Hecke Operators for Period Functions for Congruence Subgroups

Tobias Mühlenbruch*

Institut für Theoretische Physik
Technische Universität Clausthal
38678 Clausthal-Zellerfeld
Germany
tobias.muehlenbruch@tu-clausthal.de

Abstract. In [8] and [3] period functions for the full modular group and the congruence subgroups were introduced. It was shown that the space of period functions is in $1-1$ correspondence with the space of Maass cusp forms.
In this article we summarise the construction of Hecke operators on the period functions, based on [14]. In particular we compute the matrix representation of the second Hecke operator on period functions for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ respectively the congruence subgroup $\Gamma_0(2)$.

1 Notation

To state our main result and to sketch the content of each section we have to fix some notations used throughout the text. Let $\mathbb{N}$ denote the set of positive integers and let $n \in \mathbb{N}$ throughtout this paper. We denote by $\mathrm{M}_n(2,\mathbb{Z})$ (respectively $\mathrm{M}_*(2,\mathbb{Z})$) the set of 2×2 matrices with integer entries and determinant n (respectively positive determinant). Let $\mathcal{R}_n := \mathbb{Z}[\mathrm{M}_n(2,\mathbb{Z})]$ (respectively $\mathcal{R} := \mathbb{Z}[\mathrm{M}_*(2,\mathbb{Z})]$) be the set of finite linear combinations with coefficients in $\mathbb{Z}$ of elements of $\mathrm{M}_n(2,\mathbb{Z})$ (respectively $\mathrm{M}_*(2,\mathbb{Z})$). Similarly, we denote by $\mathrm{M}_n^+(2,\mathbb{Z})$ (respectively $\mathrm{M}_*^+(2,\mathbb{Z})$) the set of 2×2 matrices with nonnegative integer entries and determinant n (respectively positive determinant). Denote furthermore by $\mathcal{R}_n^+ := \mathbb{Z}[\mathrm{M}_n^+(2,\mathbb{Z})]$ (respectively $\mathcal{R}^+ := \mathbb{Z}[\mathrm{M}_*^+(2,\mathbb{Z})]$) the set of finite linear combinations with coefficients in $\mathbb{Z}$ of elements of $\mathrm{M}_n^+(2,\mathbb{Z})$ (respectively $\mathrm{M}_*^+(2,\mathbb{Z})$). Note that $\mathcal{R}^{(+)} = \bigcup_{n=1}^{\infty} \mathcal{R}_n^{(+)}$ and $\mathcal{R}_n^{(+)} \cdot \mathcal{R}_m^{(+)} \subset \mathcal{R}_{nm}^{(+)}$. By definition we have $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{M}_1(2,\mathbb{Z})$. The following four elements of $\mathrm{SL}_2(\mathbb{Z})$ will play a special role in our paper:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T' := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The *Hecke congruence subgroup* $\Gamma_0(n)$ is given by

$$\Gamma_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z});\ c \equiv 0 \bmod n \right\}.$$

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The full modular group is denoted by $\Gamma(1) := \Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$. Let $\mu = \mu_n$ denote the index of $\Gamma_0(n)$ in $\Gamma(1)$ and let $\alpha_1, \ldots, \alpha_\mu$ denote representatives of the right cosets in $\Gamma_0(n)\backslash\Gamma(1)$.

We also need the set of upper triangular matrices

$$X_m = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{M}_m(2, \mathbb{Z});\ d > b \geq 0 \right\}. \tag{1}$$

Let $\bar{\mathbb{C}}$ denote the one-point compactification of $\mathbb{C}$. The map

$$\mathrm{M}_*(2, \mathbb{Z}) \times \bar{\mathbb{C}} \to \bar{\mathbb{C}}; \quad \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z \right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z := \frac{az+b}{cz+d}.$$

gives an action of the matrices on $\bar{\mathbb{C}}$. It induces the familiar *slash action* $|_s$ on functions f on $\mathbb{H} := \{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$ respectively $(0, \infty)$ formally defined by

$$f\big|_s \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = (ad - bc)^s \, (cz + d)^{-2s} \, f\left(\frac{az+b}{cz+d}\right) \tag{2}$$

for complex numbers s and certain classes of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The slash action is well defined for

(a) $s \in \mathbb{Z}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_m(2, \mathbb{Z})$, $m \in \mathbb{N}$, $z \in \mathbb{H}$ and

(b) $s \in \mathbb{C}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_m^+(2, \mathbb{Z})$, $m \in \mathbb{N}$ and $z \in (0, \infty)$

as the discussions in [6] and [12] show. We have $\left(f\big|_s\alpha\right)\big|_s\gamma = f\big|_s(\alpha\gamma)$ for all matrices α, $\gamma \in \mathrm{M}_*(2, \mathbb{Z})$ in case (a) respectively $\mathrm{M}_*^+(2, \mathbb{Z})$ in case (b). We extend the slash action linearly to formal sums of matrices.

2 Maass cusp forms and period functions

Definition 1. *A* Maass cusp form u *for the congruence subgroup* $\Gamma_0(n)$ *is a real-analytic function* $u : \mathbb{H} \to \mathbb{C}$ *satisfying:*

1. $u(gz) = u(z)$ *for all* $g \in \Gamma_0(n)$,
2. $\Delta u = s(1-s)u$ *for some* $s \in \mathbb{C}$ *where* $\Delta = -y^2(\partial_x^2 + \partial_y^2)$ *is the hyperbolic Laplace operator. We call the parameter* s *the* spectral parameter *of* u.
3. u *is of rapid decay in all cusps: for* $p \in \mathbb{Q} \cup \{\infty\}$ *and* $g \in \Gamma(1)$ *such that* $gp = \infty$ *we have* $u(g^{-1}z) = \mathrm{O}\left(\mathrm{Im}(z)^C\right)$ *as* $\mathrm{Im}(z) \to \infty$ *for all* $C \in \mathbb{R}$.

We denote the space of Maass cusp forms for $\Gamma_0(n)$ *with spectral value* s *by* $S(n, s)$.

A function $f : (0, \infty) \to \mathbb{C}$ is called *holomorphic* if it is locally the restriction of a holomorphic function.

The vector valued period functions for $\Gamma_0(n)$ are defined as follows:

Definition 2. *A* period function *for $\Gamma_0(n)$ is a function $\boldsymbol{\psi} : (0,\infty) \to \mathbb{C}^\mu$ with $\boldsymbol{\psi} = (\psi_i)_{i\in\{1,\dots,\mu\}}$ such that*

1. *ψ_i is holomorphic on $(0,\infty)$ for all $i \in \{1,\dots,\mu\}$.*
2. *$\boldsymbol{\psi}(z) = \rho(T^{-1})\,\boldsymbol{\psi}(z+1) + (z+1)^{-2s}\,\rho(T'^{-1})\,\boldsymbol{\psi}\left(\frac{z}{z+1}\right)$. We call the parameter $s \in \mathbb{C}$ the* spectral parameter *of $\boldsymbol{\psi}$. The matrix representation $\rho : \Gamma(1) \to \mathbb{C}^{\mu\times\mu}$ is induced by the trivial representation of $\Gamma_0(n)$, see Appendix B.*
3. *For each $i = 1,\dots,\mu$ ψ_i satisfies the growth condition*

$$\psi_i(z) = \begin{cases} \mathrm{O}\left(z^{\max\{0,-2\mathrm{Re}(s)\}}\right) & \text{as } z \downarrow 0 \text{ and} \\ \mathrm{O}\left(z^{\min\{0,-2\mathrm{Re}(s)\}}\right) & \text{as } z \to \infty. \end{cases}$$

Following [8] we denote the space of period functions for $\Gamma_0(n)$ with spectral value s by $\mathrm{FE}(n,s)$. We call a function $\boldsymbol{\psi}$ a *period like function* if $\boldsymbol{\psi}$ satisfies only conditions 1 and 2. The space of period like functions for $\Gamma_0(n)$ with spectral value s is denoted by $\mathrm{FE}^*(n,s)$

Remark 1. The authors of [8] have shown that $\mathrm{FE}(1,s)$ is isomorphic to $S(1,s)$. A. Deitmar and J. Hilgert generalize this result to submodular groups of finite index, and hence for the congruence subgroup $\Gamma_0(n)$, in [3].

Before we recall Hecke operators for $S(n,s)$ and construct Hecke operators for $\mathrm{FE}(n,s)$ we have to show how the spaces $S(n,s)$ and $\mathrm{FE}(n,s)$ are related.

3 Vector valued cusp forms

For each $u \in S(n,s)$ we construct a vector valued version of u which transforms under $\Gamma(1)$ with respect to the representation ρ.

Definition 3. *A* vector valued cusp form $\boldsymbol{u} : \mathbb{H} \to \mathbb{C}^\mu$ *for $\Gamma_0(n)$ with spectral value $s \in \mathbb{C}$ is a vector valued function $\boldsymbol{u} = (u_1,\dots,u_\mu)^{\mathrm{tr}}$ satisfying*

- *u_j is real-analytic for all $j \in \{1,\dots,\mu\}$,*
- *$\boldsymbol{u}(gz) = \rho(g)\,\boldsymbol{u}(z)$ for all $z \in \mathbb{H}$ and $g \in \Gamma(1)$, where ρ is the matrix representation defined in Appendix B,*
- *$\Delta u_j = s(1-s)u_j$ for all $j \in \{1,\dots,\mu\}$ and*
- *$u_j(z) = \mathrm{O}\left(\mathrm{Im}(z)^C\right)$ as $\mathrm{Im}(z) \to \infty$ for all $C \in \mathbb{R}$ and $j \in \{1,\dots,\mu\}$.*

We denote the space of all vector valued cusp forms with spectral parameter s for $\Gamma_0(n)$ by $S_{\mathrm{ind}}(n,s)$.

To each $u \in S(n,s)$ we associate the vector valued function $\Pi(u)$ given by

$$\Pi : S(n,s) \to S_{\mathrm{ind}}(n,s); \quad u \mapsto \Pi(u) := \left(u\big|_0\alpha_1,\dots,u\big|_0\alpha_n\right)^{\mathrm{tr}}. \tag{3}$$

It was shown in [14] that Π is bijective. This proves the following

Lemma 1. *The spaces $S_{\mathrm{ind}}(n,s)$ and $S(n,s)$ are isomorphic.*

4 The period functions of vector valued cusp forms

We identify $\pm\infty$ with the cusp $i\infty$. The action of $\Gamma(1)$ on $\mathbb{H}$ extends naturally to $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$.

We use the definition in [14] of a *path* L connecting points $z_0, z_1 \in \mathbb{H}^*$. Basically we understand by a *path* L connecting points $z_0, z_1 \in \mathbb{H}^*$ a piecewise smooth curve which lies inside $\mathbb{H}$ except possibly for finitely many points. If a point z of the path L is in $\mathbb{Q} \cup \{\infty\}$ then locally the path lies inside an arc bounded by geodesics starting in z. For distinct $z_0, z_1 \in \mathbb{H}^* \setminus \mathbb{H}$ the *standard path* L_{z_0,z_1} is the geodesic connecting z_0 and z_1.

Definition 4. *For $\boldsymbol{u} \in S_{\mathrm{ind}}(n,s)$ and $L_{0,\infty}$ the standard path the integral transform $P : S_{\mathrm{ind}}(n,s) \to \big(C^\omega(0,\infty)\big)^\mu$, with $C^\omega(0,\infty)$ the space of holomorphic functions on $(0,\infty)$, is defined as*

$$\big(P\boldsymbol{u}\big)_j(\zeta) = \int_{L_{0,\infty}} \eta\big(u_j, R_\zeta^s\big) \qquad \text{for } \zeta > 0,\ j \in \{1,\ldots,\mu\}. \tag{4}$$

Formally, we write (4) as

$$\big(P\boldsymbol{u}\big)(\zeta) = \int_{L_{0,\infty}} \eta\big(\boldsymbol{u}, R_\zeta^s\big). \tag{5}$$

Remark 2. – Note that $R_\zeta(z)$ denotes a function of two variables ζ and z.
- The existence of the integral transform is shown in [14].
- The 1-forms $\eta(u_i, R_\zeta^s)$ are closed, see Appendix C. Hence

$$\int_{L'} \eta\big(u_i, R_\zeta^s\big) = \int_{L_{0,\infty}} \eta\big(u_i, R_\zeta^s\big) = \big[Pu\big]_i(\zeta)$$

  for arbitrary paths L' homotopic to $L_{0,\infty}$.
- The fact that $R_\zeta^s(z)$ is holomorphic in ζ, see (22), implies that $\big[P\boldsymbol{u}\big]_j(\zeta)$ is a holomorphic function in ζ.

The function $P\boldsymbol{u}$ in (5) has the following properties [14]:

Lemma 2. *For $\zeta > 0$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ with $a,b,c,d \geq 0$ the function $P\boldsymbol{u}$ with $\boldsymbol{u} \in S_{\mathrm{ind}}(n,s)$ satisfies*

$$(c\zeta + d)^{-2s}\,\rho(\gamma^{-1})\,P\boldsymbol{u}(\gamma\zeta) = \int_{L_{\gamma^{-1}0,\gamma^{-1}\infty}} \eta\big(\boldsymbol{u}, R_\zeta^s\big). \tag{6}$$

Lemma 3. *For $\boldsymbol{u} \in S_{\mathrm{ind}}(n,s)$ each component of the function $P\boldsymbol{u}$ satisfies the growth conditions*

$$(P\boldsymbol{u})_i(\zeta) = \begin{cases} \mathrm{O}\left(\zeta^{\max\left(0,-2\mathrm{Re}(s)\right)}\right) & \text{as } \zeta \downarrow 0 \text{ and} \\ \mathrm{O}\left(\zeta^{\min\left(0,-2\mathrm{Re}(s)\right)}\right) & \text{as } \zeta \to \infty. \end{cases}$$

Lemma 2 and Lemma 3 imply the following:

Proposition 1. *For $\boldsymbol{u} \in S_{\mathrm{ind}}(n,s)$ the function $P\boldsymbol{u}$ is a period function.*

Moreover, [14] shows the following

Proposition 2. *For $s \in \mathbb{C} \setminus \mathbb{Z}$, $\mathrm{Re}(s) > 0$ the operator*

$$P: \quad S_{\mathrm{ind}}(n,s) \to \mathrm{FE}(n,s); \quad \boldsymbol{u} \mapsto P\boldsymbol{u}$$

is bijective.

5 Hecke operators for Maass cusp forms

We define the Hecke operators on $S(n,s)$ for a fixed $s \in \mathbb{C}$ similarly as in [1] for modular forms.

Definition 5. *Denote by $T(p)$ and $U(q)$ for $\gcd(p,n) = 1$, $q|n$ and p,q prime the following elements in $\mathcal{R}_p$ respectively $\mathcal{R}_q$:*

$$T(p) = \sum_{\substack{ad=p \\ 0 \le b < d}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad \textit{and} \quad U(q) = \sum_{0 \le b < q} \begin{pmatrix} 1 & b \\ 0 & q \end{pmatrix}. \tag{7}$$

The induced maps $S(n,s) \to S(n,s)$; $u \mapsto u\big|_0 T(p)$ respectively $u \mapsto u\big|_0 U(q)$ are called the p^{th} and q^{th} Hecke operator H_p respectively H_q on $S(n,s)$

Obviously, the Hecke operators H_p and H_q depend on n.

Remark 3. The Hecke operators H_p and H_q use a nonstandard normalization.

6 Hecke operators for vector valued cusp forms

For each $g \in \Gamma(1)$ we define the map $\sigma_g : X_m \to X_m$ by $A\,g\,(\sigma_g(A))^{-1} \in \Gamma(1)$ for all $A \in X_m$. It was shown in [14] that σ_g is bijective and satisfies $\sigma_g^{-1} = \sigma_{g^{-1}}$.

Also for $A \in X_m$ we define the map

$$\phi_A = \phi_{A,n}: \quad \{1,\ldots,\mu_n\} \to \{1,\ldots,\mu_n\}; \quad i \mapsto \phi_A(i) \tag{8}$$

such that

$$A\alpha_i \in \Gamma_0\,(n)\alpha_{\phi_A(i)}\,\sigma_{\alpha_i}(A). \tag{9}$$

Usually we write $\phi_{A,n} = \phi_A$ omitting the index n since n is fixed in the entire discussion. Note that the map ϕ_A is not bijective, see e.g. the example given in Table 1.

To derive a formula for the Hecke operators acting on $S_{\mathrm{ind}}(n,s)$ we have to write the vector valued cusp form $\Pi\big(u\big|_0 \sum_A A\big)$ in terms of a linear action of a certain matrix sum on the vector valued cusp form $\Pi(u)$.

For prime p, q with $\gcd(p,n) = 1$ and $q|n$ the p^{th} (respectively q^{th}) Hecke operator H_p (respectively H_q) is given by the action of $T(p)$ (respectively $U(q)$) on the space of cusp forms which we write as

$$S(n,s) \to S(n,s); \quad u \mapsto u\big|_0 \sum_A A = \sum_{A \in \mathcal{A}} u\big|_0 A$$

with $\sum_A A = T(p)$ and $\mathcal{A} = X_p$ (respectively $\sum_A A = U(q)$ and $\mathcal{A} = X_q \setminus \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)$).

Consider the j^{th} component of the vector valued cusp form $\mathit{\Pi}\big(u\big|_0 \sum_A A\big)$. By (9) write this component as

$$\left(u\big|_0 \sum_A A\right)\big|_0 \alpha_j = u\big|_0 \sum_A (A\alpha_j) = u\big|_0 \sum_A \alpha_{\phi_A(j)} \sigma_{\alpha_j}(A) \tag{10}$$

for indices $\phi_A(j) \in \{1, \ldots, \mu_n\}$ and upper triangular matrices $\sigma_{\alpha_j}(A)$.

This allows us to define Hecke operators for vector valued cusp forms.

Definition 6. *For $n, m \in \mathbb{N}$, m prime and $s \in \mathbb{C}$ put $\sum_A A := T(m)$ if $m \not| n$ and put $\sum_A A := U(m)$ if $m \mid n$. The m^{th}* Hecke operator *$H_{n,m}$ on $\boldsymbol{u} \in S_{\text{ind}}(n,s)$ is defined as*

$$\big(H_{n,m}\boldsymbol{u}\big)_j \mapsto \sum_A u_{\phi_A(j)}\big|_0 \sigma_{\alpha_j}(A) \quad \textit{for } j \in \{1, \ldots, \mu_n\}. \tag{11}$$

In (11) we sum over all $A \in X_m$ if $m \not| n$ and $A \in X_m \setminus \left\{\left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right)\right\}$ if $m \mid n$.

The m^{th} Hecke operator $H_{n,m}$ on $S_{\text{ind}}(n,s)$ corresponds to the m^{th} Hecke operator H_m on $S(n,s)$ [14]:

Proposition 3. *$\mathit{\Pi}(H_m u) = H_{n,m}\mathit{\Pi}(u)$.*

7 Left neighbor sequences

We recall the left neighbor sequences as introduced in [14]. The necessary facts on Farey sequences can be found in Appendix A.

Definition 7. *The* left neighbor map $\mathrm{LN} : \mathbb{Q} \cup \{+\infty\} \to \mathbb{Q} \cup \{-\infty\}$ *such that $\mathrm{LN}(q)$ is the left neighbor of q in the Farey sequence $F_{\mathrm{lev}(q)}$, that is*

$$\mathrm{LN}(q) = \max\{r \in F_{\mathrm{lev}(q)};\, r < q\}. \tag{12}$$

Lemma 4. *For $q \in \mathbb{Q} \cup \{+\infty\}$ and $\mathrm{lev}(q) > 0$ we have $\mathrm{lev}\big(\mathrm{LN}(q)\big) < \mathrm{lev}(q)$.*

Definition 8. *Let be $q \in \mathbb{Q} \cup \{+\infty\}$ and $L = L_q \in \mathbb{N}$ such that*

$$\mathrm{LN}^L(q) = -\infty \qquad \textit{and} \qquad \mathrm{LN}^l(q) > -\infty \quad \textit{for all } l = 1, \ldots, L-1.$$

The left neighbor sequence $\mathrm{LNS}(q)$ *of q is the finite sequence*

$$\mathrm{LNS}(q) = \big(\mathrm{LN}^L(q), \mathrm{LN}^{L-1}(q), \ldots, \mathrm{LN}^1(q), q\big),$$

where we use the notation $\mathrm{LN}^l(q) := \underbrace{\mathrm{LN}\big(\mathrm{LN}(\cdots \mathrm{LN}(q))\cdots\big)}_{l \text{ times}}$.

The number $L = L_q$ in Definition 8 is unique.

To LNS(q) we construct an element $M(q) \in \mathcal{R}_1$:

Definition 9. *To each rational $q \in [0,1)$ consider* $\mathrm{LNS}(q) = \left(\frac{a_0}{b_0}, \ldots, \frac{a_L}{b_L}\right)$ *with* $\gcd(a_l, b_l) = 1$ *and $b_l \geq 0$, $l = 0, \ldots, L$. We define $M(q) = \sum_{l=1}^{L} m_l \in \mathcal{R}_1$ by*

$$M(q) = \begin{pmatrix} -a_0 & a_1 \\ -b_0 & b_1 \end{pmatrix}^{-1} + \ldots + \begin{pmatrix} -a_{l-1} & a_l \\ -b_{l-1} & b_l \end{pmatrix}^{-1} + \ldots + \begin{pmatrix} -a_{L-1} & a_L \\ -b_{L-1} & b_L \end{pmatrix}^{-1}. \tag{13}$$

Lemma 5. *Let $0 \leq q < 1$ rational and $M(q) = \sum_{l=1}^{L} \begin{pmatrix} * & * \\ c_l & d_l \end{pmatrix}$. We have $c_l\zeta + d_l > 0$ for all $\zeta > q$ and $l \in \{1, \ldots, L\}$.*

Lemma 6. *Let $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{M}_*(2, \mathbb{Z})$ be such that $a, b \in \mathbb{N}$, $0 \leq b < d$ and $M\left(\frac{b}{d}\right) = \sum_{l=1}^{L} m_l$. Then*

- $\det m_l = 1$,
- *the matrices $m_l A$ contain only nonnegative integer entries and*
- *the entries of $m_l A = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ satisfy $a' > c' \geq 0$ and $d' > b' \geq 0$.*

Lemma 7. *For rational $q \in [0,1)$ put $M(q) = \sum_l m_l$ as in Definition 9. The two paths $L_{q,\infty}$ and $\bigcup_l L_{m_l^{-1}0, m_l^{-1}\infty}$ have the same initial and end point.*

8 Hecke operators for period functions for $\Gamma(1)$

We consider first the simpler case $n = 1$.

Definition 10. *For $m \in \mathbb{N}$ define*

$$\tilde{H}(m) := \sum_{A \in X_m} M\bigl(A0\bigr)A \in \mathcal{R}_m^+. \tag{14}$$

For $s \in \mathbb{C}$ the formal sum $\tilde{H}(m)$ defines an operator $\tilde{H}_m$ on $C^\omega(0,\infty)$ by

$$\tilde{H}_{1,m} f := f\big|_s \tilde{H}(m) \qquad (f \in C^\omega(0,\infty)). \tag{15}$$

Remark 4.
- Lemmas 6 and 5 imply in particular that $\bigl(\tilde{H}_{1,m} f\bigr)(\zeta)$ is well defined for all $\zeta > 0$.
- Lemma 6 also shows that $\{m_l A;\ A \in X_m,\ M(A0) = \sum_{l=1}^{L} m_l\}$, containing all matrices in $\tilde{H}(m)$, is a subset of

  $$S_m = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix};\ a > c \geq 0, d > b \geq 0 \right\} \subset \mathrm{M}_n^+(2, \mathbb{Z}).$$

  It is shown in [6] that both sets are equal, implying $\tilde{H}(m) = \sum_{B \in S_m} B$. (The authors in [6] assume that $\gcd(a,b,c,d) = 1$ but this restriction is not necessary.)

Before we show that $\tilde{H}_{1,m}$ are indeed the Hecke operators on $\mathrm{FE}(1,s)$ we need some technical lemmas [14]:

Lemma 8. *Let $u \in S(1,s)$ be a cusp form and Pu as in Definition 4. For $A = \left(\begin{smallmatrix} * & * \\ 0 & d \end{smallmatrix}\right) \in X_m$ and $M(A0) = \sum_{l=1}^{L} m_l \in \mathcal{R}_1$ we have*

$$m^s d^{-2s} \int_{L_{A0,A\infty}} \eta\big(u, R^s_{A\zeta}\big) = \sum_{l=1}^{L} \big(Pu\big|_s m_l A\big)(\zeta) = \big(Pu\big|_s M(A0)A\big)(\zeta) \quad (16)$$

for all $\zeta > 0$.

Lemma 9. *Let Pu be the period function of the cusp form $u \in S(1,s)$. For any $m \in \mathbb{N}$ the operator $\tilde{H}_{1,m}$ satisfies*

$$\Big(\tilde{H}_{1,m}(Pu)\Big)(\zeta) = \int_0^{i\infty} \eta\big(H_m u, R^s_\zeta\big) \qquad \text{for } \zeta > 0. \quad (17)$$

Proof. Use Lemma 8 together with the transformation property (23). □

The relation between H_m on $S(1,s)$ and $\tilde{H}_{1,m}$ on $\mathrm{FE}(1,s)$ is given by

Proposition 4. *For $u \in S(1,s)$ the period function $Pu \in \mathrm{FE}(1,s)$ satisfies the identity*

$$\big(Pu\big)\big|_s \tilde{H}(m) = P\big(u\big|_0 H(m)\big).$$

Proof. This follows immediately from Lemma 9. □

Remark 5. Proposition 4 shows $\tilde{H}_{1,m}$ are indeed Hecke operators on $\mathrm{FE}(1,s)$. They are the same, [9], [5], as the operators constructed in [6] using only properties of the period functions respectively transfer operators for the groups $\Gamma_0\,(m)$. Another derivation of $\tilde{H}_{1,m}$ for $\Gamma(1)$ is given also in [13] using a criterion found by Choie and Zagier in [2]. L. Merel gives also a similar representation in [10].

9 Hecke operators for period functions for $\Gamma_0\,(n)$

In this section we extend the above derivation of the Hecke operators for period functions for $\Gamma(1)$ to the congruence subgroups $\Gamma_0\,(n)$.

First, we extend Lemma 8:

Lemma 10. *For $A = \left(\begin{smallmatrix} * & * \\ 0 & d \end{smallmatrix}\right) \in X_m$ put $M(A0) = \sum_{l=1}^{L} m_l \in \mathcal{R}_1$. If $P\boldsymbol{u}$ is a period function of $\boldsymbol{u} \in S_{\mathrm{ind}}(n,s)$, then*

$$m^s d^{-2s} \int_{L_{A0,A\infty}} \eta\big(\boldsymbol{u}, R^s_{A\zeta}\big) = \sum_{l=1}^{L} \rho(m_l^{-1}) \Big(P\boldsymbol{u}\big|_s m_l A\Big)(\zeta) \qquad \text{for } \zeta > 0. \quad (18)$$

In the following we denote the i^{th} component of the vector $\boldsymbol{u}$ by $[\boldsymbol{u}]_i$.

Lemma 11. *Let $\alpha_1, \dots, \alpha_\mu$ be representatives of the right coset of $\Gamma_0(n)$ in $\Gamma(1)$ where $\mu = [\Gamma(1) : \Gamma_0(n)]$. Let $P\boldsymbol{u}$ be the period function of $\boldsymbol{u} \in S_{\mathrm{ind}}(n,s)$. For $A \in X_m$, $j \in \{1, \dots, \mu_n\}$ let be $M(\sigma_{\alpha_j}(A)0) = \sum_{l=1}^{L} m_l \in \mathcal{R}_1$. Then the following identity holds for all $j \in \{1, \dots, \mu\}$ and $\zeta > 0$:*

$$\int_{L_{0,\infty}} \eta\big([\boldsymbol{u}]_{\phi_A(j)}\big|_0 \sigma_{\alpha_j}(A), R_\zeta^s\big) = \sum_{l=1}^{L} \left[\rho(m_l^{-1})\, P\boldsymbol{u}\right]_{\phi_A(j)} \big|_s m_l \sigma_{\alpha_j}(A)\,(\zeta). \quad (19)$$

Proof. Write $\boldsymbol{u} = (u_j)_j$. Using Lemma 14 and Property (23) of R_ζ we find for any $j \in \{1, \dots, \mu_n\}$ and $\zeta > 0$:

$$\begin{aligned}\int_{L_{0,\infty}} \eta\big(u_{\phi_A(j)}\big|_0 \sigma_{\alpha_j}(A), R_\zeta^s\big) \ &= \int_{L_{\sigma_{\alpha_j}(A)0,\sigma_{\alpha_j}(A)\infty}} \eta\big(u_{\phi_A(j)}, R_\zeta^s\big|_0 \big(\sigma_{\alpha_j}(A)\big)^{-1}\big) \\ &= m^s d_j^{-2s} \int_{L_{\sigma_{\alpha_j}(A)0,\sigma_{\alpha_j}(A)\infty}} \eta\Big(u_{\phi_A(j)}, R^s_{\sigma_{\alpha_j}(A)\zeta}\Big)\end{aligned}$$

where $\sigma_{\alpha_j}(A) = \begin{pmatrix} * & * \\ 0 & d_j \end{pmatrix}$ is again in X_m. Take $M(\sigma_{\alpha_j}(A)0) = \sum_{l=1}^{L} m_l \in \mathcal{R}_1$ and apply Lemma 10. We have

$$\begin{aligned}\int_{L_{0,\infty}} \eta\big(u_{\phi_A(j)}\big|_0 \sigma_{\alpha_j}(A), R_\zeta^s\big) &= \Big[\int_{L_{0,\infty}} \eta\big(\boldsymbol{u}\big|_0 \sigma_{\alpha_j}(A), R_\zeta^s\big)\Big]_{\phi_A(j)} \\ &= \left[\sum_{l=1}^{L} \rho(m_l^{-1}) \left(P\boldsymbol{u}\Big|_s m_l \sigma_{\alpha_j}(A)\right)(\zeta)\right]_{\phi_A(j)}.\end{aligned}$$

□

Remark 6. For $u \in S(n,s)$ and $A \in X_m$ Relation (9) implies that

$$u\big|_0 A\alpha_j = u\big|_0 \alpha_{\phi_A(j)} \sigma_{\alpha_j}(A).$$

Hence equation (19) can be written as

$$\int_{L_{0,\infty}} \eta\big(\Pi(u\big|_0 A), R_\zeta^s\big) = \sum_{l=1}^{L} \left[\rho(m_l^{-1})\, P\Pi(u)\right]_{\phi_A(j)} \big|_s m_l \sigma_{\alpha_j}(A)\,(\zeta)$$

Lemma 11 allows us to derive an explicit formula for the action of the Hecke operators on the period functions of $\Gamma_0(n)$ induced from the action of these operators on $S_{\mathrm{ind}}(n,s)$ for this group.

Proposition 5. *Let $\alpha_1, \dots, \alpha_\mu$ be representatives of the right cosets of $\Gamma_0(n)$ in $\Gamma(1)$. Let $P\boldsymbol{u}$ be the period function of $\boldsymbol{u} \in S_{\mathrm{ind}}(n,s)$. For m prime take $\mathcal{A} \subset X_m$ such that $T(m) = \sum_{A \in \mathcal{A}} A$ if $\gcd(m,n) = 1$ respectively $U(m) = \sum_{A \in \mathcal{A}} A$ if $m \mid n$. The m^{th} Hecke operator $\tilde{H}_{n,m}$ acting on $P\boldsymbol{u}$ is given by*

$$\Big[\tilde{H}_{n,m}\big(P\boldsymbol{u}\big)\Big]_j = \sum_{A \in \mathcal{A}} \sum_{l=1}^{L} \left[\rho(m_l^{-1})\, P\boldsymbol{u}\right]_{\phi_A(j)} \big|_s \big(m_l \sigma_{\alpha_j}(A)\big). \quad (20)$$

Remark 7. We emphasize that the constant L in (20) depends on A and that $\mathcal{A} = X_m$ for $\gcd(m,n) = 1$ respectively $\mathcal{A} = X_m \setminus \left\{ \left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix} \right) \right\}$ for $m \mid n$.

Proof (of Proposition 5). The m^{th} Hecke operator $H_{n,m}$ acts on $\boldsymbol{u}$ as

$$[H_{n,m}\boldsymbol{u}]_j = \sum_A u_{\phi_A(j)}\big|_0 \sigma_{\alpha_j}(A) \qquad \text{for } k \in \{1, \ldots, \mu\}.$$

Applying Lemma 11 to both sides then gives formula (20). □

Remark 8. In his diploma thesis [4], M. Fraczek uses a similar approach to compute a representation of the Fricke operator on $\mathrm{FE}(n,s)$. Part of his thesis was also to write a C-program computing the representation of $\tilde{H}_{n,m}$. Moreover the approach was also used in [5] to understand Hecke-like operators on $\mathrm{FE}(n,s)$ constructed in [6].

A On Farey sequences

We recall Farey sequences and some of its properties as presented in [7] and [14]. We adhere to the convention to denote infinity in rational form as $\infty = \frac{1}{0}$ and $-\infty = \frac{-1}{0}$ and to denote rationals $\frac{p}{q}$ with coprime $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Definition 11. *For $n \in \mathbb{N}$ the* Farey sequence F_n *of* level n *is the sequence*

$$F_n := \left(\frac{u}{v};\, u, v \in \mathbb{Z}, |u| \leq n, 0 \leq v \leq n \right).$$

ordered by the standard order $<$ of $\mathbb{R}$. We define F_0 as

$$F_0 := \left(\frac{-1}{0}, \frac{0}{1}, \frac{1}{0} \right).$$

The level function $\operatorname{lev} : \mathbb{Q} \cup \{\pm\infty\} \to \mathbb{Z}$ *is defined by*

$$\operatorname{lev}\left(\frac{a}{b}\right) = \begin{cases} 0 & \text{if } \frac{a}{b} \in \{\frac{-1}{0}, \frac{0}{1}, \frac{1}{0}\} \text{ and} \\ \max\{|a|, |b|\} & \text{otherwise.} \end{cases}$$

The number $\operatorname{lev}(q)$ is just the level of the Farey sequence in which the number q appears for the first time.

Let $\frac{a}{c}$ and $\frac{b}{d}$ be two neighbors in the Farey sequence F_n. Then the square matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ satisfies $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) = \pm 1$.

Lemma 12. *Let $\frac{a}{c}$ and $\frac{b}{d}$ be two neighbors of the Farey sequence F_n. Then*

$$\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) = -1 \quad \Longleftrightarrow \quad \frac{a}{c} < \frac{b}{d}.$$

Remark 9. Our applications of the Farey sequences deal mostly with the case $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) = -1$. However, we prefer matrices in $\Gamma(1)$. For this we replace $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ by $\left(\begin{smallmatrix} -a & b \\ -c & d \end{smallmatrix} \right)$ which obviously does not change $\frac{a}{c}$ and $\frac{b}{d}$.

Lemma 13. *For $\frac{a}{c}$ and $\frac{b}{d}$ with $a, b, c, d \in \mathbb{Z}$, $c, d \geq 0$ and $ad - bc = \pm 1$ define $n := \max\left\{ \operatorname{lev}\left(\frac{a}{c}\right), \operatorname{lev}\left(\frac{b}{d}\right) \right\}$. Then $\frac{a}{c}$ and $\frac{b}{d}$ are neighbors in F_n.*

B Induced representations

Let G be a group and H be a subgroup of G of finite index $\mu = [G : H]$. For each representation $\chi : H \to \mathrm{End}(V)$ we consider the induced representation $\chi_H : G \to \mathrm{End}(V_G)$, where

$$V_G := \{f : G \to V;\ f(hg) = \chi(h) f(g) \quad \text{for all } g \in G, h \in H\}$$

and

$$\big(\chi_H(g) f\big)(g') = f(g'g) \qquad \text{for all } g, g' \in G.$$

For $V = \mathbb{C}$ and χ the trivial representation we call the induced representation χ_H the *right regular representation*. In fact, in this case V_G is the space of left H-invariant functions on G or, what is the same, functions on $H\backslash G$, and the action is by right translation in the argument. One can identify V_G with V^μ using a set $\{\alpha_1, \ldots, \alpha_\mu\}$ of representatives for $H\backslash G$, i.e., $H\backslash G = \{H\alpha_1, \ldots, H\alpha_\mu\}$. Then

$$V_G \to V^\mu \quad \text{with} \quad f \mapsto \big(f(\alpha_1), \ldots, f(\alpha_\mu)\big)$$

is a linear isomorphism which transports χ_H to the linear G-action on V^μ given by

$$g \cdot (v_1, \ldots, v_\mu) = \big(\chi(\alpha_1 g \alpha_{k_1}^{-1}) v_{k_1}, \ldots, \chi(\alpha_\mu g \alpha_{k_\mu}^{-1}) v_{k_\mu}\big)$$

where $k_j \in \{1, \ldots, \mu\}$ is the unique index such that $H\alpha_j g = H\alpha_{k_j}$. To see this, one simply calculates

$$\big(\chi_H(g) f\big)(\alpha_j) = f(\alpha_j g) = f(\alpha_j g \alpha_{k_j}^{-1} \alpha_{k_j}) = \chi(\alpha_j g \alpha_{k_j}^{-1})\big(f(\alpha_{k_j})\big).$$

In the case of the right regular representation the identification $V_G \cong \mathbb{C}^\mu$ gives a matrix realization

$$\rho(g) = \big(\delta_H(\alpha_i g \alpha_j^{-1})\big)_{1 \le i,j \le \mu}$$

where $\delta_H(g) = 1$ if $g \in H$ and $\delta_H(g) = 0$ otherwise. In particular, the matrix $\rho(g)$ is a permutation matrix.

We take $G = \Gamma(1)$, $H = \Gamma_0(n)$ and $\alpha_1, \ldots, \alpha_\mu \in \Gamma(1)$ as representatives of the $\Gamma_0(n)$ orbits in $\Gamma(1)$. The *matrix representation* $\rho : \Gamma(1) \to \mathbb{C}^{\mu \times \mu}$ is

$$\rho(g) := \Big(\delta_{\Gamma_0(n)}(\alpha_i\, g\, \alpha_j^{-1})\Big)_{1 \le i,j \le \mu} \qquad \text{for all } g \in \Gamma(1). \tag{21}$$

We easily check that ρ satisfies $\rho(g')\,\rho(g) = \rho(g'g)$ for all $g, g' \in \Gamma(1)$.

C The function $R_\zeta(z)$ and the 1-form $\eta(\cdot,\cdot)$

We define the function $R_\zeta(z) : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ as

$$R_\zeta(z) = \frac{y}{(x - \zeta)^2 + y^2}. \tag{22}$$

Note that $R_\zeta(z)$ is a nonstandard notation for a function of the two variables $\zeta \in \mathbb{C}$ and $z \in \mathbb{H}$.

For $\zeta \in \mathbb{R}$ we have $R_\zeta(z) = \frac{\mathrm{Im}(z)}{|z-\zeta|^2}$.

It is shown in [8], [12] and also in [14] that $R_\zeta(z)$ satisfies the transformation formula

$$\frac{|\det g|}{|c\zeta + d|^2} R_{g\zeta}(gz) = R_\zeta(z) \tag{23}$$

for all $g \in \mathrm{M}_1(2, \mathbb{Z})$ and real ζ. Moreover, it was also shown there that $R_\zeta^s(z)$ is an eigenfunction of the hyperbolic Laplace operator with eigenvalue $s(1-s)$ for all complex s.

As in [8], we define the 1-form $\eta(u,v)$ for two smooth functions u, v on $\mathbb{H}$:

$$\eta(u,v) := \big(v\partial_y u - u\partial_y v\big)dx + \big(u\partial_x v - v\partial_x u\big)dy. \tag{24}$$

The following lemma is shown in [8]:

Lemma 14. *If u and v are eigenfunctions of Δ with the same eigenvalue, then the 1-form $\eta(u,v)$ is closed. If $z \mapsto g(z)$ is any holomorphic change of variables, then the 1-form satisfies $\eta(u \circ g, v \circ g) = \eta(u,v) \circ g$.*

D Example: $\tilde{H}_{1,2}$

For suitable complex s and $u \in S(1,s)$ let Pu be the period function of u given in §4. Consider the 2^{nd} Hecke operator H_2 on $S(1,s)$ given by

$$u \mapsto H_2 u = u\big|_0 T(2).$$

In this example we compute the matrix representation $\tilde{H}(2)$ of the 2^{nd} Hecke operator $\tilde{H}_{1,2}$ on $\mathrm{FE}(1,s)$.

Recall that the set X_2, see (1), and the element $T(2)$, see (7), are given by

$$X_2 = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\} \quad \text{and} \quad T(2) = \sum_{A \in X_2} A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right). \tag{25}$$

Given the Farey sequences F_0, F_1 and F_2 we compute $\mathrm{LNS}(A0)$ and the $M(A0)$ for all $A \in X_2$, see §7:

$$\mathrm{LNS}(0) = \left(\frac{-1}{0}, \frac{0}{1}\right), \quad \mathrm{LNS}\big(\tfrac{1}{2}\big) = \left(\frac{-1}{0}, \frac{0}{1}, \frac{1}{2}\right), \tag{26}$$

$$M(0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \quad \text{and} \quad M\big(\tfrac{1}{2}\big) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right). \tag{27}$$

For $P\left(u\big|_0 A\right)(\zeta)$, $\zeta > 0$, with $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right) \in X_2$ we find

$$P\left(u\big|_0 A\right)(\zeta) = \int_{L_{0,\infty}} \eta(u\big|_0 A, R_\zeta^s)(z) = \int_{L_{A0,A\infty}} \eta(u, R_\zeta^s\big|_0 A^{-1})(z)$$

where the slash operator acts on the z variable of $R_\zeta(z)$. The transformation property (23) gives

$$P\left(u\big|_0 A\right)(\zeta) = 2^s \int_{L_{A0,A\infty}} \eta\big(u, R^s_{A\zeta}\big)(z) = 2^s \int_{L_{A0,A\infty}} \eta\left(u, R^s_{\frac{\zeta+1}{2}}\right)(z).$$

We would like to write $\int_{L_{A0,A\infty}} \eta(u, R^s_{A\zeta})(z) = \sum_l Pu(m_l A\zeta)$. This can be done using $\mathrm{LNS}\big(\frac{1}{2}\big)$ and the related formal sum $M\big(\frac{1}{2}\big)$. We have

$$L_{A0,A\infty} = L_{\frac{1}{2},\infty} = L_{0,\infty} \cup L_{\frac{1}{2},0} = L_{m_1^{-1}0,m_1^{-1}\infty} \cup L_{m_2^{-1}0,m_2^{-1}\infty}$$

with $m_1 = I \in \Gamma(1)$ and $m_2 = \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma(1)$. Hence, $P\left(u\big|_0 A\right)(\zeta)$ can be written as

$$\begin{aligned} P\left(u\big|_0 A\right)(\zeta) &= 2^s \int_{L_{A0,A\infty}} \eta(u, R^s_{A\zeta})(z) \\ &= 2^s \int_{L_{m_1^{-1}A0,m_1^{-1}A\infty}} \eta(u, R^s_{m_1 A\zeta})(z) \\ &\quad +2^s(\zeta+1)^{-2s} \int_{L_{m_2^{-1}A0,m_2^{-1}A\infty}} \eta(u, R^s_{m_2 A\zeta})(z) \\ &= (Pu)\big|_s (m_1 A + m_2 A)(\zeta) = (Pu)\big|_s \big(M(A0)A\big) \end{aligned}$$

where $M\big(\frac{1}{2}\big) = m_1 + m_2$ with $m_1 = I$ and $m_2 = \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ as in (27).

We perform the analogous computation for the other matrices in X_2 and find that

$$P\big(u\big|_0 T(2)\big) = \big(Pu\big)\big|_s \tilde{H}(2) \tag{28}$$

where the formal matrix sum $\tilde{H}(2)$ has the form

$$\begin{aligned} \tilde{H}(2) &= \sum_{A\in X_2} M(A0)A = M(0)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) + M\big(\tfrac{1}{2}\big)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right) + M(0)\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ &= I\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) + \left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right]\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right) + I\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right). \end{aligned} \tag{29}$$

E Example: $\tilde{H}_{2,2}$

For suitable complex s and $u \in S(2,s)$ let $P\Pi u$ be the period function of u given in §4. Consider the 2nd Hecke operator H_2 on $S(2,s)$ given by

$$u \mapsto H_2 u = u\big|_0 U(2).$$

In this example, we compute the matrix representation $\tilde{H}(2)$ of the 2nd Hecke operator $\tilde{H}_{2,2}$ on $\mathrm{FE}(2,s)$.

According to (7) we have

$$U(2) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right) = A_1 + A_2 \in \mathcal{R}_2$$

using $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)$ and $A_3 = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

We already computed $M(A_1 0) = M(A_3 0) = M(0)$ and $M(A_2 0) = M\left(\frac{1}{2}\right)$ in (26).

Put

$$\alpha_1 = I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \alpha_2 = S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \quad \text{and} \quad \alpha_3 = ST = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right) \tag{30}$$

as representatives of the right cosets in $\Gamma_0(2)\backslash\Gamma(1)$. The values of the functions $\phi_A = \phi_{A,2}$ and σ_{α_i} defined in §6 are given in Table 1.

The matrix representation ρ, see (21), has the values

$$\rho(I) = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} \qquad \text{and} \quad \rho\left(\left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)^{-1}\right) = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}. \tag{31}$$

Computing $\tilde{H}_{2,2}$ via (20), we find for the first component

$$\begin{aligned}
\left[\tilde{H}_{2,2}\big(P\boldsymbol{u}\big)\right]_1 &= \sum_{A\in\{A_1,A_2\}} \sum_{l=1}^{L_{\sigma_{\alpha_1}(A)}} \left[\rho(m_l^{-1})\, P\boldsymbol{u}\right]_{\phi_A(1)} \big|_s \big(m_l \sigma_{\alpha_1}(A)\big) \\
&= \left[\rho(I)\, P\boldsymbol{u}\right]_{\phi_{A_1}(1)} \big|_s \big(I\sigma_{\alpha_1}(A_1)\big) + \left[\rho(I)\, P\boldsymbol{u}\right]_{\phi_{A_2}(1)} \big|_s \big(I\sigma_{\alpha_1}(A_2)\big) \\
&\qquad + \left[\rho\left(\left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)^{-1}\right)\, P\boldsymbol{u}\right]_{\phi_{A_2}(1)} \big|_s \big(\left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)\sigma_{\alpha_1}(A_2)\big) \\
&= [P\boldsymbol{u}]_1 \big|_s I A_1 + [P\boldsymbol{u}]_1 \big|_s I A_2 + [P\boldsymbol{u}]_2 \big|_s \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right) A_2 \\
&= [P\boldsymbol{u}]_1 \big|_s \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) + [P\boldsymbol{u}]_1 \big|_s \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right) + [P\boldsymbol{u}]_2 \big|_s \left(\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right).
\end{aligned} \tag{32}$$

Similarly, we find for the second and third component

$$\left[\tilde{H}_{2,2}\big(P\boldsymbol{u}\big)\right]_2 = [P\boldsymbol{u}]_2 \big|_s \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) + [P\boldsymbol{u}]_1 \big|_s \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right) + [P\boldsymbol{u}]_2 \big|_s \left(\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right) \text{ and} \tag{33}$$

$$\left[\tilde{H}_{2,2}\big(P\boldsymbol{u}\big)\right]_3 = [P\boldsymbol{u}]_2 \big|_s \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) + [P\boldsymbol{u}]_1 \big|_s \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right). \tag{34}$$

Remark 10.
- Note that the third component of $\tilde{H}_{2,2}\big(P\boldsymbol{u}\big)$ has only two terms compared to the three terms of the other components. This is related to the fact that A_2 does not occur in the image of $\sigma_{\alpha_3}(\{A_1, A_2\})$.
- The operator $\tilde{H}_{2,2}$ does not use the third component of the vector valued period form since the index 3 does not appear in the image of ϕ_A in Table 1.
- M. Fraczek, [4], wrote a C-program for the computation of $\tilde{H}_{n,m}$.

j	$\phi_{A_1}(j)$	$\sigma_{\alpha_j}(A_1)$	$\phi_{A_2}(j)$	$\sigma_{\alpha_j}(A_2)$	$\phi_{A_3}(j)$	$\sigma_{\alpha_j}(A_3)$
1	1	A_1	1	A_2	1	A_3
2	2	A_3	1	A_2	2	A_1
3	2	A_3	1	A_1	2	A_2

Table 1. The values of the functions $\phi_A = \phi_{A,2}$ and $\sigma_{\alpha_j}(A)$ for $m = n = 2$ and $A \in X_2$. The representatives α_j of the cosets of $\Gamma_0(2)\backslash\Gamma(1)$ are given in (30).

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