Number theoretical peculiarities in the dimension theory of dynamical systems

J. Neunhäuserer[1]
Fachbereich Mathematik, Technische Universität Dresden
Zellescher Weg. 12-14, 01069 Dresden, Germany
e-mail: neuni@math.tu-dresden.de

Abstract

We show that dimensional theoretical properties of dynamical systems can considerably change because of number theoretical peculiarities of some parameter values
AMS subject classification 2000: 37C45, 37A45

1 Introduction

In the last decades there has been an enormous interest in geometrical invariants of dynamical systems especially in the Hausdorff dimension of invariant sets like attractors, reppelers or hyperbolic sets and ergodic measures on these sets. A dimension theory of dynamical systems was developed and now a days the Hausdorff dimension seems to have its place beside classical invariants like entropy or Lyapunov exponents.[2]
There are two main principles that form a kind of a guide line through the dimension theory of dynamical systems. The first states the identity of Hausdorff and box-counting dimension of invariant sets. The second one is the variational principle for Hausdorff dimension which states that the Hausdorff dimension of a given invariant set can be approximated by the Hausdorff dimension of ergodic measures on these set or in a stronger form states the existence of an ergodic measure of full Hausdorff dimension on an given invariant set. In many situations these principle are essential to determine the Hausdorff dimension of an invariant set and for relating this quantity to other characteristics of the dynamics like entropy, Lyapunov

[1]Supported by "DFG-Schwerpunktprogramm - Dynamik: Analysis, effiziente Simulation und Ergodentheorie".

[2]We refer appendix A of this work and to the book of Falconer [6] for an introduction to dimension theory and recommend the book of Pesin [18] for the dimension theory of dynamical systems.

exponents and pressure.
For conformal repellers we know that the identity of Hausdorff and box-counting dimension holds and that there exists an ergodic measure of full Hausdorff dimension (see chapter 7 of [18]). For hyperbolic sets of diffeomorphisms the variational principle for Hausdorff dimension does not hold in general (see [16]). But again if the system is conformal restricted to stable resp. unstable manifolds there exits an ergodic measure of full dimension for the restrictions and the identity of box-counting and Hausdorff dimension of the hyperbolic set holds (see again chapter 7 of [18]). In the non conformal situation there is no general theory this days that allows us to determine the dimensional theoretical properties of a given dynamical system. But there are a lot of results for special classes of systems that state that the variational principle or the identity of box-counting and Hausdorff dimension or both hold at least generically in the sense of Lebesgue measure on the parameter space (see for instance [7], [24], [17], [27], [26]). In this paper we focus at such classes of systems.
We will show that in situation were there generically exists an ergodic measure of full Hausdorff dimension the variational principle for Hausdorff dimension may not hold in general because of number theoretical peculiarities of some parameter values (see Theorem 2.1 below). Furthermore we will show that the identity of Box-Counting and Hausdorff dimension may drop because of number theoretical peculiarities in situations were this identity generically holds (see Theorem 2.2 below). Our example for the first phenomena is the Fat Baker's transformation and our example for the second phenomena is a class of self-affine reppelers. Both classes of systems are very simple but it seems obvious to us that the same phenomena appear as well in more complicated examples also this would be of course even harder to proof.
All our results are related to a special class of algebraic integers namely Pisot-Vijayarghavan numbers[3] (short: PV numbers) and they are in some sense the consequence of a generalisation of results of Erdös [5], Garsia ([8], [9]) and Alexander and Yorke [1] on the singularity and dimension of invenitly convolved measures. We think that from the viewpoint of geo-

[3] A Pisot-Vijayarghavan number is an algebraic integer with all its algebraic conjugates inside the unit circle (see appendix B)

metric measure theory and algebraic number theory this generalisation is interesting in itself (see Theorem 4.1 below).
The rest of the paper is organised as follows. In section two we define the systems we study, state our main Theorems 2.1 and 2.2 about these systems and comment on our results. In section three we introduce coding maps for our systems and find representations of all ergodic measures using these codings. In section four we define a class of Borel probability measures associated with a PV numbers (Erdös measures), introduce a kind of entropy related to this measure (Garsia entropy) and state our main Theorem 4.1 about the singularity and the Hausdorff dimension of Erdös measures. The proof of Theorem 4.1 is given in section five, the proof of Theorem 2.1 is contained in section six and the proof of Theorem 2.2 can be found in section seven. All our proofs consist of several propositions which may be interesting in them self. In appendix A we collect some basic definitions and facts in dimension theory and in appendix B we we define PV numbers and state the properties of these algebraic integers that we need in our work.

Acknowlegdment: I wish to thank Jörg Schmeling who helped me a lot to find the results presented here.

2 Basic definitions and main results

For $\beta \in (0.5, 1)$ we define the **Fat Baker's transformation** f_β: $I\!R \times [-1,1] \to I\!R \times [-1,1]$ by

$$f_\beta(x,y) = \begin{cases} (\beta x + (1-\beta), 2y-1) & \text{if} \quad y \geq 0 \\ (\beta x - (1-\beta), 2y+1) & \text{if} \quad y < 0. \end{cases}$$

This map was introduced by Alexander and Yorke in [1]. It is called Fat Baker's transformation because if we set $\beta = 0.5$ we get the classical Baker's transformation.
It is obvious that the attractor of f_β is the whole square $[-1,1]^2$ which has Hausdorff and box-counting dimension two. We always restrict f_β to its attractor.

Now we state our main result about the Fat Baker's transformation.

Theorem 2.1 *If $\beta \in (0.5, 1)$ is the reciprocal of a PV number then the variational principle for Hausdorff dimension does not hold for $([-1,1]^2, f_\beta)$ i.e. $\{\dim_H \mu \colon \mu\ f_\beta\text{-ergodic}\} < 2$.*

Remark 2.1 Theorem 2.1 is an extension of the result of Alexander and York [1] that states that the Sinai-Ruelle-Bowen measures for $([-1,1]^2, f_\beta)$ does not have full Rényi dimension.

Remark 2.2 It follows from [1] together with Solomyak's theorem about Bernoulli convolutions [25] that for almost all $\beta \in (0.5, 1)$ the Sinai-Ruelle-Bowen measures for $([-1,1]^2, f_\beta)$ has full dimension. Thus our theorem shows that in situations where there generically is an ergodic measure of full dimension the variational principle for Hausdorff dimension may not hold in general because of special number theoretical properties of some parameter values. As far as we know our theorem provides the first example of this type.

Now we come to our second class of examples. For $\beta \in (0.5, 1)$ and $\tau \in (0, 0.5)$ we define two affine contractions on $[-1,1]^2$ by

$$T_1^{\beta,\tau}(x,z) = (\beta x + (1-\beta), \tau z + (1-\tau))$$

$$T_{-1}^{\beta,\tau}(x,z) = (\beta x - (1-\beta), \tau z - (1-\tau)).$$

From [10] we know that there is a unique compact self-affine subset $\Lambda_{\beta,\tau}$ of $[-1,1]^2$ satisfying

$$\Lambda_{\beta,\tau} = T_1^{\beta,\tau}(\Lambda_{\beta,\tau}) \cup T_{-1}^{\beta,\tau}(\Lambda_{\beta,\tau}).$$

Let $T_{\beta,\tau}$ be the smooth expanding transformation on $T_1^{\beta,\tau}([-1,1]^2) \cup T_{-1}^{\beta,\tau}([-1,1]^2)$ defined by

$$T_{\beta,\tau}(x) = (T_i^{\beta,\tau})^{-1}(x) \quad \text{if} \quad x \in T_i^{\beta,\tau}([-1,1]^2) \quad \text{for} \quad i = 1, -1.$$

Obviously the set $\Lambda_{\beta,\tau}$ is an invariant repeller for the transformation $T_{\beta,\tau}$. We call the system $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$ a **self-affine repeller**.

Let us state our main result about the systems $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$.

Theorem 2.2 *Let $\beta \in (0.5, 1)$ be the reciprocal of a PV number. For all $\tau \in (0, 0.5)$ we have $\dim_H \Lambda_{\beta,\tau} < \dim_B \Lambda_{\beta,\tau}$. Moreover if τ is sufficient small there can not be a Bernoulli measure of full dimension for the system $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$.*

Remark 2.3 We know from [17] that for almost all $\beta \in (0, 5, 1)$ and all $\tau \in (0, 0.5)$ the identity

$$\dim_H \Lambda_{\beta,\tau} = \dim_B \Lambda_{\beta,\tau} = \frac{\log 2\beta}{\log \tau} + 1$$

holds and that there is a Bernoulli measure of full dimension for $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$. Thus Theorem 2.2 shows that dimensional theoretical properties of dynamical systems can considerably change because of number theoretical peculiarities.

Remark 2.4 That the identity of Hausdorff and box-counting dimension may drop because of number theoretical peculiarities was shown before by Przytycki and Urbanski [21] in the context of Weierstrass like functions. Pollicott and Wise [22] claimed (without a proof) that the first statement of our theorem follows for small τ from the work of Przytycki and Urbanski. We were not able to see that this is true and thus wrote down an independent proof which gives explicit upper bounds on $\dim_H \Lambda_{\beta,\tau}$ (see section seven).

Remark 2.5 We do not know if there exists an ergodic measure of full Hausdorff dimension for the systems $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$ and we can not calculate $\dim_H \Lambda_{\beta,\tau}$ in the case that $\beta \in (0.5, 1)$ is the reciprocal of a PV number. The second statement of our theorem only shows that it is not possible to calculate $\dim_H \Lambda_{\beta,\tau}$ by means of Bernoulli measures in this situation.

3 Coding maps and representation of ergodic measures

We first introduce here the symbolic spaces which we use for our coding. Let $\Sigma = \{-1, 1\}^{\mathbb{Z}}$ and $\Sigma^+ = \{-1, 1\}^{\mathbb{N}_0}$. By pr_+ we denote the projection from Σ onto Σ^+. With a natural product metric Σ (resp. Σ^+) becomes a perfect, totally disconnected and compact metric space. For $u, v \in \mathbb{Z}$ (resp. $u, v \in \mathbb{N}$) and $t_0, t_1 \ldots, t_u \in \{-1, 1\}$ we define a cylinder set in Σ (resp. Σ^+) by

$$[t_0, t_1 \ldots, t_u]_v := \{(s_k)\colon s_{v+k} = t_k \text{ for } k = 0, \ldots, u\}.$$

The cylinder sets form a basis for the metric topology on Σ (resp. Σ^+). The forward shift map σ on Σ (resp. Σ^+) is given by $\sigma((s_k)) = (s_{k+1})$. The backward shift σ^{-1} is defined on Σ and given by $\sigma((s_k)) = (s_{k-1})$. By b^p for $p \in (0, 1)$ we denote the Bernoulli measure on Σ (resp. Σ^+), which is the product of the discrete measure giving 1 the probability p and -1 the probability $(1 - p)$. We write b for the equal-weighted Bernoulli measure $b^{0.5}$. The Bernoulli measures are ergodic with respect to forward and backward shifts (see [4]).

We are now prepared to define the Shift coding for the Fat Baker's transformation $([-1, 1]^2, f_\beta)$. Define a continuous map $\hat{\pi}_\beta$ from Σ onto $[-1, 1]^2$ by

$$\hat{\pi}_\beta(\underline{i}) = ((1 - \beta) \sum_{k=0}^{\infty} i_k \beta^k, \sum_{k=1}^{\infty} i_{-k} (1/2)^k).$$

A simple check shows that

$$f_\beta \circ \hat{\pi}_\beta(\underline{i}) = \hat{\pi}_\beta \circ \sigma^{-1}(\underline{i}) \quad \forall \underline{i} \in \bar{\Sigma} = (\Sigma \backslash \{(s_k)\colon \exists k_0 \forall k \leq k_0 : s_k = 1\}) \cup \{(1)\}.$$

Note that if μ is a σ-invariant Borel probability measure on Σ we have $\mu(\bar{\Sigma}) = 1$. From this fact by applying standard techniques in ergodic theory it is possible to show that the map

$$\mu \longmapsto \mu_\beta := \mu \circ \hat{\pi}_\beta^{-1}$$

from the space of σ-ergodic Borel probability measures on Σ is continuous with respect to the weak* topology and is onto the space of f_β-ergodic Borel probability measures on $[-1, 1]^2$. Moreover the system $([-1, 1], f_\beta, \mu_\beta)$ is a measure theoretical factor of $(\Sigma, \sigma^{-1}, \mu)$

Now we introduce a shift coding for the self-affine repeller $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$. Consider the homeomorphism $\pi_{\beta,\tau}\colon \Sigma^+ \to \Lambda_{\beta,\tau}$ given by

$$\pi_{\beta,\tau}(\underline{i}) = ((1-\beta)\sum_{k=0}^{\infty} i_k\beta^k, (1-\tau)\sum_{k=0}^{\infty} i_k\tau_k).$$

It is easy to see that $\pi_{\beta,\tau} \circ \sigma = T_{\beta,\tau} \circ \sigma$. Thus the systems $(\Lambda_{\beta,\tau}, T_{\beta,\tau})$ is homoeomorph conjugated to (Σ, σ) and the map

$$\mu \longmapsto \mu_{\beta,\tau} := \mu \circ \pi_{\beta,\tau}^{-1}$$

is a homeomorphism with respect to the weak* from the space of σ-ergodic Borel probability measures on Σ^+ onto the space of $T_{\beta,\tau}$-ergodic Borel probability measures on $\Lambda_{\beta,tau}$.

4 Erdös measures and Garsia entropy

For $\beta \in (0.5, 1)$ define a continuous map from Σ^+ onto $[-1, 1]$ by

$$\pi_\beta(\underline{i}) = (1-\beta)\sum_{k=0}^{\infty} i_k\beta^k.$$

Given a Borel probability measure ν on Σ^+ we define a Borel probability measure on $[-1, 1]$ by $\nu_\beta = \nu \circ \pi_\beta^{-1}$. If we choose the Bernoulli measure b^p on Σ^+ for a $p \in (0, 1)$ then b^p_β is a self-similar measure which is usually a called Bernoulli convolution. There are a lot of results in the literature about Bernoulli convolutions and we can not cite all these works here. Instead we like to refer to the nice overview article "Sixty years of Bernoulli convolutions" by Peres, Schlag and Solomyak [20].

In our work we are not only interested in Bernoulli convolutions but in all measures ν_β where ν is a σ invariant Borel probability measure Σ^+ and $\beta \in (0.5, 1)$ is the reciprocal of a PV number (see appendix B). We call a measure of this type an **Erdös** measure.

Now we will introduce a special kind of entropy related to Erdös measure. What we will do here is generalisation of the approach of Garsia

([8],[9]) for Bernoulli convolutions to all Erdös measures. Let $\sim_{n,\beta}$ be the equivalence relation on Σ^+ given by

$$\underline{i} \sim_{n,\beta} \underline{j} \Leftrightarrow \sum_{k=0}^{n-1} i_k\beta^k = \sum_{k=0}^{n-1} j_k\beta^k$$

and define a partition $\Pi_{n,\beta}$ of Σ^+ by $\Pi_{n,\beta} = \Sigma^+/\sim_{n,\beta}$. Recall that entropy of a partition Π with respect to a Borel probability measure ν on Σ^+ is

$$H_\nu(\Pi) = -\sum_{P\in\Pi} \nu(P)\log\nu(P).$$

We denote the join of two partitions Π_1 and Π_2 by $\Pi_1 \vee \Pi_2$. This is the partition consisting of all sections $A\cap B$ for $A \in \Pi_1$ and $B \in \Pi_2$. It is easy to see that the $\Pi_{n,\beta} \vee \sigma^{-n}(\Pi_{m,\beta})$ is finer than the partition $\Pi_{n+m,\beta}$ and hence the sequence $H_\nu(\Pi_{n,\beta})$ is sub-additive for a shift invariant measure ν on Σ^+. We can thus define the **Garsia entropy** $G_\beta(\nu)$ for a shift invariant Borel probability measure ν on Σ^+ by

$$G_\beta(\nu) := \lim_{n\longrightarrow\infty} \frac{H_\nu(\Pi_{n,\beta})}{n} = \inf_n \frac{H_\nu(\Pi_{n,\beta})}{n}.$$

The limit exists and is equal to the infimum since the sequence $H_\nu(\Pi_{n,\beta})$ is sub-additive. Another simple consequence of the sub-additivity of this sequence is that the map

$$\nu \longmapsto G_\beta(\nu)$$

upper-semi-continuous with respect to the weak* topology on the space of σ invariant Borel probability measures on Σ^+.

We are now prepared to state our main theorem about Erdös measures and Garsia entropy.

Theorem 4.1 *Let $\beta \in (0.5, 1)$ be the reciprocal of a PV number. For all σ-ergodic Borel probability measures ν on Σ^+ the following equivalence holds*

$$\nu_\beta \textit{ is singular} \Leftrightarrow G_\beta(\nu) < -\log\beta \Leftrightarrow \dim_H \nu_\beta < 1.$$

Moreover the set of σ-ergodic measures Borel probability measures ν on Σ^+ such that ν_β is singular is open in the weak topology and contains the Bernoulli measures b^p for $p \in (0, 1)$.*

Remark 4.1 It has been shown by Erdös [5] that the equal-weighted Bernoulli convolution b_β is singular if $\beta \in (0.5, 1)$ is the reciprocal of a PV number. Using this result Garsia [9] proved $G_\beta(b) < -\log \beta$ and from this Alexander and Yorke [1] deduced that the Rénji dimension of b_β is less than one. In the proof of Theorem 4.1 we will adopt ideas of all of these authors. In our generalisation from the equal-weighted Bernoulli measure to all σ-invariant measures we had do deal with some difficulties which are mainly of technical nature (see section four).

Remark 4.2 The PV case is exceptional. It was shown by Solomyak [25] that for almost all $\beta \in (0.5, 1)$ the Bernoulli convolution b_β is absolutely continuous with density in L^2.

5 Proof of Theorem 4.1

The proof of Theorem 4.1 follows from three propositions and is given at the end of this section

Proposition 5.1 *If $\beta \in (0, 5, 1)$ is the reciprocal PV number then the measures b^p_β are singular for all $p \in (0, 1)$.*

Proof The measure b^p_β is given by the infinite convolution of the discrete measures $b^{p,n}_\beta$, which give $(1-\beta)\beta^n$ the probability p and $-(1-\beta)\beta^n$ the probability $(1-p)$. From [11] we know that the Fourier transformation of a convolution is the product of the Fourier transformation of the convolved measures. Consequently the Fourier transformation ϕ of b^p_β is given by:

$$\phi(b^p_\beta, \omega) = \prod_{n=0}^{\infty} (\cos((1-\beta)\beta^n\omega) + (2p-1)\sin((1-\beta)\beta^n\omega)).$$

We see that

$$|\phi(b^p_\beta, \omega)| = \prod_{n=0}^{\infty} |(\cos((1-\beta)\beta^n\omega) + (2p-1)\sin((1-\beta)\beta^n\omega))|$$

$$\geq \prod_{n=0}^{\infty} |\cos((1-\beta)\beta^n\omega)|.$$

Now let $\omega_k = 2\pi\beta^{-k}/(1-\beta)$. We have

$$|\phi(b^p_\beta, \omega_k)| \geq \prod_{n=0}^{\infty} |\cos(2\pi\beta^{n-k})| = \prod_{n=0}^{k} |\cos(2\pi\beta^{n-k})| \prod_{n=k+1}^{\infty} |\cos(2\pi\beta^{n-k})|$$

$$= C \prod_{n=0}^{k} |\cos(2\pi\beta^{-n})|$$

where C is a constant independent of k and not zero. Now let β be the reciprocal of a PV number. From proposition B1 of appendix B we know that there is a constant $0 < \theta < 1$ such that $||\beta^{-n}||_{\mathbb{Z}} \leq \theta^n \ \forall n \geq 0$ where $||.||_{\mathbb{Z}}$ denotes the distance to the nearest integer. This implies $|\phi(b^p_\beta, \omega_k)| \geq \hat{C} > 0$ for all $k > 0$. Thus we have that $|\phi(b^p_\beta, \omega)|$ does not tend to zero with $\omega \longrightarrow \infty$. Hence by Riemann-Lebesgue lemma b^p_β can not be absolutely continuous if β is the reciprocal of a PV number. But it follows from the theory of infinity convolutions developed by Jessen and Winter [11] that b^p_β is of pure type that means either absolutely continuous. This completes the proof. □

Remark 5.1 This proof is nothing but an obvious extension of Erdös [5] original argument.

Proposition 5.2 *Let $\beta \in (0.5, 1)$ be the reciprocal of a PV number and ν be a shift invariant Borel probability measure on Σ^+. If ν_β is singular then $G_\beta(\nu) < -\log\beta$ holds.*

Proof Fix β. Define π_n from Σ^+ to $[-1,1]$ by $\pi_n((s_k)) = \sum_{k=0}^{n-1} s_k(1-\beta)\beta^k$ and let $\nu_n = \nu \circ \pi_n^{-1}$. Let $\sharp(n)$ be the number of distinct points of the form $\sum_{k=0}^{n-1} \pm(1-\beta)\beta^k$ and $\omega(n)$ be the minimal distance between two of those points. Furthermore denote the points by $x_i^n \quad i = 1 \dots \sharp(n)$ and let m_i^n be the ν measure of the corresponding elements in $\Pi_{n,\beta}$, which means $m_i^n = \nu_n(x_i^n)$.

We first state a property of PV numbers we will have to use here, see proposition B2 of appendix B:

$$\beta^{-1} \text{ is PV number } \Rightarrow \exists \ \ \bar{c} : \omega(n) \geq \bar{c}\beta^n.$$

Since $(\sharp(n)-1)\omega(n) \leq 2$ we get $\sharp(n) \leq 4\omega(n)^{-1} \leq c\beta^{-n}$ with $c := 4\bar{c}^{-1}$.

Now we assume that ν_β is singular. It follows that there exists a constant C such that:
$\forall \epsilon > 0 \ \exists$ disjoint intervals $(a_1, b_1), \ldots, (a_u, b_u)$ with

$$\sum_{l=1}^{u}(b_l - a_l) < \epsilon \quad \text{and} \quad \nu_\beta(O) > C \quad \text{where} \quad O := \bigcup_{l=0}^{u}(a_l, b_l).$$

With out loss of generality we may assume $\nu_\beta(a_l) = \nu_\beta(b_l) = 0$ for $l = 1 \ldots u$. It is obvious that the discreet distribution ν_n converges weakly to ν_β. Thus we have: $\exists n_1(\epsilon) \ \ \forall n > n_1(\epsilon) \ : \ \nu_n(O) > C$. We now expand the intervals a little bit, so that their length is a multiple of $\omega(n)$.

$$k_{l,n} := \max\{k \colon k\omega(n) \leq a_l\} \quad a_{l,n} := k_{l,n}\omega(n)$$

$$\bar{k}_{l,n} := \min\{k \colon b_l \leq k\omega(n)\} \quad b_{l,n} := \bar{k}_{l,n}\omega(n)$$

Since $\omega(n) \longrightarrow 0$ we have:
$\exists n_2(\epsilon) > n_1(\epsilon) \ \ \forall n > n_2(\epsilon) : \quad (a_{l,n}, b_{l,n})$ disjunct for $l = 1 \ldots u$ and

$$\sum_{l=1}^{u}(b_{l,n} - a_{l,n}) < \epsilon \quad \text{and} \quad \nu_n(\bar{O}) > C \quad \text{where} \quad \bar{O} = \bigcup_{l=0}^{u}(a_{l,n}, b_{l,n}).$$

Let $\hat{\sharp}(n)$ be the number of distinct points x_i^n in $\bar{O}$. Since in one interval $(a_{l,n}, b_{l,n})$ there are at most $\bar{k}_{l,n} - k_{l,n}$ points x_i^n we have $\omega(n)\hat{\sharp}(n) \leq \epsilon$ and hence $\hat{\sharp}(n) \leq \epsilon c\beta^{-n}$.
For all $n > n_2(\epsilon)$ we can now estimate:

$$H_\nu(\Pi_{n,\beta}) = -\sum_{i=1}^{\sharp(n)} m_i^n \log m_i^n = -\sum_{x_i^n \in \bar{O}} m_i^n \log m_i^n - \sum_{x_i^n \notin \bar{O}} m_i^n \log m_i^n$$

$$\begin{aligned}
&\leq \nu_n(\bar{O}) \log \frac{\hat{\sharp}(n)}{\nu_n(\bar{O})} + (1 - \nu_n(\bar{O})) \log \frac{\sharp(n) - \hat{\sharp}(n)}{1 - \nu_n(\bar{O})} \\
&\leq \nu_n(\bar{O}) \log \hat{\sharp}(n) + (1 - \nu_n(\bar{O})) \log \sharp(n) + \log 2 \\
&\leq \nu_n(\bar{O}) \log \epsilon c\beta^{-n} + (1 - \nu_n(\bar{O})) \log c\beta^{-n} + \log 2 \\
&\leq n \log \beta^{-1} + C \log \epsilon + \log c + \log 2.
\end{aligned}$$

If ϵ is small enough we have $H_\nu(\Pi_{n,\beta})/n < \log \beta^{-1}$ for all $n \geq n_2(\epsilon)$. Using the sub-additivity of $H_\nu(\Pi_{n,\beta})$ we get our result. $\square$

Remark 5.2 Garsia sketched a proof of this proposition for the equal weighted Bernoulli measure in [8] . Our poof is a more detailed and extended version of Garsia's argumentation.

Proposition 5.3 *If ν is a shift ergodic Borel probability measure on Σ^+ and $\beta \in (0.5, 1)$ we have*

$$\dim_H \nu_\beta \leq G_\beta(\nu)/-\log \beta.$$

Proof Because we will operate with Rényi dimension $\dim_R$ (see appendix A) we are interested in an upper bound on the quantity

$$h_\nu(\epsilon) = \inf\{H_\nu(\Pi) \colon \Pi \text{ a partition with } \operatorname{diam}\Pi \leq \epsilon\}$$

by the entropy of the partitions $\Pi_{n,\beta}$ of Σ^+. We proof the following statement

$$h_{\nu_\beta}(2\beta^n) \leq H_\nu(\Pi_{n,\beta}).$$

Fix $\beta \in (0.5, 1)$, $\tau \in (0, 0.5)$, a measure ν on Σ^+ and $n \in \mathbb{N}$. We use the convention that the first coordinate axis is called x-axis and pr_X denotes the projection on this axis.
We define a partition of $\Lambda_{\beta,\tau}$ by $\wp_n = \pi_{\beta,\tau}(\Pi_{n,\beta})$. By definition we have

$$H_\nu(\Pi_{n,\beta}) = H_{\nu_{\beta,\tau}}(\wp_n).$$

We should say something about the structure of $\wp_n$. The image of a cylinder set $[i_0, \ldots, i_{n-1}]_0$ in Σ^+ under $\pi_{\beta,\tau}$ is the part of $\Lambda_{\beta,\tau}$ lying in the rectangle $T^{\beta,\tau}_{i_{n-1}} \circ \ldots \circ T^{\beta,\tau}_{i_0}(Q)$ of x-length $2\beta^n$. It is not difficult to check that two cylinder sets lie in the same element of $\Pi_{n,\beta}$ if and only if the corresponding rectangles lie above each other. So the projection of an element in $\wp_n$ onto the x-axis has length $2\beta^n$.
The projection onto the x-axis of two elements in $\wp_n$ may overlap. Starting with $\wp_n$, we want to construct inductively a partition $\bar{\wp}_n$ of $\Lambda_{\beta,\tau}$ with non-overlapping projections, in a way that does neither increase length of the

projections nor entropy. Let $N(\wp)$ be the number of pairs of elements in a partition $\wp$ that do have overlapping projections onto the x-axis. We now construct a finite sequence $\wp_n^k$ of partitions. First let $\wp_n^0 = \wp_n$. Now let $\wp_n^k$ be constructed and $N(\wp_n^k) > 0$. Let P_1 and P_2 be two elements of $\wp_n^k$ with overlapping projections. Without loss of generality we may assume $\nu_{\beta,\tau}(P_1) \geq \nu_{\beta,\tau}(P_2)$ and define:

$$\hat{P}_1 = P_1 \cup (P_2 \cap (pr_X P_1 \times [-1,1])) \qquad \hat{P}_2 = P_2 \backslash (pr_X P_1 \times [-1,1]).$$

We have $\hat{P}_1 \dot{\cup} \hat{P}_2 = P_1 \dot{\cup} P_2$, $P_1 \subseteq \hat{P}_1$ and $\hat{P}_2 \subseteq P_2$. Thus we know: $\nu_{\beta,\tau}(P_1) + \nu_{\beta,\tau}(P_2) = \nu_{\beta,\tau}(\hat{P}_1) + \nu_{\beta,\tau}(\hat{P}_2)$ and $\nu_{\beta,\tau}(\hat{P}_1) \geq \nu_{\beta,\tau}(P_1) \geq \nu_{\beta,\tau}(P_2) \geq \nu_{\beta,\tau}(\hat{P}_2)$. Since the function $-x \log x$ is concave, this implies:

$$-(\nu_{\beta,\tau}(\hat{P}_1) \log \nu_{\beta,\tau}(\hat{P}_1) + \nu_{\beta,\tau}(\hat{P}_2) \log \nu_{\beta,\tau}(\hat{P}_2)) \leq$$

$$-(\nu_{\beta,\tau}(P_1) \log \nu_{\beta,\tau}(P_1) + \nu_{\beta,\tau}(P_2) \log \nu_{\beta,\tau}(P_2)).$$

Hence if we substitute $\hat{P}_1$, $\hat{P}_2$ for P_1, P_2, we get a partition $\wp_n^{k+1}$ of $\Lambda_{\beta,\tau}$ with non-increased entropy. From the definition of $\hat{P}_1$ and $\hat{P}_2$ we see that $pr_X \hat{P}_1 = pr_X P_1$, $pr_X \hat{P}_2 \subseteq pr_X P_2$ and that the projections of $\hat{P}_1$ and $\hat{P}_2$ onto the x-axis do not overlap. So the length of the projections are obviously not increased. Furthermore we observe that there cannot be any new overlaps of the projections of $\hat{P}_1$ or $\hat{P}_2$ with the projections of other elements in $\wp_n^k$, that do not appear, when we consider P_1 or P_2. Hence $N(\wp_n^{k+1}) < N(\wp_n^k)$.

So after a finite number of steps we get a partition $\bar{\wp}_n$ with

$$H_{\nu_{\beta,\tau}}(\wp_n) \geq H_{\nu_{\beta,\tau}}(\bar{\wp}_n),$$

non-overlapping projections onto the x-axis and diam $pr_X \bar{\wp}_n \leq 2\beta^n$. $pr_X \bar{\wp}_n$ is a partition of the interval $[-1,1]$ and we have

$$H_{\nu_\beta}(pr_X \bar{\wp}_n) = H_{\nu_{\beta,\tau}}(\bar{\wp}_n),$$

since the measure ν_β is the projection of $\nu_{\beta,\tau}$ onto the x-axis. The proof of our claim is complete:

$$h_{\nu_\beta}(2\beta^n) \leq H_{\nu_\beta}(pr_X \bar{\wp}_n) = H_{\nu_{\beta,\tau}}(\bar{\wp}_n) \leq H_{\nu_{\beta,\tau}}(\wp_n) = H_\nu(\Pi_{n,\beta}).$$

We are now able to estimate the Rényi dimension

$$\overline{\dim}_R \nu_\beta = \overline{\lim}_{\epsilon\longrightarrow\infty} \frac{h_{\nu_\beta}(\epsilon)}{\log \epsilon^{-1}} = \overline{\lim}_{n\longrightarrow\infty} \frac{h_{\nu_\beta}(2\beta^n)}{\log 0.5\beta^{-n}} = \overline{\lim}_{n\longrightarrow\infty} \frac{h_{\nu_\beta}(2\beta^n)}{n \log \beta^{-1}}$$

$$\leq \lim_{n\longrightarrow\infty} \frac{H_\nu(\Pi_{n,\beta})}{n\log\beta^{-1}} = \frac{G_\beta(\nu)}{\log\beta^{-1}}.$$

Using part (3) of proposition A1 from appendix A we get

$$\forall\delta > 0 \exists X : \nu_\beta(X) > 0 \text{ and } \underline{d}(x,\nu_\beta) \leq G_\beta(\nu)/\log\beta^{-1} + \delta \quad \forall x \in X.$$

But the measure ν_β is exact dimensional, because it is the transversal measure in the context of the ergodic dynamical system $(\Lambda_{\beta,\tau}, T_{\beta,\tau}, \nu_{\beta,\tau})$. This fact was observed by Ledrappier and Porzio, see [14]. So our estimate must hold ν_β-almost everywhere and by part (2) of proposition A2 we get $\dim_H \nu_\beta \leq G_\beta(\nu)/\log\beta^{-1} + \delta$ for all $\delta > 0$. This proves the proposition. □

Remark 5.3 Let us remark that Alexander and Yorke [1] proved the identity $\dim_R b_\beta = G_\beta(b)/\log\beta^{-1}$ for the equal-weighted infinitely convolved Bernoulli measure b_β. In their proof they used the self-similarity of this measure. In our general situation we could not appeal to self-similarity and thus had to develop a different technique.

Proof of Theorem 4.1 Under the assumptions of our theorem we have

$$\nu_\beta \text{ is singular} \Rightarrow^{5.2} G_\beta(\nu) < \log\beta^{-1} \Rightarrow^{5.3} \dim_H \nu_\beta < 1 \Rightarrow \nu_\beta \text{ is singular}.$$

These implications prove the first statement of Theorem 4.1. Now choose an singular Erdös measure ξ_β. We have $G_\beta(\xi) < \log\beta^{-1}$. By upper-semi-continuity of G we get $G_\beta(\nu) < \log\beta^{-1}$ and hence $\dim \nu_\beta < 1$ for all ν in a hole weak* neighbourhood of ξ. Thus the set $\{\nu\colon \nu_\beta \text{ is singular}\}$ is open in the weak* topology. The set contains all Bernoulli measure by Proposition 4.1. □

6 Proof of Theorem 2.1

The proof of Theorem 2.1 follows from and Theorem 4.1 and two propositions providing upper estimates on the Hausdorff dimension of all ergodic measures μ_β for the Fat Baker's transformation f_β. It can be found at the end of this section.

Proposition 6.1 *If μ is a shift ergodic Borel probability measure on Σ and $\beta \in (0, 5)$ we have*

$$\dim_H \mu_\beta \le 1 + \le G_\beta(pr_+(\mu)) / -\log \beta$$

where pr_+ denotes the projection from Σ onto Σ^+.

Proof By Proposition A2 and the definition of the Hausdorff dimension of a measure we have $\dim_H \mu_\beta \le 1 + \dim_H pr_X \mu_\beta$ where pr_X denotes the projection onto the first coordinate axis. Just by definition of the involved measures we have $pr_X \mu_\beta = (pr_+\mu)_\beta$ and hence $\dim_H \mu_\beta \le 1 + \dim_H (pr_+\mu)_\beta$. The proposition follows now immediately from Proposition 5.3. $\square$

Proposition 6.2 *If μ is a shift ergodic Borel probability measure on Σ and $\beta \in (0, 5)$ we have*

$$\dim_H \mu_\beta \le 1 + \le h_\mu(\sigma) / \log 2$$

where $h_\mu(\sigma)$ is the usual measure-theoretic entropy of the shift (Σ, σ, μ).

Proof The proof of this proposition is a little bit difficult. We want to use the general theory relating the dimension of ergodic measure to entropy and Lyapunov exponents (see [13] and [3]). Usually this theory is stated in the context of diffeomorphisms but the Fat Baker's transformation is not invertible and has a singularity. To deal with the first problem we define for $\beta \in (0.5, 1)$ and $\tau \in (0, 0.5)$ a lift $\hat{f}_{\beta,\tau}\colon [-1,1]^3 \to [-1,1]^3$ of the Fat Baker's transformation f_β by

$$\hat{f}_\beta(x,y,z) = \{ \begin{array}{ll} (\beta x + (1-\beta), 2y - 1, \tau z + (1-\tau)) & \text{if} \quad y \ge 0 \\ (\beta x - (1-\beta), 2y + 1, \tau z - (1-\tau)) & \text{if} \quad y < 0. \end{array}$$

This maps is invertible and its projection onto the (x,y)-plane is f_β. Moreover it is easy to see that $f_{\beta,\tau}$ has an attractor $\hat{\Lambda}_{\beta,\tau}$ which is given by the product of the self-affine set $\Lambda_{\beta,\tau}$ in the (x,z)-plane with the interval

$[-1,1]$ on the y-axis. Let us introduce a Shift coding $\hat{\pi}_{\beta,\tau} : \Sigma \longmapsto \hat{\Lambda}_{\beta,\tau}$ for the system $(\hat{\Lambda}_{\beta,\tau}, \hat{f}_{\beta,\tau})$ by

$$\hat{\pi}_{\beta,\tau}(\underline{i}) = ((1-\beta)\sum_{k=0}^{\infty} i_k\beta^k, \sum_{k=1}^{\infty} i_{-k}(1/2)^k, (1-\tau)\sum_{k=0}^{\infty} i_k\tau_k).$$

Given a σ-ergodic measure on Σ we define a $\hat{f}_{\beta,\tau}$-ergodic measure $\hat{\mu}_{\beta,\tau}$ on $\hat{\Lambda}_{\beta,\tau}$ by $\hat{\mu}_{\beta,\tau} = \mu \circ \hat{\pi}_{\beta,\tau}^{-1}$. Section four of [17] contains a proof of the fact that we are allowed to apply the general results found in [13] and [3] to the system $(\hat{\Lambda}_{\beta,\tau}, \hat{f}_{\beta,\tau}, \hat{\mu}_{\beta,\tau})$ also this system has a singularity. We do not want to reproduce the argument here. We only like to mention that main idea is that the set of points that approaches the singularity of $(\hat{\Lambda}_{\beta,\tau}, \hat{f}_{\beta,\tau}, \hat{\mu}_{\beta,\tau})$ with exponential speed has zero measure and thus Lyapunov charts exist almost everywhere for $(\hat{\Lambda}_{\beta,\tau}, \hat{f}_{\beta,\tau}, \hat{\mu}_{\beta,\tau})$. From Theorem C and Theorem F of [13] we have by this fact

$$\dim_H \hat{\mu}_{\beta,\tau} \leq \frac{h_{\hat{\mu}_{\beta,\tau}}(\hat{f}_{\beta,\tau})}{\log 2} + \dim \hat{\mu}^s_{\beta,\tau}.$$

where $\dim \hat{\mu}^s_{\beta,\tau}$ is the local dimension of the conditional measures of $\hat{\mu}_{\beta,\tau}$ on the partition $\{[-1,1] \times \{y\} \times [-1,1]\colon\ y \in [-1,1]\}$ in the stable direction of $\hat{f}_{\beta,\tau}$ and $h_{\hat{\mu}_{\beta,\tau}}(\hat{f}_{\beta,\tau})$ is the measure theoretical entropy of the system $(\hat{\Lambda}_{\beta,\tau}, \hat{f}_{\beta,\tau}, \hat{\mu}_{\beta,\tau})$. Since the conditional measures are just by definition concentrated on the set $\{(x,y,z)\colon\ (x,z) \in \Lambda_{\beta,\tau} \quad y \in [-1,1]\}$ we have $\dim \mu^s_{\beta,\tau} \leq \dim_B \Lambda_{\beta,\tau}$ and from [22] we know $\dim_B \Lambda_{\beta,\tau} = \log(2\beta/\tau)/\log(1/\tau)$. Furthermore it is easy to see that the systems $(\hat{\Lambda}_{\beta,\tau}, \hat{f}_{\beta,\tau}, \hat{\mu}_{\beta,\tau})$ and (Σ, σ, μ) are measure theoretical conjugated and thus $h_{\hat{\mu}_{\beta,\tau}}(\hat{f}_{\beta,\tau}) = h_\mu(\sigma)$. Hence we have

$$\dim_H \hat{\mu}_{\beta,\tau} \leq \frac{h_\mu(\sigma)}{\log 2} + \frac{\log(2\beta/\tau)}{\log(1/\tau)}.$$

Now note that $\mu_{\beta,\tau}$ projects to μ_β and hence $\dim_H \mu_\beta \leq \dim_H \hat{\mu}_{\beta,\tau}$ for all $\tau \in (0, 0.5)$. Thus we get

$$\dim_H \hat{\mu}_{\beta,\tau} \leq \frac{h_\mu(\sigma)}{\log 2} + \frac{\log(2\beta/\tau)}{\log(1/\tau)} \forall \tau \in (0, 0.5).$$

With $\tau \longrightarrow 0$ our proof is complete. $\square$

Proof of Theorem 2.1 From Theorem 4.1 and the upper-semi-continuity of G_β we get $G_\beta(pr^+\mu)/\log\beta^{-1} \le c_1 < 1$ for all μ in hole weak* neighbourhood U of b in space of σ-ergodic Borel probability measures on Σ. Hence by Proposition 6.1 $\dim_H \bar{\mu}_\beta \le c_1 + 1 < 2$ holds for all μ in U. On the other hand we have by well known properties of the measure theoretical entropy, $h_\mu(\sigma)/\log 2 \le c_2 < 1$ on the complement of U (see [4]). From Proposition 6.1 we thus get $\dim_H \mu_\beta \le c_2 + 1 < 2$ for all μ in the complement of U. Putting these facts together we obtain

$$\dim_H \mu_\beta \le \max\{c_1, c_2\} + 1 < 2 = \dim_H [-1,1]^2.$$

But we know that all ergodic measures for the system $([-1,1]^2, f_\beta)$ are of the form μ_β for some σ-ergodic Borel probability measures μ on Σ. and the proof is complete. $\Box$

7 Proof of Theorem 2.2

The proof of Theorem 2.2 has a lot of ingredencies, a formula for $\dim_B \Lambda_{\beta,\tau}$ found in [22], a formula for $\dim_H b^p_{\beta,\tau}$ found in [17], Theorem 4.1 and the following two proposition giving upper bounds on $\dim_H \Lambda_{\beta,\tau}$.

Proposition 7.1 *If $\beta \in (0.5, 1)$ is the reciprocal of an PV number and $\tau \in (0, 0.5)$ we have*

$$\dim_H \Lambda_{\beta,\tau} \le \frac{\log(\sum_{P\in\Pi_{n,\beta}} (\sharp P)^{\frac{\log\beta}{\log\tau}})}{n\log\beta^{-1}} \qquad \forall n \ge 1$$

where $\Pi_{n,\beta}$ is the partition of Σ^+ defined in section four and $\sharp P$ denotes the number of cylinder sets of length n contained in an element of this partition.

Proof Fix a reciprocal of a PV number $\beta \in (0.5, 1)$ and $\tau \in (0, 0.5)$. Let $n \ge 1$ and set

$$u_n = \frac{\log(\sum_{P\in\Pi_{n,\beta}} (\sharp P)^{\frac{\log\beta}{\log\tau}})}{n\log\beta^{-1}}.$$

Consider the set of cylinders in Σ^+ given by $C_n = \{[\tilde{s}_1\tilde{s}_2\ldots\tilde{s}_m]_0\colon \tilde{s}_i \in \{-1,1\}^n\ i = 1\ldots m\}$. Define a set function η on C_n by

$$\eta([\tilde{s}]_0) = \frac{\sharp P(\tilde{s})^{\log\beta/\log\tau}}{\sharp P(\tilde{s})}\beta^{nu_n} \quad \text{and}$$

$$\eta([\tilde{s}_1\tilde{s}_2\dots\tilde{s}_m]_0) = \eta([\tilde{s}_1]_0)\cdot\eta([\tilde{s}_2]_0)\cdot\ldots\cdot\eta([\tilde{s}_m]_0)$$

where $\tilde{s}, \tilde{s}_1, \dots \tilde{s}_m$ are elements of $\{-1,1\}^n$ and $P(\tilde{s})$ denotes the element of the partition $\Pi_{n,\beta}$ containing the cylinder $[\tilde{s}]_0$.

Note the facts that C_n is a basis of the metric topology of Σ^+ and that $\sum_{\tilde{s}\in\{-1,1\}^n}\eta([\tilde{s}]_0) = 1$ by the definition of u_n. Thus we can extend η to a Borel probability measure on Σ^+ and $\eta_{\beta,\tau} := \eta\circ\pi_{\beta,\tau}^{-1}$ defines a Borel probability measure on $\Lambda_{\beta,\tau}$.

Given $m \geq 1$ we set $q(m) = \lceil m(\log\beta/\log\tau)\rceil$. Given a $\tilde{s}_i \in \{-1,1\}^n$ for $i = 1\dots m$ we define a subset of $\Lambda_{\beta,\tau}$ by

$$R_{\tilde{s}_1\dots\tilde{s}_n} = \{(\sum_{i=0}^{\infty} s_i(1-\beta)\beta^i, \sum_{i=0}^{\infty} t_i(1-\tau)\tau^i) \colon s_i, t_i \in \{-1,1\}$$

$$(s_{(i-1)n},\dots,s_{in-1}) = \tilde{s}_i \quad i = 1\dots m \quad \text{and}$$

$$(t_{(i-1)n},\dots,t_{in-1}) = \tilde{s}_i \quad i = 1\dots q(m)\}.$$

We see that $R_{\tilde{s}_1\dots\tilde{s}_m}$ is "almost" a square in $\Lambda_{\beta,\tau}$ of side length β^{mn}. More precise we have:

$$c_1\beta^{mn} \leq \mathrm{diam} R_{\tilde{s}_1\dots\tilde{s}_m} \leq c_2\beta^{mn} \qquad (1)$$

where the constants c_1, c_2 are independent of the choice of $\tilde{s}_i$.

Now let as examine the $\eta_{\beta,\tau}$ measure of the sets $R_{\tilde{s}_1\dots\tilde{s}_m}$.
Assume that $\tilde{t}_i \sim_{n,\beta} \tilde{s}_i$ for $i = q(m)+1\dots m$ where $\sim_{n,\beta}$ is the equivalence relation introduced in section four. The rectangles $\pi_{\beta,\tau}([\tilde{s}_1\dots\tilde{s}_{q(m)}\tilde{t}_{q(m)+1}\dots\tilde{t}_m]_0)$ are all disjoint and lie above each other in the set $R_{\tilde{s}_1\dots\tilde{s}_m}$. Hence we have

$$\eta_{\beta,\tau}(R_{\tilde{s}_1\dots\tilde{s}_m}) \geq \eta(\bigcup_{\tilde{t}_i\sim_{n,\beta}\tilde{s}_i \quad i=q(m)+1\dots m} \pi_{\beta,\tau}([\tilde{s}_1\dots\tilde{s}_{q(m)}\tilde{t}_{q(m)+1}\dots\tilde{t}_m]_0) =$$

$$= \sum_{\tilde{t}_i\sim_{n,\beta}\tilde{s}_i \quad i=q(m)+1\dots m} \eta([\tilde{s}_1\dots\tilde{s}_{q(m)}\tilde{t}_{q(m)+1}\dots\tilde{t}_m]_0).$$

Using the fact $\tilde{s} \sim_{n,\beta} \tilde{t} \Rightarrow \sharp P(\tilde{s}) = \sharp P(\tilde{t}) \Rightarrow \eta([\tilde{s}]_0) = \eta([\tilde{t}]_0)$ this last expression equals

$$\prod_{i=1}^{m} \eta([\tilde{s}_i]_0) \sum_{\tilde{t}_i \sim_{n,\beta} \tilde{s}_i \quad i=q(m)+1\ldots m} 1$$

$$= \prod_{i=1}^{m} \frac{\sharp P(\tilde{s}_i)^{\log\beta/\log\tau}}{\sharp P(\tilde{s}_i)} \beta^{mnu_n} \sum_{\tilde{t}_i \sim_{n,\beta} \tilde{s}_i \quad i=q(m)+1\ldots m} 1$$

$$= \frac{\prod_{i=1}^{m} \sharp P(\tilde{s}_i)^{\log\beta/\log\tau}}{\prod_{i=1}^{q(m)} \sharp P(\tilde{s}_i)} \beta^{mnu_n} = (\phi_{\tilde{s}_1\ldots\tilde{s}_m} \beta^{nu_n})^m$$

where

$$\phi_{\tilde{s}_1\ldots\tilde{s}_m} = \left(\frac{\prod_{i=1}^{m} \sharp P(\tilde{s}_i)^{\log\beta/\log\tau}}{\prod_{i=1}^{q(m)} \sharp P(\tilde{s}_i)}\right)^{1/m}.$$

Now fix an $\epsilon > 0$ We use the sets $R_{\tilde{s}_1\ldots\tilde{s}_m}$ to construct a good cover of $\Lambda_{\beta,\tau}$ in the sense for Hausdorff dimension. To this end set

$$R_m := \{R_{\tilde{s}_1\ldots\tilde{s}_m} \colon \phi_{\tilde{s}_1\ldots\tilde{s}_m} \geq \beta^{n\epsilon}\}.$$

We have an upper bound on the cardinality of R_m. If $R \in R_m$ then $\eta_{\beta,\tau}(R) \geq \beta^{mn(u_n+\epsilon)}$ and since $\eta_{\beta,\tau}$ is a probability measure we see:

$$\mathrm{card}(R_m) \leq \beta^{-mn(u_n+\epsilon)} \qquad (2).$$

Now let $R(M) = \bigcup_{m \geq M} R_m$. We want to prove that $R(M)$ is a cover of $\Lambda_{\beta,\tau}$ for all $M \geq 1$.

For $\underline{s} = (s_k) \in \Sigma^+$ we define the function ϕ_m by $\phi_m(\underline{s}) = \phi_{s_0\ldots s_{mn-1}}$. In addition we need two auxiliary functions on Σ^+:

$$f_m(\underline{s}) = \frac{\prod_{i=0}^{m} \sharp P((s_{(i-1)n}, \ldots, s_{in-1}))^{1/m}}{\prod_{i=0}^{q(m)} \sharp P((s_{(i-1)n}, \ldots, s_{in-1}))^{1/q(m)}},$$

$$g_m(\underline{s}) = \left(\prod_{i=1}^{q(m)} \sharp P((s_{(i-1)n}, \ldots, s_{in-1}))\right)^{1/q(m)(\log\beta\log\tau - q(m)/m)}.$$

Since $1 \leq \sharp P(\tilde{s}) \leq 2^n$ we have $1 \leq g_m(\underline{s}) \leq 2^{n(\log\beta/\log\tau - q(m)/m)}$. Thus by the definition of $q(m)$ we have $g_m(\underline{s}) \longrightarrow 1$. Moreover we have $\overline{\lim}_{m\longrightarrow\infty} f_m(\underline{s}) \geq 1$ because $\prod_{i=0}^{t} \sharp P((s_{i-1}n, \ldots, s_{in-1}))^{1/t} \geq 1 \ \ \forall t \geq 1$. A simple calculation shows $\phi_m(\underline{s}) = (f_m(\underline{s}))^{\log\beta/\log\tau} g_m(\underline{s})$. The properties of f and g thus imply:

$$\overline{\lim}_{m\longrightarrow\infty} \phi_m(\underline{s}) \geq 1 \qquad \forall\ \underline{s} \in \Sigma^+.$$

This will help us to show that $R(M)$ is a cover of $\Lambda_{\beta,\tau}$. For all $\underline{s} = (s_k) \in \Sigma^+$ there is an $m \geq M$ such that $\phi_m(\underline{s}) \geq \beta^{n\epsilon}$ and thus $\pi_{\beta,\tau}(\underline{s}) \in R_{s_0,\ldots,s_{mn-1}} \in R(M)$. Since $\pi_{\beta,\tau}$ is onto $\Lambda_{\beta,\tau}$ we see that $R(M)$ is indeed a cover of $\Lambda_{\beta,\tau}$.

We are now able to complete the proof. For every $\epsilon > 0$ and every $M \in I\!N$ we have:

$$\sum_{R\in R(M)} (\mathrm{diam} R)^{u_n+2\epsilon} = \sum_{m\geq M} \sum_{R\in R_m} (\mathrm{diam} R)^{u_n+2\epsilon}$$

$$\leq^{(1)} \sum_{m\geq M} \sum_{R\in R_m} (c_2\beta^{mn})^{u_n+2\epsilon} = \sum_{m\geq M} \mathrm{card}(R_m)(c_2\beta^{mn})^{u_n+2\epsilon}$$

$$\leq^{(2)} c_2^{u_n+2\epsilon} \sum_{m\geq M} \beta^{mn\epsilon}.$$

The last expression goes to zero with $M \longrightarrow 0$. By the definition for Hausdorff dimension we thus get $\dim_H \Lambda_{\beta,\tau} \leq u_n + 2\epsilon$ and since ϵ is arbitrary, we have $\dim_H \Lambda_{\beta,\tau} \leq u_n$. □

Remark 7.1 Some ideas we used here are to due the prove of McMullen's theorem on self-affine carpets [15] by Pesin in [18].

Now we use strategies developed in the proof of Proposition 5.2 to get:

Proposition 7.2 *If $\beta \in (0.5, 1)$ is the reciprocal of a PV number and $\tau \in (0, 0.5)$ we have*

$$\exists\ N \in I\!N \ \ \forall\ n > N \qquad \frac{\log(\sum_{P\in\Pi_{n,\beta}} (\sharp P)^{\frac{\log\beta}{\log\tau}})}{n\log\beta^{-1}} < \frac{\log(2\beta/\tau)}{\log(1/\tau)}.$$

Proof Fix a reciprocal of a PV number β. Consider the proof of Proposition 5.2 for the equal weighted Bernoulli measure b. Recall that we denote by x_i^n $i = 1 \ldots \sharp(n)$ the distinct points of the form $\sum_{k=0}^{n-1} \pm(1-\beta)\beta^k$ and by m_i^n the b measure of corresponding element P_n^i from the partition $\Pi_{n,\beta}$.

By the singularity of b_β we have more than we used in the proof of 5.2: $\forall C \in (0,1)$ $\forall \epsilon > 0$ $\exists$ disjoint intervals $(a_1, b_1), \ldots, (a_u, b_u)$ with

$$\sum_{l=1}^{u} (b_l - a_l) < \epsilon \quad \text{and} \quad b_\beta(O) > C \quad \text{where} \quad O := \bigcup_{l=0}^{u} (a_l, b_l).$$

By the same arguments we used in the proof of Proposition 5.2 we conclude:
$\exists c > 0$ $\forall C \in (0,1)$ $\forall \epsilon > 0$ $\exists N = N(\epsilon, C)$ $\forall n \geq N$:

$$\sum_{x_i^n \in \bar{O}} m_i^n > C \text{ and } \hat{\sharp}(n) := \operatorname{card}\{x_i^n \in \bar{O}\} \leq \epsilon c \beta^{-n}.$$

Since $m_i^n = b(P_n^i) = \sharp P_n^i / 2^n$, where $\sharp P$ denotes the number of cylinder sets of length n contained in P, it follows that there is a subset $\hat{\Pi}_{n,\beta}$ of $\Pi_{n,\beta}$ with $\hat{\sharp}(n)$ elements such that

$$\sum_{P \in \hat{\Pi}_{n,\beta}} \sharp P \geq C 2^n$$

We estimate:

$$\sum_{P \in \Pi_{n,\beta}} (\sharp P)^{\log\beta/\log\tau} = \sum_{P \in \hat{\Pi}_{n,\beta}} (\sharp P)^{\log\beta/\log\tau} + \sum_{P \in \Pi_{n,\beta} \setminus \hat{\Pi}_{n,\beta}} (\sharp P)^{\log\beta/\log\tau}$$

$$\leq \hat{\sharp}(n)^{1-\log\beta/\log\tau} \Big(\sum_{P \in \hat{\Pi}_{n,\beta}} \sharp P \Big)^{\log\beta/\log\tau}$$

$$+ (\sharp(n) - \hat{\sharp}(n))^{1-\log\beta/\log\tau} \Big(\sum_{P \in \Pi_{n,\beta} \setminus \hat{\Pi}_{n,\beta}} \sharp P \Big)^{\log\beta/\log\tau}$$

$$\leq (\epsilon c \beta^{-n})^{1-\log\beta/\log\tau} 2^{n\log\beta/\log\tau} + (c\beta^{-n})^{1-\log\beta/\log\tau} ((1-C)2)^{n\log\beta/\log\tau}$$

$$= \beta^{n(\log\beta/\log\tau - 1)} 2^{n\log\beta/\log\tau} ((\epsilon c)^{1-\log\beta/\log\tau} + c^{1-\log\beta/\log\tau} (1-C)^{\log\beta/\log\tau}).$$

Now choose ϵ and C such that

$$((\epsilon c)^{1-\log\beta/\log\tau} + c^{1-\log\beta/\log\tau}(1-C)^{\log\beta/\log\tau}) < 1.$$

For all $n \geq N(\epsilon, C)$ we have:

$$\frac{\log(\sum_{P\in\Pi_{n,\beta}}(\sharp P)^{\frac{\log\beta}{\log\tau}})}{n\log\beta^{-1}}$$

$$< \frac{\log(2\beta/\tau)}{\log(1/\tau)} + \frac{\log((\epsilon c)^{1-\log\beta/\log\tau} + c^{1-\log\beta/\log\tau}(1-C)^{\log\beta/\log\tau})}{n\log\beta^{-1}}.$$

The last term in this sum is negative and hence our proof is complete. □

Proof of 2.2 From [22] we know that the box-counting dimension of $\Lambda_{\beta,\tau}$ is given by $\log(2\beta/\tau)/\log(1/\tau)$. Thus Proposition 7.1 and 7.2 immediately imply $\dim_H \Lambda_{\beta,\tau} < \dim_B \Lambda_{\beta,\tau}$ if $\beta \in (0.5, 1)$ is the reciprocal of a PV number. This is first statement of Theorem 2.2. Now the second statement remains to prove. The following dimension formula for the Bernoulli measures $b^p_{\beta,\tau}$ on $\Lambda_{\beta,\tau}$ is a corollary of Theorem II of [17]

$$\dim_H b^p_{\beta,\tau} = \frac{p\log p + (1-p)\log(1-p)}{\log\tau} + (1 - \frac{\log\beta}{\log\tau})\dim_H b^p_\beta.$$

Thus we have by Theorem 4.1 $\dim_H b^p_{\beta,\tau} < 1$ for all $p \in (0,1)$ if $\beta \in (0.5, 1)$ is the reciprocal of a PV number and τ is small enough. But on the other hand we have $\dim_H \Lambda_{\beta,\tau} \geq 1$ since the projection of $\Lambda_{\beta,\tau}$ on the first coordinate axis is the whole interval $[-1, 1]$. This proofs the second statement of our Theorem 2.2. □

Appendix A: General definitions and facts in dimension theory

We will here first define the most important quantities in dimension theory and then collect some basic facts. We refer to the book of Falconer [6] and the book of Pesin [18] for a more detailed discussion of dimension theory.

Let $q \in I\!N$ and $Z \subseteq I\!R^q$. For a real number $s > 0$ we define the **s-dimensional Hausdorff measure** $H^s(Z)$ of Z by

$$H^s(Z) = \lim_{\lambda \longrightarrow 0} \inf\{\sum_{i \in I} (\mathrm{diam} U_i)^s \colon Z \subseteq \bigcup_{i \in I} U_i \text{ and } \mathrm{diam}(U_i) \leq \lambda\}$$

where I is a countable index set. The **Hausdorff dimension** $\dim_H Z$ of Z is given by

$$\dim_H Z = \sup\{s \colon H^s(Z) = \infty\} = \inf\{s \colon H^s(Z) = 0\}.$$

Let $N_\epsilon(Z)$ be the minimal number of balls of radius ϵ that are needed to cover Z. We define the **upper box-counting dimension** $\overline{\dim}_B$ resp. **lower box-counting dimension** $\underline{\dim}_B$ of Z by

$$\overline{\dim}_B Z = \overline{\lim}_{\epsilon \longrightarrow 0} \frac{\log N_\epsilon(Z)}{-\log \epsilon} \qquad \underline{\dim}_B Z = \underline{\lim}_{\epsilon \longrightarrow 0} \frac{\log N_\epsilon(Z)}{-\log \epsilon}.$$

If the limit it is called the **box-counting dimension** $\dim_B$ of Z. We remark that these quantities are not changed if we replace $N_\epsilon(Z)$ by the mi nimal number of squares parallel to the axis with side length ϵ that are needed to cover Z. Furthermore we note that limit in the definition exists, if it exists for some exponential decreasing sequence.

Now let μ be a Borel probability measure on $I\!R^q$. We define the Hausdorff dimension of μ by

$$\dim_H \mu = \inf\{\dim_H Z \colon \mu(Z) = 1\}.$$

We introduce one more notion of dimension for the measure μ. Let $h_\mu(\epsilon) = \inf\{H_\mu(\Pi) \colon \Pi \text{ a partition with } \mathrm{diam}\Pi \leq \epsilon\}$ where $H_\mu(\Pi)$ is the usual entropy of Π. We define the **upper Rényi dimension** $\overline{\dim}_R$ resp. **lower Rényi dimension** $\underline{\dim}_R$ of μ by

$$\overline{\dim}_R \mu = \overline{\lim}_{\epsilon \longrightarrow 0} \frac{h_\mu(\epsilon)}{-\log \epsilon} \qquad \underline{\dim}_R \mu = \underline{\lim}_{\epsilon \longrightarrow 0} \frac{h_\mu(\epsilon)}{-\log \epsilon}.$$

If the limit exists it is called **Rényi dimension** $\dim_R$ of μ. The **upper local dimension** $\overline{d}(x, \mu)$ resp. **lower local dimension** $\underline{d}(x, \mu)$ of the measure μ in a point x is defined by

$$\overline{d}(x, \mu) = \overline{\lim}_{\epsilon \longrightarrow 0} \frac{\mu(B_\epsilon(x))}{\log \epsilon} \qquad \underline{d}(x, \mu) = \underline{\lim}_{\epsilon \longrightarrow 0} \frac{\mu(B_\epsilon(x))}{\log \epsilon}.$$

One basic fact we like to mention here is that dimensional theoretical quantities are not increased by projections or more general Lipschitz maps. This is immediate from the definitions. Basic relations between the dimensions introduced here are stated in the following proposition.

Proposition A1 *For all $Z \subseteq \mathbb{R}^q$ and all Borel probability measures μ on $\mathbb{R}^q$ we have:*
(1) $\dim_H Z \leq \underline{\dim}_B Z \leq \overline{\dim}_B Z$
(2) $\underline{d}(x,\mu) \leq c$ *$\mu-$almost everywhere* $\Rightarrow$ $\dim_H \mu \leq c$.
(3) $\underline{d}(x,\mu) \geq c$ *$\mu-$almost everywhere* $\Rightarrow$ $\dim_H \mu \geq c$ *and* $\underline{\dim}_R \mu \geq c$.
(4) $\overline{d}(x,\mu) = \underline{d}(x,\mu) = c$ *$\mu-$almost everywhere* $\Rightarrow$ $\dim_H \mu = \dim_R \mu = c$.

The first inequality is obvious. A proof of the other statements is contained in the work of Young [28]. If the condition in part (4) holds, the measure μ is called **exact dimensional** and the common value of the dimensions is denoted by $\dim \mu$.

We need one other basic fact in our work which follows from Proposition 7.4 of [6]..

Proposition A2 *If $Z \subseteq \mathbb{R}^q$ and I is an interval then* $\dim_H(Z \times I) = \dim_H +1$.

Appendix B: Pisot-Vijayarghavan numbers

A **Pisot-Vijayarghavan number** (short: PV number) is by definition the root of an algebraic equitation whose algebraic conjugates lie all inside the unit circle in the complex plane. Salem [23] showed that the set of PV numbers is a closed subset of the reals and that 1 is an isolated element. In our context we are interested in numbers $\beta \in (0.5, 1)$ such that β^{-1} is a PV number. We list some examples including all reciprocals of PV numbers with minimal polynomial of degree two and three and a sequence of such numbers decreasing to 0.5.

x^2+x-1	$(\sqrt{5}-1)/2$
x^3+x^2+x-1	$0.5436898\ldots$
x^3+x^2-1	$0.754877\ \ldots$
x^3+x-1	$0.6823278\ldots$
x^3-x^2+2x-1	$0.5698403\ldots$
x^4-x^3-1	$0.7244918\ldots$
$x^n+x^{n-1}\ldots+x-1$	$r_n \longrightarrow 0.5$

Table 1: Reciprocals of PV numbers

An important property of PV numbers is that their powers are near integers. More precise:

Proposition B1*If α is a PV number then there is a constant $0<\theta<1$ such that $||\alpha^n||_{\mathbb{Z}} \leq \theta^n\ \forall n \geq 0$ where $||.||_{\mathbb{Z}}$ denotes the distance to the nearest integer.*

This statement can be found in [5]. There is an another property of PV numbers that is of great importance for us. For $\beta \in (0,1)$ we denote by $\sharp_\beta(n)$ the number of distinct points of the for $\sum_{k=0}^{n-1} \pm\beta^k$ and by $\omega_\beta(n)$ the minimal distance between two of those points.

Proposition B2 *If $\beta \in (0.5,1)$ is the reciprocal of a PV number then there are constants $\bar{c}>0$ and $\bar{C}>0$ such that $\omega_\beta(n) \geq \bar{c}\beta^n$ and $\sharp_\beta(n) \geq \bar{C}\beta^{-n}$ holds for all $n \geq 0$.*

For the first inequality we refer to Lemma 1.6 of [9]. For the second inequality see formula (15) in [21]. Finally we like to mention that there is a whole book about Pisot and Salem numbers [2]. Certainly the reader will find much more information about the role of these numbers in algebraic number theory and Fourier analysis in this book than we provided here for our purposes.

References

[1] J.C. Alexander, J.A. Yorke, Fat Baker s transformation's, Ergodic Thy. Dyn. Sys. 4, 1-23, 1984.

[2] M.J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, J.P. Schreiber, Pisot and Salem numbers, Birkhauser Verlag Basel, 1992.

[3] L. Barreira, Ya. Pesin and J. Schmeling, Dimension and product structure of hyperbolic measures, Annals of Math., 149:3, 755-783, 1999.

[4] M. Denker, C. Grillenberger, K.Sigmund, Ergodic Theory on Compact Spaces, Lecture Notes in Math. 527, Springer Verlag Berlin, 1976.

[5] Erdös, On a family of symmetric Bernoulli convolutions, , Amer. J. Math 61, 974-976, 1939.

[6] K. Falconer, Fractal Geometry - Mathematical Foundations and Applications, Wiley, New York, 1990.

[7] K. Falconer, The Hausdorff dimension of self-affine fractals, Math. Proc. Camb. Phil. Soc. 103, 339-350, 1988.

[8] A.M. Garsia, Entropy and singularity of infinite convolutions, Pac. J. Math. 13, 1159-1169, 1963.

[9] A.M. Garsia, Arithmetic properties of Bernoulli convolutions, Trans. Amer. Math. Soc. 162, 409-432, 1962.

[10] J.E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30, 271-280, 1981.

[11] B. Jessen and A. Winter, Distribution functions and the Riemann zeta function, Trans. Amer. Math. Soc. 38, 48-88, 1935.

[12] A. Katok and B. Hasselblatt, Introduction to Modern theory of dynamical Systems, Cambridge University press, 1995.

[13] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphism, Ann. Math. 122, 509-574, 1985.

[14] F. Ledrappier and A. Porzio, A dimension formula for Bernoulli convolutions, J. Stat. Phy. 76, no. 5/6, 1307-1326, 1994.

[15] C. McMullen, The Hausdorff dimension of general Sierpinski carpets. Nagoya Math. J., 96, 1-9, 1984.

[16] A. Manning and H. McCluskey, Hausdorff dimension for horseshoes, Ergodic Thy. Dyn. Sys. 3, 251-260, 1983.

[17] J. Neunhäuserer, Properties of some overlapping self-similar and some self-affine measures, Schwerpunktprogramm der deutschen Forschungsgemeinschaft: DANSE, Preprint 35/99; to appear in: Acta Mathematica Hungarica 2002.

[18] Ya. Pesin, Dimension Theory in Dynamical Systems - Contemplary Views an Applications, University of Chicago Press, 1997.

[19] Y. Peres and B. Solomyak, Self-similar measures and intersection of Cantor sets, Trans. Amer. Math. Soc 350, no. 10, 4065-4087, 1998

[20] Y. Peres, B. Solomyak and W. Schlag, Sixty years of Bernoulli convolutions, Fractals and stochastic II, Progress in Probability 46, 95-106, Birkhauser, 2000.

[21] F. Przytychi and M. Urbanski, On Hausdorff dimension of some fractal sets, Studia. Math. 54, 218-228, 1989.

[22] M. Pollicott and H. Weiss, The dimension of self-affine limit sets in the plane, J. Stat. Phys. 77, 841-860, 1994.

[23] R. Salem, A remarkable class of algebraic integers, proof of a conjecture by Vijayarghavan, Duke Math. J., 103-108, 1944.

[24] J. Schmeling, A dimension formula for endomorphisms - The Belykh family, Ergod. Th. Dyn. Sys 18, 1283-1309, 1998.

[25] B. Solomyak, On the random series $\sum \pm\lambda^i$ (an Erdös problem), Ann. Math. 142, 1995.

[26] B. Solomyak, Measures and Dimensions for some Fractal Families, Proc. Cambridge Phil. Soc., 124/3, 531-546, 1998.

[27] B. Solomyak and K. Simon, Dimension of horseshoes in $\mathbb{R}^3$, Ergodic Theory and Dyn. Sys. 19, 1345-1363, 1999.

[28] L.-S. Young, Dimension, entropy and Lyapunov exponents, Ergod. Thy. Dyn. Sys. 2, 109-124, 1982.