underline{s}\wedge\underline{t}|+1} c^{\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})} (1 + \sum_{k=1}^{\infty} a_k(\underline{s},\underline{t})\gamma^k).$$

For the last equation we used the fact that $\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s}) = \sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{t})$.[5] Now setting $g_{\underline{s},\underline{t}}(\gamma) = 1 + \sum_{k=1}^{\infty} a_k(\underline{s},\underline{t})\gamma^k$ and $C_2 = 1/2(s_{|\underline{s}\wedge\underline{t}|} - t_{|\underline{s}\wedge\underline{t}|})(1+c)$ we have the formula

$$\phi_{\underline{s},\underline{t}}(\gamma) = C_2 \gamma^{|\underline{s}\wedge\underline{t}|+1} c^{\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})} g_{\underline{s},\underline{t}}(\gamma).$$

Here the absolute value of C_2 does not depend on $\underline{s}$ and $\underline{t}$. We now claim that the absolute value of the coefficients of the power series $g_{\underline{s},\underline{t}}$ is less or equal to one:

$$|a_k(\underline{s},\underline{t})| \leq 1 \qquad \forall\ k > 0 \text{ and } \underline{s},\underline{t} \in \Sigma^+.$$

Since $\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s}) = \sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{t})$ we can write

$$|a_k(\underline{s},\underline{t})| = \frac{(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s}))_k c^{\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s})} - (\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t}))_k c^{\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t})}}{1+c}.$$

[5] We use the convention that $\sharp_n(\underline{s}) = 0$ if $n < 0$

But we have

$$|(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s}))_k c^{\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s})} - (\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t}))_k c^{\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t})}| \leq |c^{\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s})}| + |c^{\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t})}| \leq 1 + c$$

by the definition of $|\underline{s} \wedge \underline{t}|$, which proves our claim.

4. Step: The transversality condition

Now we need some pure analytical tools to continue with the proof. We say that the **ρ-transversality** condition holds for a C^1 function g on a closed interval I if $g(x) < \rho \Rightarrow g^{'}(x) > \rho\ \forall x \in I$. This means that the graph of the function g crosses all horizontal lines that it meets below height λ transversally with slope at most $-\rho$. Obviously the transversality condition holds for some ρ on an interval I if and only if g has no double zero on the interval I $(g(x) = 0 \Rightarrow g^{'}(x) \neq 0\ \forall x \in I)$.

If we have the ρ-transversality condition for g on I then

$$\ell\{x \in I||g(x)| \leq r\} \leq 2r\rho^{-1} \qquad \forall r > 0.$$

This is easy to see. If $r \geq \rho$ then the claim is obvious. If $r < \rho$ then g is monotonous decreasing with $g^{'} < -\rho$ on the set $\{x \in I||g(x)| \leq r\}$ by ρ-transversality. But this immediately yields the assertion.

From [19] we know that

$$O := \inf\{x|x \text{ is a double zero of a power series } f = 1 + \sum_{k=1}^{\infty} a_k x^k \text{ with } |a_k| \leq 1\}$$
$$\approx 0.649138.$$

It follows that there is a ρ such that the ρ-transversality condition holds for all power series $f = 1 + \sum_{k=1}^{\infty} a_k x^k$ with $|a_k| \leq 1$ on the Interval $[0, O]$. Especially ρ transversality holds for all power series $g_{\underline{s},\underline{t}}$ defined in the third step of our proof on $[0, 0.649]$. Thus we get

$$\ell\{\gamma \in [\gamma_0, 0.649]|\ |\phi_{\underline{s},\underline{t}}(\gamma)| \leq r\}$$
$$\leq \ell\{\gamma \in [\gamma_0, 0.649]|\ |g_{\underline{s},\underline{t}}(\gamma)| \leq r|C_2|^{-1}\gamma^{-|\underline{s}\wedge\underline{t}|-1}c^{-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}\}$$
$$\leq 2\rho^{-1}r|C_2|^{-1}\gamma_0^{-|\underline{s}\wedge\underline{t}|-1}c^{-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})} = C_3 r\gamma_0^{-|\underline{s}\wedge\underline{t}|-1}c^{-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})} \text{ with } C_3 = 2\rho^{-1}|C_2|^{-1}.$$

5. Step: Integrating

We put our estimates of step two and four together and obtain

$$\Im(\gamma_0) \leq C_4 \int_{\Sigma+}(\int_{\Sigma+} \gamma_0^{-|\underline{s}\wedge\underline{t}|-1}c^{-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}\ db^p(\underline{t}))^{q-1}db^p(\underline{s})$$

where $C_4 = C_1 C_3^{q-1} 2^{1-q}$. Now we integrate:

$$\int_{\Sigma^+} \gamma_0^{-|\underline{s}\wedge\underline{t}|-1} c^{-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}\, db^p(\underline{t}) = \sum_{n=0}^{\infty} \gamma_0^{-n-1} c^{\sharp_{n-1}(\underline{s})} b^p(\{\underline{t}\in\Sigma^+|\ |\underline{s}\wedge\underline{t}| = n\})$$

$$= \sum_{n=0}^{\infty} \gamma_0^{-n-1} c^{-\sharp_{n-1}(\underline{s})} p^{n-\sharp_{n-1}(\underline{s})}(1-p)^{\sharp_{n-1}(\underline{s})}(s_n(1/2-p)+1/2).$$

Using the inequality $(\sum x_i)^\alpha \le \sum x_i^\alpha$ for $\alpha = q-1 \le 1$ we continue with

$$\Im(\gamma_0) \le C_4 \sum_{n=0}^{\infty} \int_{\Sigma^+} (\gamma_0^{-n-1} c^{-\sharp_{n-1}(\underline{s})} p^{n-\sharp_{n-1}(\underline{s})}(1-p)^{\sharp_{n-1}(\underline{s})}(s_n(1/2-p)+1/2))^{q-1} db^p(\underline{s})$$

$$= C_4 \sum_{n=0}^{\infty} \gamma_0^{(-n-1)(q-1)}((1-p)^{q-1}p + p^{q-1}(1-p)) \sum_{k=0}^{n} (c^{-k}(1-p)^k p^{n-k})^{q-1} b^p\{\underline{s}\in\Sigma^+|\sharp_{n-1}(\underline{s}) = k\}$$

$$= C_4((1-p)^{q-1}p + p^{q-1}(1-p)) \sum_{n=0}^{\infty} \gamma_0^{(-n-1)(q-1)} \sum_{k=0}^{n} \binom{n}{k} ((1-p)^q c^{1-q})^k p^{q(n-k)}$$

$$= C_4((1-p)^{q-1}p + p^{q-1}(1-p))\gamma_0^{1-q} \sum_{n=0}^{\infty} (\gamma_0^{-(q-1)}((1-p)^q c^{1-q} + p^q)^n.$$

The sum in the last expression converges if and only if $\gamma_0 > \gamma_0(c,q,p) = (p^q + c^{1-q}(1-p)^q)^{\frac{1}{q-1}}$. So $\Im(\gamma_0) < \infty$ holds for all $\gamma_0 > \gamma_0(c,q,p)$. This proofs our claim.

Part II: Singularity

Fix $\beta_1, \beta_2, p \in (0,1)$. We define a metric δ^{β_1,β_2} on Σ^+ by

$$\delta^{\beta_1,\beta_2}(\underline{s},\underline{t}) = \beta_1^{|\underline{s}\wedge\underline{t}|-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})} \beta_2^{\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}.$$

We first show that

$$d^{\beta_1,\beta_2}(\underline{s}, b^p) := \lim_{\epsilon\longrightarrow 0} \frac{\log B_\epsilon^{\beta_1,\beta_2}(\underline{s})}{\log \epsilon} = \frac{p\log p + (1-p)\log(1-p)}{p\log\beta_1 + (1-p)\log\beta_2}$$

holds b^p-almost everywhere. Here d^{β_1,β_2} is the local dimension of the measure b^p with respect to metric δ^{β_1,β_2} and accordingly $B_\epsilon^{\beta_1,\beta_2}$ is a ball of radius ϵ with respect to this metric.
Let σ be the shift map on Σ^+, i.e. $\sigma((s_k)) = (s_{k+1})$ and defiene a cylinder set $[s_0,\dots,s_n]_0$ by

$$[s_0,\dots,s_n]_0 := \{(t_k)\in\Sigma^+ | t_k = s_k \quad k = 0\dots n\}$$

It is well known that the system (Σ^+, σ, b^p) is ergodic. Hence by applying Birkhoff's ergodic Theorem (see theorem 4.1.2. of [10]) to the function

$$h(\underline{s}) = \{ \begin{array}{lll} \log\beta_1 & \text{if} & s_0 = 1 \\ \log\beta_2 & \text{if} & s_0 = -1 \end{array}$$

we obtain

$$\lim_{n\longrightarrow\infty}\frac{1}{n+1}\log \mathrm{diam}_{\beta_1,\beta_2}([s_0,\dots,s_n]_0)=\lim_{n\longrightarrow\infty}\frac{1}{n+1}\sum_{k=0}^{n+1}h(\sigma^k(\underline{s}))=\int h\ db^p(\underline{s})$$

$$=p\log\beta_1+(1-p)\log\beta_2 \qquad b^p-\text{almost everywhere.}$$

By Shannon-McMillan-Breiman Theorem (see theorem 13.4. of [3]) we have

$$\lim_{n\longrightarrow\infty}-\frac{1}{n+1}\log b^p([s_0,\dots,s_n]_0)=h_{b^p}(\sigma) \qquad b^p\text{-almost everywhere,}$$

where $h_{b^p}(\sigma)$ is the metric entropy of the shift map σ with respect to the ergodic measure b^p. With the help of the well known formula $h_\sigma(b^p)=-(p\log p+(1-p)\log(1-p))$ (see section 4.4. of [10]) we get

$$d^{\beta_1,\beta_2}(\underline{s},b^p)=\lim_{n\longrightarrow\infty}\frac{\log b^p([s_0,\dots,s_n]_0)}{\mathrm{diam}_{\beta_1,\beta_2}([s_0,\dots,s_n]_0)}=\frac{p\log p+(1-p)\log(1-p)}{p\log\beta_1+(1-p)\log\beta_2}.$$

Now we claim that the map π_{β_1,β_2} is Lipschitz with respect to the metric δ^{β_1,β_2}:

$$|\pi_{\beta_1,\beta_2}(\underline{s})-\pi_{\beta_1,\beta_2}(\underline{t})|\le\sum_{k=|\underline{s}\wedge\underline{t}|}^{\infty}|s_k\beta_1^{k-\sharp_k(\underline{s})+1}\beta_2^{\sharp_k(\underline{s})}-t_k\beta_1^{k-\sharp_k(\underline{t})+1}\beta_2^{\sharp_k(\underline{t})}|$$

$$=\beta_1^{|\underline{s}\wedge\underline{t}|-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}\beta_2^{\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}$$

$$\sum_{k=0}^{\infty}|s_{k+|\underline{s}\wedge\underline{t}|}\beta_1^{k-\sharp_k(\sigma^{|\underline{s}\wedge-\underline{t}|}(\underline{s}))+1}\beta_2^{\sharp_k(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s}))}-t_{k+|\underline{s}\wedge\underline{t}|}\beta_1^{k-\sharp_k(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t}))+1}\beta_2^{\sharp_k(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t}))}|$$

$$\le\delta^{\beta_1,\beta_2}(\underline{s},\underline{t})\frac{2}{1-\max\{\beta_1,\beta_2\}}.$$

Recall that we have by definition $\pi_{\beta_1,\beta_2}(b^p)=b^p_{\beta_1,\beta_2}$. Hence we get that

$$\overline{d}(x,b^p_{\beta_1,\beta_2}):=\overline{\lim_{\epsilon\longrightarrow 0}}\frac{\log b^p_{\beta_1,\beta_2}(B_\epsilon(x))}{\log\epsilon}\le\frac{p\log p+(1-p)\log(1-p)}{p\log\beta_1+(1-p)\log\beta_2}$$

holds $b^p_{\beta_1,\beta_2}$-almost everywhere. By proposition 4.2. of [23] this implies the upper bound on the box-counting dimension of the measures $b^p_{\beta_1,\beta_2}$ stated in Theorem I. The singularity assertion is just an obvious consequence of this upper bound.

4. Proof of Theorem II

We want to apply the general dimension theory of ergodic measures developed in [11] and in some sense completed in [2] in order to find an expression of the dimension of the self-affine measures $\hat{b}^p_\vartheta$ in terms of entropy, Lyapunov exponents and the dimension of the self-similar measures $b^p_{\beta_1,\beta_2}$. In order to do so we define a auxillar map. Let $C_\vartheta = [\frac{-\beta_2}{1-\beta_2}, \frac{\beta_1}{1-\beta_1}] \times [\frac{-\tau_2}{1-\tau_2}, \frac{\tau_1}{1-\tau_1}] \times [-1,1]$ and consider the map $f_\vartheta : C_\vartheta \longrightarrow C_\vartheta$ given by

$$f_\vartheta(x,y,z) = \begin{cases} (\beta_1 x + \beta_1, \tau_1 y + \tau_1, 2z-1) & \text{if} \quad z \geq 0.5, \\ (\beta_2 x - \beta_2, \tau_2 y - \tau_2, 2z+1) & \text{if} \quad z < 0.5. \end{cases}$$

This map has an attractor A_ϑ which is given by the product of the self-affine set Λ_ϑ with the interval $[-1,1]$. We define a shift coding for the dynamical system $(A_\vartheta, f_\vartheta)$ in the following way: Let $\iota : \{-1,1\}^{\mathbb{Z}^-} \longmapsto [-1,1]$ be given by $\iota((s_k)) = \sum_{k=1}^\infty s_{-k}2^{-k}$ and define $\bar{\pi}_\vartheta : \Sigma \longmapsto A_\vartheta$ by $\bar{\pi}_\vartheta((s_k)_{k\in\mathbb{Z}} = (\hat{\pi}_\vartheta((s_k)_{k\in\mathbb{N}_0}), \iota((s_k)_{k\in\mathbb{Z}^-})$. This coding map can be shown to have the following properties:

(1) $\bar{\pi}_\vartheta$ is continuous and onto A_ϑ.
(2) $\bar{\pi}_\vartheta$ is a bijection from $\bar{\Sigma} = (\Sigma \backslash \{(s_k) | \exists k_0 \forall k \leq k_0 : s_k = 1\}) \cup \{(1)\}$ onto Λ_ϑ.
(3) $\bar{\pi}_\vartheta$ conjugates the backward shift map σ^{-1} and f_ϑ on $\bar{\Sigma}$.

Define measures $\bar{b}^p_\vartheta$ on A_ϑ by $\bar{b}^p_\vartheta := \bar{\pi}_\vartheta(b^p) = b^p \circ \bar{\pi}_\vartheta^{-1}$. These measures are ergodic with respect to f_ϑ by the properties of the coding map. Moreover we have Lyapunov exponents for the dynamical systems $(A_\vartheta, f_\vartheta, \bar{b}^p_\vartheta)$: Let $E^s = \text{span}\{(1,0,0),(0,1,0)\}$, $E^{ss} = \text{span}\{(0,1,0)\}$ and $E^u = \text{span}\{(0,0,1)\}$. If we have $p\log\tau_1 + (1-p)\log\tau_2 \leq p\log\beta_1 + (1-p)\log\beta_2$ then

$$\lim_{n\longrightarrow\infty} \frac{1}{n} \log ||D_{\mathbf{x}} f^n_\vartheta v|| = \log 2 \qquad \forall v \in E^u,$$

$$\lim_{n\longrightarrow\infty} \frac{1}{n} \log ||D_{\mathbf{x}} f^n_\vartheta v|| = \begin{cases} p\log\beta_1 + (1-p)\beta_2 & \text{if} \quad v \in E^s \backslash E^{ss}, \\ p\log\tau_1 + (1-p)\log\tau_2 & \text{if} \quad v \in E^{ss} \end{cases}$$

holds $\bar{b}^p_\vartheta$-almost everywhere. This is easy to see using Birkhoff's ergodic Theorem (see Theorem 4.1.2. of [10]). To apply the theory of Ledrappier and Young [11] the existence of Lyapunov exponents is not enough. In addition one needs appropriate Lyapunov charts (see section 4 of [21]). Ledrappier and Young considered C^2 diffeomorphisms in order to guarantee the existence of such charts. But from Schmeling and Troubetzkoy (see section 3 of [21]) we know that Lyapunov charts exist for a hyperbolic system with a singularity if the set of points that does not approach the singularity with exponential speed has full measure. More precise following [21] we have to show that:

$$\bar{b}^p_\vartheta(\{\mathbf{x} \in A_\vartheta | \exists l > 0 \ \forall n > 0 \ \ d(f^n(\mathbf{x}), S) > (1/l)e^{-\epsilon n}\}) = 1 \quad \forall \epsilon > 0$$

where $S = [\frac{-\beta_2}{1-\beta_2}, \frac{\beta_1}{1-\beta_1}] \times [\frac{-\tau_2}{1-\tau_2}, \frac{\tau_1}{1-\tau_1}] \times \{0\}$ is the singularity of the system. Let us proof this:

Fix $\epsilon > 0$. First note that it is sufficient if we show

$$\bar{b}^p_\vartheta(\{\mathbf{x} \in A_\vartheta | \exists (n_k)_{k\in\mathbb{N}} \longrightarrow \infty \ \forall k > 0 \ d(f^{n_k}(\mathbf{x}), S) \leq e^{-\epsilon n_k}\}) = 0$$

because if we have for a point $\mathbf{x}$ that there n_0 such that forall $n > n_0$ $d(f^n(\mathbf{x}), S) > e^{-\epsilon n}$ holds then there exists $l > 0$ such that $d(f^n(\mathbf{x}), S) > (1/l)e^{-\epsilon n}$ forall $n > 0$.

By the properties of the coding map this assertion is equivalent to the following statement about the symbolic system (Σ,σ^{-1},b^p):

$$b^p(N)=0 \text{ where } N:=\{\underline{s}\in\hat{\Sigma}|\exists(n_k)_{k\in\mathbb{N}}\longrightarrow\infty\ \forall k>0\ \tilde{d}(\sigma^{-n_k}(\underline{s}),\tilde{S})\le e^{-\epsilon n_k}\}$$

and $\tilde{S}=\{\underline{s}\in\Sigma|s_{-1}=1 \text{ and } s_k=-1\ \forall k<-1\}$ is the pre-image of the singularity under the coding map $\bar{\pi}_\vartheta$.
If $\underline{s}\in N$ we have $\tilde{d}(\sigma^{-n_k}(\underline{s}),\tilde{S})\le e^{-\epsilon n_k}\forall k>0$. By the definition of the metric $\tilde{d}$ this implies that $\sigma^{-n_k}(\underline{s})$ is contained in a cylinder set $[\underbrace{-1,-1,\dots,-1,1}_{\lceil c\epsilon n_k\rceil}]_{-\lceil c\epsilon n_k\rceil-1}$ forall $k>0$. where $\lceil x\rceil$ denotes the smallest integer bigger than x and the constant c is independent of ϵ, n_k and $\underline{s}$. This shows that $(\sigma^i(\underline{s}))_{-2}\neq 1$ for $i=n_k,\dots,n_k+\lceil c\epsilon n_k\rceil-1$ forall $k>0$. Thus we have

$$N\subseteq\{\underline{s}|\exists(n_k)_{k\in\mathbb{N}}\longrightarrow\infty\ \forall k>0\ :\ (\sigma^i(\underline{s}))_{-2}\neq 1\quad i=n_k,\dots,n_k+\lceil c\epsilon n_k\rceil-1\}.$$

Applying lemma 7.1. of [22] to the ergodic system (Σ,σ^{-1},b^p) we obtain $b^p(N)=0$.

Now we know that we are allowed to apply the results of [11] in our context. Define partitions W^s in the stable and W^{ss} in the strong stable direction of f_ϑ by the partition elements $W^s(z)=[\frac{-\beta_2}{1-\beta_2},\frac{\beta_1}{1-\beta_1}]\times[\frac{-\tau_2}{1-\tau_2},\frac{\tau_1}{1-\tau_1}]\times\{z\}$ and $W^{ss}(x,z)=\{x\}\times[\frac{-\tau_2}{1-\tau_2},\frac{\tau_1}{1-\tau_1}]\times\{z\}$. We claim that the conditional measures with respect to the measures $\bar{b}^p_\vartheta$ on the partition W^s are given by the self-affine measure $\hat{b}^p_\vartheta$ and that the transversal measures of the nested partition (W^s,W^{ss}) in the sense of [11] 11.4 are given by the self-similar measure $b^p_{\beta_1,\beta_2}$. Let $\ell^p=b^p\circ\iota^{-1}$. From the product structure of the map $\bar{\pi}_\vartheta$ it follows easily that $\bar{b}^p_\vartheta=\hat{b}^p_\vartheta\times\ell^p$. This implies our first claim. We have just by definition $\hat{\pi}_\vartheta=(\pi_{\beta_1,\beta_2},\pi_{\tau_1,\tau_2})$. Thus the projection of $\hat{b}^p_\vartheta$ onto the first coordinate axis is the measure b^p_ϑ. But this projection forms the transversal measures by the definition of our partitions.
Now Theorem C of [11] gives us

$$\dim_B\hat{b}^p_\vartheta=\dim_H\hat{b}^p_\vartheta=\frac{h_{b^p_\vartheta}(f_\vartheta)}{-\Xi^{ss}(b^p_\vartheta)}+(1-\frac{\Xi^s(b^p_\vartheta)}{\Xi^{ss}(b^p_\vartheta)})\dim_H b^p_\vartheta.$$

Here $\Xi^s(b^p_\vartheta)$ is the stable Lyapunov exponent given by $p\log\beta_1+(1-p)\log\beta_2$ and $\Xi^{ss}(b^p_\vartheta)$ is the strong stable Lyapunov exponent given by $p\log\tau_1+(1-p)\log\tau_2$. The metric entropy of the system $(A_\vartheta,f_\vartheta,\bar{b}^p_\vartheta)$ is $-(p\log p+(1-p)\log(1-p))$ because the system is measure theoretical conjugated to the Bernoulli shift (Σ,σ^{-1},b^p) via $\bar{\pi}_\vartheta$. Hence the formula stated in Theorem II is proved.

5. Proof of Theorem III

Let $P=\{(\beta_1,\beta_2,\tau_1,\tau_2)\in(0,0.649)^2\times(0,1)^2|\beta_1+\beta_2\ge 1,\quad \tau_1+\tau_2<1\}$. Given $\vartheta=(\beta_1,\beta_2,\tau_1,\tau_2)\in P$ let $d=d(\vartheta)$ be the solution of $\beta_1\tau_1^x+\beta_2\tau_2^x=1$ and let $p=p(\vartheta)=\beta_1\tau_1^d$. First of all we show that the box-counting dimension of Λ_ϑ is given by $d+1$ for all $\vartheta\in P$. This is just a nice exercise:

Given a real number $r > 0$ we define a set of finite sequences by

$$X_r := \{(s_1, \ldots, s_k) | \min\{\tau_1, \tau_2\} r \leq \tau_{s_1}\tau_{s_2} \ldots \tau_{s_k} < r \text{ where } s_j \in \{1,2\} \; \forall j = 1 \ldots k\}.$$

Notice that the sequences in X_r have not the same length. Let $\bar{k}(r)$ be the maximal length of a sequence in X_r. We observe that for every sequence $(s_j) \in \{1,2\}^{\bar{k}(r)}$ there is an unique k such that $(s_1, \ldots, s_k) \in X_r$. Thus we get

$$\sum_{(s_1,\ldots,s_k)\in X_r} \beta_{s_1}\beta_{s_2}\ldots\beta_{s_k}(\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k})^d$$

$$= \sum_{(s_1,\ldots,s_k)\in X_r} \beta_{s_1}\beta_{s_2}\ldots\beta_{s_k}(\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k})^d(\beta_1\tau_1^d + \beta_2\tau_2^d)^{\bar{k}(r)-k}$$

$$= \sum_{(s_1,\ldots,s_{\bar{k}(r)})\in\{1,2\}^{\bar{k}(r)}} \beta_{s_1}\beta_{s_2}\ldots\beta_{s_{\bar{k}(r)}}(\tau_{s_1}\tau_{s_2}\ldots\tau_{s_{\bar{k}(r)}})^d = (\beta_1\tau_1^d + \beta_2\tau_2^d)^{\bar{k}(r)} = 1. \qquad (1)$$

Beside equation (1) we need one more fact. Let v be the unique positive number satisfying $\tau_1^v + \tau_2^v = 1$. Since $\tau_1 + \tau_2 < 1$ we have $v \leq 1 \leq d+1$. Consequently

$$\sum_{(s_1,\ldots,s_k)\in X_r} (\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k})^{d+1} \leq \sum_{(s_1,\ldots,s_k)\in X_r} (\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k})^v = 1. \qquad (2)$$

Now we define a cover of Λ_ϑ by

$$C_r = \{\pi_\vartheta([\kappa(s_1), \ldots, \kappa(s_k)]_0) | (s_1, \ldots, s_k) \in X_r\}$$

where $\kappa(1) = 1$ and $\kappa(2) = -1$. Since $\{[\kappa(s_1), \ldots, \kappa(s_k)]_0 | (s_1, \ldots, s_k) \in X_r\}$ is a cover of Σ^+ we have that C_r is in fact a cover of Λ_ϑ.
An element of C_r is a rectangle parallel to the axis with x-length $2\beta_{s_1}\beta_{s_2}\ldots\beta_{s_k}$ and y-length $2\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k}$. We cover each of this rectangles by squares parallel to the axis of side length $2\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k}$. We choose the squares in a row such that they only intersect in their boundary. So we get for each rectangle a covering by $\lceil\frac{\beta_{s_1}\beta_{s_2}\ldots\beta_{s_k}}{\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k}}\rceil$ squares. In this way we obtain a new cover $\hat{C}_r$ of Λ_ϑ, which consists of squares with side length in $(2\min\{\tau_1,\tau_2\}r, 2r]$. Furthermore the number $\hat{N}(r)$ of elements in $\hat{C}_r$ is given by

$$\hat{N}(r) = \sum_{(s_1,\ldots,s_k)\in X_r} \lceil\frac{\beta_{s_1}\beta_{s_2}\ldots\beta_{s_k}}{\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k}}\rceil.$$

Now we have the upper estimate

$$\hat{N}(r)r^{d+1} \leq \min\{\tau_1,\tau_2\}^{-(d+1)} \sum_{(s_1,\ldots,s_k)\in X_r} \lceil\frac{\beta_{s_1}\beta_{s_2}\ldots\beta_{s_k}}{\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k}}\rceil(\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k})^{d+1}$$

$$\leq \min\{\tau_1,\tau_2\}^{-(d+1)}\Big(\sum_{(s_1,\dots,s_k)\in X_r}\beta_{s_1}\beta_{s_2}\dots\beta_{s_k}(\tau_{s_1}\tau_{s_2}\dots\tau_{s_k})^d+\sum_{(s_1,\dots,s_k)\in X_r}(\tau_{s_1}\tau_{s_2}\dots\tau_{s_k})^{d+1}\Big)$$
$$\leq^{(1)/(2)} 2\min\{\tau_1,\tau_2\}^{-(d+1)}$$

and the lower estimate

$$\hat{N}(r)r^{d+1}\geq\sum_{(s_1,\dots,s_k)\in X_r}\Big\lceil\frac{\beta_{s_1}\beta_{s_2}\dots\beta_{s_k}}{\tau_{s_1}\tau_{s_2}\dots\tau_{s_k}}\Big\rceil(\tau_{s_1}\tau_{s_2}\dots\tau_{s_k})^{d+1}$$
$$\geq\sum_{(s_1,\dots,s_k)\in X_r}\beta_{s_1}\beta_{s_2}\dots\beta_{s_k}(\tau_{s_1}\tau_{s_2}\dots\tau_{s_k})^d=^{(1)}1.$$

Now let $N(r)$ be the minimal cardinality of an arbitrary cover of Λ_ϑ with squares parallel to the axis of side length $2r$. Obviously we have $N(r)\leq\hat{N}(r)$ but we need another argument for an opposite estimate.
Let R be a rectangle in the cover C_r. We see that the projection of $\Lambda_\vartheta\cap C_r$ on the x-axis has the full x-length of the rectangle since we assumed $\beta_1+\beta_2\geq 1$. This implies that the intersection of each square in $\hat{C}_r$ with Λ_ϑ is not empty. Thus if we have a cover of Λ_ϑ each element of $\hat{C}_r$ has to be intersected by at least one element of the cover. But one square with side length $2r$ can not intersect more than $9\min\{\tau_1,\tau_2\}^{-2}$ squares in $\hat{C}_r$ because the squares in $\hat{C}_r$ have side length bigger than $2\min\{\tau_1,\tau_2\}r$ and intersect, if at all, only in the boundary. It follows that $N(r)\geq 1/9\min\{\tau_1,\tau_2\}^2\hat{N}(r)$.
Putting our estimates together we obtain

$$\frac{1}{9}\min\{\tau_1,\tau_2\}^2\leq N(r)r^{d+1}\leq 2\min\{\tau_1,\tau_2\}^{-(d+1)}$$

which implies our claim.

Now we go on with the main proof. Note that $p\log p+(1-p)\log(1-p)\geq p\log\tau_1+(1-p)\log\tau_2$ holds for all $p\in(0,1)$ since $\tau_1+\tau_2<1$. This implies the inequality $p\log\beta_1+(1-p)\log\beta_2\geq p\log\tau_1+(1-p)\log\tau_2$ using the definition of p. Hence we can apply Theorem II and get after a short calculation $\dim_H b^p_{\beta_1,\beta_2}=1\Rightarrow\dim_H\hat{b}^p_\vartheta=d+1$. Thus it only remains to show that for almost all $\vartheta\in P$ we have $\dim_H b^p_{\beta_1,\beta_2}=1$. We now show this.

Note that if the relation $\log\tau_2\log p/\beta_1=\log\tau_1\log(1-p)/\beta_2:=\bar{d}$ holds for some $\tau_1,\tau_2\in(0,1)$ with $\tau_1+\tau_2<1$ then

$$(p\beta_1)^p((1-p)\beta_2)^{1-p}=(\beta_1\beta_2\tau_1^{\bar{d}})^p(\beta_1\beta_2\tau_2^{\bar{d}})^{1-p}=\beta_1\beta_2\tau_1^{\bar{d}p}\tau_2^{\bar{d}(1-p)}<\beta_1\beta_2.$$

Thus it follows from Theorem I that for all $p\in(0,1)$ there exists a set $A(p)\subseteq(0,0.649)^2$ with $\ell^2(A(p))=\ell^2((0,0.649)^2)$ such that for all $(\beta_1,\beta_2)\in A(p)$ and all $\tau_1,\tau_2>0$ with $\tau_1+\tau_2<1$ and $\log\tau_2\log p/\beta_1=\log\tau_1\log(1-p)/\beta_2$ we have $\dim_H b^p_{\beta_1,\beta_2}=1$. Let $G(\tau_1)$ be given by the following union:

$$\bigcup_{p\in(0,1)}\{(\beta_1,\beta_2,\tau_2)|(\beta_1,\beta_2)\in A(p)\ ,\ \tau_1+\tau_2<1\ ,\ \log\tau_2\log p/\beta_1=\log\tau_1\log(1-p)/\beta_2\}.$$

It is easy to see that the union

$$\bigcup_{p\in(0,1)} \{(\beta_1,\beta_2,\tau_2)|(\beta_1,\beta_2)\in(0,0.649)\ ,\ \tau_1+\tau_2<1\ ,\ \log\tau_2\log p/\beta_1=\log\tau_1\log(1-p)/\beta_2\}$$

equals the set $\{(\beta_1,\beta_2,\tau_2)|(\beta_1,\beta_2)\in P^2_{trans}\ ,\ \tau_1+\tau_2<1\}$. By the Theorem of Fubini we thus have $\ell^3(G(\tau_1))=\ell^3\{(\beta_1,\beta_2,\tau_2)|(\beta_1,\beta_2)\in(0,0.649)^2\ ,\ \tau_1+\tau_2<1\}$. Now let

$$G=\bigcup_{\tau_1\in(0,1)}\{(\beta_1,\beta_2,\tau_1,\tau_2)|(\beta_1,\beta_2,\tau_2)\in G(\tau_1)\}.$$

Note that we have $G\subseteq P$ and $\ell^4(G)=\ell^4(P)$. Moreover by definition we have $\dim_H b^p_{\beta_1,\beta_2}=1$ for all $\vartheta\in G$ if p fulfills $\log\tau_2\log p/\beta_1=\log\tau_1\log(1-p)/\beta_2$. But if $p=\beta_1\tau_1^d$ and $\beta_1\tau_1^d+\beta_2\tau_2^d=1$ then this relation holds. Thus the proof is complete.

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