An application of Jacquet-Langlands correspondence to transfer operators for geodesic flows on Riemann surfaces

Arash Momeni[1] and Alexei Venkov[2]
[1] Department of Statistical Physics and Nonlinear Dynamics
Institute of Theoretical Physics,
Clausthal University of Technology, 38678 Clausthal-Zellerfeld, Germany. E-Mail: arash.momeni@tu-clausthal.de
[2]Institute for Mathematics, University of Aarhus, 8000 Aarhus C, Denmark. E-Mail: venkov@imf.au.dk

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Abstract

In the paper as a new application of the Jacquet-Langlands correspondence we connect the transfer operators for different cofinite Fuchsian groups by comparing the corresponding Selberg zeta functions.

1 Transfer operator for cofinite Fuchsian groups and Selberg's zeta function

In this section we give a short summary of the generalization of Mayer's theory [1] by following Morita [2]. In Mayer's theory a transfer operator $\mathcal{L}_s$ is introduced as a special case of Ruelle's operator for a dynamical system, for instance for the group $SL(2,\mathbb{Z})$ this is the twice iterated Gauss map $T = T_G^2$ acting on the unit interval and the weight function $-\beta log|T'(z)|$. Then as a result of the one-to-one correspondence between the closed geodesics on the modular surface $M = H\backslash SL(2,\mathbb{Z})$ and the primitive periodic orbits of T the Selberg zeta function can be written as a Fredholm determinant of the transfer operator [3].

Morita in [2] takes instead of $PSL(2,\mathbb{R})$ and the Poincare upper half plane H the group $PSU(1,1)$ and as homogeneous space the unit disc $\mathbb{D}$ which are isomorphic to $PSL(2,\mathbb{R})$ and H respectively. Let Γ be a cofinite Fuchsian group. The canonical fundamental domain of Γ is a polygon with a finite even number of sides s_i, each of which extends to a circular arc $C(s_i)$ perpendicular to the unit circle S^1, the boundary of $\mathbb{D}$. To every side s_i a generator $g(s_i)$ is assigned which identifies the sides mutually. Now the action of the generators on the boundary points S^1 defines a Markov map [4]

$$T_\Gamma x = g_i x \quad x \in S^1 \tag{1}$$

where one chooses for each x the corresponding g_i according to the location of x between the footpoints of $C(s_i)$'s on S^1 as explained in [2].

By letting the group Γ act on sides s_i and taking from them the arcs passing through vertexes of the fundamental polygon which cut the boundary, S^1 is partitioned into a set of intervals $\mathcal{P}' = \{I_i\}_i$. Like for the modular group and the Gauss map with the corresponding partition [5], the map T_Γ with respect to the partition $\mathcal{P}'$ satisfies the properties of orbit equivalence, piecewise monotonicity, the Markov property and transitivity. As in [2] by using the intervals in $\mathcal{P}'$ it can be constructed a finite partition $\mathcal{R} = \{A(1), A(2), ..., A(q)\}$ and its refinement $\mathcal{P} = \{J\}$ finite or infinite, such that the union of each of them differs from the other up to a set of zero Lebesgue measure. Furthermore to each $J \in \mathcal{P}$ there is a homomorphism T_J from J onto an interval $A \in \mathcal{R}$. Finally a map T is defined approximately everywhere on $X = \bigsqcup_{A\in\mathcal{R}} A$ such that $T|_{intJ} = T_J|_{intJ}$. We call the set of partitions $\mathcal{R}$ and $\mathcal{P}$ together with the map T a Markov system $\mathcal{T}_\Gamma = (\mathcal{R}, \mathcal{P}, T)$.

Now we set

$$G_J^s(w) = |T_J'(w)|^{-s} \quad w \in S^1, \quad s \in \mathbb{C}, \tag{2}$$

then the transfer operator is defined as

$$(L_{\mathcal{T}_\Gamma}(s)f)_i(z) = \sum_{J\in\mathcal{P}:\tau(J)=i} G_J^s(T_J^{-1}z) f_{\iota(J)}(T_J^{-1}z) \tag{3}$$

on direct sum of $q(= card\ \mathcal{R})$ Banach spaces chosen in such a way that (3) defines a nuclear operator and for $J \in \mathcal{P}$, $\iota(J)$ denotes the index with $int\ J \subset int\ A(\iota(J))$ and $\tau(J)$ denotes the index with $T_J J = A(\tau(J))$. See the Convention 2.6 of [2]. For a general cofinite Fuchsian group Γ, like the case of $SL(2,\mathbb{Z})$ [1] regarding the correspondence between the closed geodesics on the fundamental surface of the group and the periodic orbits of T we achieve a determinant expression for the corresponding Selberg zeta function. We summarize the final result as the following theorem of Morita [2].

Theorem 1. *Let Γ be a cofinite Fuchsian group. We denote $HC(\Gamma)$ to be the set of all primitive hyperbolic conjugacy classes in Γ. Then the Selberg zeta function $Z_\Gamma(s)$ for the group Γ has a determinant representation given by*

$$Z_\Gamma(s)\Xi(s)^2 = Det(I - L_{\mathcal{T}_\Gamma}(s)) \tag{4}$$

where $\Xi(s) = \prod_{k=0}^{\infty}\prod_{c\in HC_1(\Gamma)}(1 - exp(-(s+k)l(c)))$ for a certain finite subset $HC_1(\Gamma)$ of $HC(\Gamma)$. The function $\Xi(s)$ is meromorphic in $\mathbb{C}$ and it is analytic in $\{s \in \mathbb{C} : Re(s) > 0\}$ without zeros.

We recall some notations and definitions we used in Theorem 1. According to Selberg for $Res > 1$ the Selberg zeta function is given by absolutely convergent infinite product similar to Euler products of zeta function from number theory

$$Z_\Gamma(s) = \prod_{k=0}^{\infty} \prod_{\{P\}_\Gamma} (1 - \mathcal{N}(P)^{-k-s}) \tag{5}$$

where P runs through all primitive hyperbolic conjugacy classes in $HC(\Gamma)$, where a norm $\mathcal{N}(P) > 1$. P primitive plays the role of prime number or prime

ideal. Using his trace formula Selberg proved that
(1) $Z_\Gamma(s)$ has analytic(meromorphic) continuation to the whole $s \in \mathbb{C}$
(2) $Z_\Gamma(s)$ satisfies the functional equation

$$Z_\Gamma(1-s) = \Psi(s) Z_\Gamma(s) \tag{6}$$

with known function Ψ
(3) the nontrivial zeros of $Z_\Gamma(s)$ are related to eigenvalues and resonances of automorphic Laplacian $A(\Gamma)$ for the group Γ [14], see also [6].

There is another equivalent definition of the Selberg zeta function related to the dynamical system of geodesic flows on Riemann surfaces with constant negative curvature. This definition is important for theorem 1 above:

$$Z_\Gamma(s) = \prod_{k=0}^{\infty} \prod_{c \in HC(\Gamma)} (1 - exp(-(s+k)l(c))) \tag{7}$$

In formula above we use c instead of P like in [2]. Recall that an element c of $PSU(1,1)$ is hyperbolic if it has exactly two fixed points on the boundary S^1. c is a primitive element of Γ if it is not a positive power of other elements in Γ. Let $w_1,\ w_2 \in S^1$ are fixed points of c. This defines the geodesic line connecting $w_1,\ w_2$. The projection of this geodesic to the quotient space $\mathbb{D}\backslash\Gamma$ is a closed geodesic and $l(c)$ is the hyperbolic length of the corresponding prime closed geodesic. We have

$$\mathcal{N}(c) = e^{l(c)} \tag{8}$$

2 Jacquet-Langlands correspondence

In [10] an explicit integral operator lift with Siegel theta function as the kernel, between Maass forms of unit group of an indefinite quaternion division algebra and congruence subgroups of the modular group is constructed. This is indeed a special case of Jacquet-Langlands correspondence which Hejhal reproved by using classical arguments [10]. To introduce this correspondence we follow [10], [8] and [7]. We start by recalling the unit group of quaternion algebra.

2.1 Unit group of quaternion algebra

In this part we follow [11]. We call a ring B with unity an algebra of dimension n over a field F, if the following three conditions are satisfied:

(1) $F \subset B$ and the unity of F coincides with the unity of B;
(2) all elements of F commute with all elements of B;
(3) B is a vector space over F of dimension n.

Let B be an algebra over F. The center of B is defined as the set of all commuting elements of the algebra and is denoted by $Z(B)$. We call B a central algebra when $Z(B) = F$. The algebra B is called simple if it is simple as a ring, namely, if B has no two-sided ideals except for $\{0\}$ and B itself. We call B a division algebra if every nonzero element of B is invertable. Now we can give the definition of the quaternion algebra.
A central simple algebra B of dimension 4 over a field F is called a quaternion algebra over F. Furthermore, if B is a division algebra, we call B a division

quaternion algebra. Let B be a quaternion algebra over a field F. As a result of Wedderburn's theorem there are only two possibilities [12]

(1) B is a division quaternion algebra

(2) B splits over F that is, B is isomorphic to $M_2(F)$

Using the following results [15] one can define a norm for B. 1)If F is algebraically closed, then $M_2(F)$ is the unique quaternion algebra over F up to isomorphism. 2) Let K be any extension over F. Then $B \otimes_F K$ is a quaternion algebra over K. We say that B is ramified or splits over K if $B \otimes_F K$ is a division quaternion algebra or is isomorphic to $M_2(K)$, respectively. Now for the quaternion algebra B over F let $\overline{F}$ be the algebraic closure of F. According to the results above $B \otimes_F \overline{F}$ is a quaternion algebra over the algebraically closed field $\overline{F}$ and therefore $B \otimes_F \overline{F}$ is isomorphic to $M_2(\overline{F})$. Now we can define (reduced) norm and (reduced) trace for elements of B by

$$N_B(\beta) = det(\beta) \qquad tr_B(\beta) = tr(\beta) \tag{9}$$

where $det(\beta)$ and $tr(\beta)$ are determinant and trace of β as an element of $M_2(\overline{F})$. In [15] it is proved that both $N_B(\beta)$ and $tr_B(\beta)$ belong to F.

For an algebra B not necessarily of quaternion type over the field of rational numbers $F = \mathbb{Q}$ or its p-adic extensions $F = \mathbb{Q}_p$ the concept of an order is defined as a subset $\mathcal{O}$ of B if the following two conditions are satisfied:
(1) $\mathcal{O}$ is a subring containing the unity of B
(2) $\mathcal{O}$ is finitely generated over $\mathbb{Z}$ (or $\mathbb{Z}_p$) and contains a basis of B over F.
An order of B is called maximal if it is maximal with respect to inclusion.

From now we restrict B to be a quaternion algebra over the field of rationals $\mathbb{Q}$. The algebra B is characterized up to an isomorphism by a positive integer $d(B)$ called (reduced) discriminant which is defined to be the product of primes p where B is ramified over $\mathbb{Q}_p$ that is $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is division quaternion algebra. Note that some authors define discriminant as the square of what we defined as $d(B)$[11],[10]. The discriminant is also defined for orders of the algebra B. Indeed conjugate orders are characterized by the discriminant. The (reduced) discriminant $d(\mathcal{O})$ of an order $\mathcal{O}$ of the algebra B is defined by

$$d(\mathcal{O}) = \sqrt{|det[tr_B(\xi_j \xi_k)]|} \tag{10}$$

where ξ_1, ξ_2, ξ_3, ξ_4 is any $\mathbb{Z}$-basis of $\mathcal{O}$. All orders with the same discriminant are conjugate. For a maximal order $\mathcal{O}_{max}$ the discriminant is equal to the discriminant of the algebra B that is $d(B) = d(\mathcal{O}_{max})$ which means all maximal orders of an algebra B are conjugate.
We call B indefinite or definite according as $B \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $M_2(\mathbb{R})$ or is a division quaternion algebra. We also recall that for each square free number $d \in \mathbb{Z}^+$, there is exactly one quaternion algebra B over $\mathbb{Q}$ with $d(B) = d$, up to isomorphisms. Fortheremore, $d(B) > 1$ if and only if B is a division algebra and B being indefinite means that $d(B)$ has even number of prime factors [17], [19]. Let B be an indefinite quaternion algebra over $\mathbb{Q}$. We fix an isomorphism of $B \otimes_{\mathbb{Q}} \mathbb{R}$ onto $M_2(\mathbb{R})$ and consider B as an subalgebra of $M_2(\mathbb{R})$ through this isomorphism. Then the norm $N_B(\beta)$ of an element β of B is nothing but the determinant of β as a matrix, by definition. Let $\mathcal{O}$ be an order of B. We put

$$\mathcal{O}^1 = \{\beta \in \ \mathcal{O}| \ N_B(\beta) = 1\} \subset SL(2, \mathbb{R}) \tag{11}$$

and call it the unit group of norm 1 of $\mathcal{O}$. Then we have the following well known result [11]

Theorem 2. *Let B be an indefinite quaternion algebra over $\mathbb{Q}$, and $\mathcal{O}$ be an order of B. Then $\mathcal{O}^1$ is a Fuchsian group of the first kind. Moreover, if B is a division quaternion algebra, then $\mathcal{O}^1 \backslash H$ is compact.*

Because of this theorem from now on we restrict ourself to an order $\mathcal{O}$ of an indefinite quaternion algebra over the field of rational numbers $\mathbb{Q}$ and its unit group $\mathcal{O}^1$. By definition we can fix an emmbeding

$$\sigma : \mathcal{O}^1 \longrightarrow M_2(\mathbb{R}) \tag{12}$$

Furthermore according to definition we have

$$\mathcal{O} \cong e_1\mathbb{Z} \oplus e_2\mathbb{Z} \oplus e_3\mathbb{Z} \oplus e_4\mathbb{Z} \cong \mathbb{Z}^4 \tag{13}$$

where e_1, e_2, e_3 and e_4 form a basis for $\mathcal{O}$ over $\mathbb{Z}$.

For this representation of the order $\mathcal{O}$ as a four dimensional vector space over $\mathbb{Z}$ the norm on $\mathcal{O}$ is defined as a quadratic form which realized as a 4×4 symmetric matrix S' which with respect to the $\mathbb{Z}$-basis $\{e_i\}_{i=1}^4$ of the order $\mathcal{O}$ is given by

$$[S']_{i,j} = tr(e_i e_j) \tag{14}$$

that is for an element $q \in \mathcal{O}$ with the vector representation $k_q = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix}$, the norm is given by

$$n(q) = \frac{1}{2} k_q^t S' k_q \tag{15}$$

2.2 Siegel theta function

In this part we introduce the Siegel theta function by following [10], [8], and [7]. For a symmetric matrix $S \in GL_n(\mathbb{R})$ the majorant P is defined to be a positive definite matrix such that $PS^{-1}P = S$.
We fix S to be

$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{16}$$

It is easy to see that the identity matrix is one of the majorants of S.
For $L_1,\ L_2 \in SL(2,\mathbb{R})$ and $m_1,\ m_2 \in M_2(\mathbb{R})$ such that

$$m_1 = L_1 m_2 L_2^{-1} \tag{17}$$

we define $A(L_1, L_2) \in M_4(\mathbb{R})$ by

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \delta_1 \end{pmatrix} = A(L_1, L_2) \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \\ \delta_2 \end{pmatrix} \tag{18}$$

where

$$m_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \; m_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \tag{19}$$

As in [10] one can see that $A(L_1, L_2)^t A(L_1, L_2)$ is a majorant of S. For w and z belonging to H where $z = x + iy$ we define

$$M_z := \begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \text{ and } P_{zw} := A(M_z^{-1}, M_w^{-1})^t A(M_z^{-1}, M_w^{-1}) \tag{20}$$

P_{zw} is a majorant of S. Let $\mathcal{O}$ be an order in an indefinite quaternion algebra over $\mathbb{Q}$. Then let S' be the symmetric matrix of the norm form on $\mathcal{O}$ with respect to a fixed $\mathbb{Z}$-basis of the order. For $q \in \mathcal{O}$ let $k_q \in \mathbb{Z}^4$ be the corresponding element in $\mathbb{Z}^4$ in this basis (see formula (13) from above subsection).

We fix an embedding $\sigma : \mathcal{O} \longrightarrow M_2(\mathbb{R})$. Since σ is linear, we have a unique $B \in GL_4(\mathbb{R})$ which for every q satisfies

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = Bk_q, \text{whenever } \sigma_q := \sigma(q) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{21}$$

Since

$$k_q^t S' k_q = 2n(q) = 2 \; det(\sigma_q) = 2(\alpha\delta - \beta\gamma) = (Bk_q)^t S(Bk_q) \tag{22}$$

we conclude that $S' = B^t SB$. For this fixed embedding of $\mathcal{O}$, we define majorants P'_{zw} of S' by $P'_{zw} := B^t P_{zw} B$ where B is the same matrix as in (21). Note that here we used the fact that P_{zw} is a majorant of S and the fact that if P is a majorant of S then $B^t PB$ is a majorant of $B^t SB$. Now fix $z_0 \in H$ and let $\tau = u + iv,\; z = x + iy \in H$. With $R := uS' + ivP'_{zz_0}$, we define the Siegel theta function $\theta(z; \tau)$ by

$$\theta(z; \tau) := Im(\tau) \sum_{k \in \mathbb{Z}^4} e^{\pi i k^t Rk} = Im(\tau) \sum_{q \in \mathcal{O}} e^{\pi i k_q^t R k_q} \tag{23}$$

The Siegel theta function has the following transformation property which is crucial for the application in the next subsection. We give it as a theorem whose proof one can find in [8]

Theorem 3. *Let $\mathcal{O}$ be an order in an indefinite quaternion algebra over $\mathbb{Q}$, with (reduced) discriminant d. Then*
(1) $\theta(\sigma_q z; \tau) = \theta(z; \tau),\; \forall q \in \mathcal{O}^1$
(2) $\theta(z; g\tau) = \theta(z; \tau),\; \forall g \in \Gamma_0(d)$

2.3 Theta-lifts

In this subsection we introduce two integral transformations providing us a lift between Maass forms for congruence subgroups and nonconstant square integrable automorphic eigenfunctions of hyperbolic Laplacian for the unit group of quaternions. As before, let $\mathcal{O}$ be an order with discriminant $d(\mathcal{O})$ in an indefinite quaternion division algebra over $\mathbb{Q}$ and $\mathcal{O}^1$ denotes the corresponding unit group. we also put $X_d := \Gamma_0(d)\backslash H$ and $X_{\mathcal{O}} := \mathcal{O}^1\backslash H$. In [8] for a fixed reference point z_0 appearing in the definition of the Siegel theta function we have the following result which we give as a theorem.

Theorem 4. *For a fixed reference point $z_0 \in H$ in the Siegel theta function the following maps*

$$\Theta : L^2_0(X_{\mathcal{O}}) \longrightarrow \mathcal{C}_{d(\mathcal{O})} \text{ and } \widetilde{\Theta} : \mathcal{C}_{d(\mathcal{O})} \longrightarrow L^2_0(X_{\mathcal{O}}) \tag{24}$$

define bounded linear operators preserving Laplace (also Hecke) eigenvalues given by

$$\Theta\varphi(\tau) := \int_{\mathcal{F}_{\mathcal{O}^1}} \theta(z;\tau)\varphi(z)d\mu(z) \tag{25}$$

and

$$\widetilde{\Theta}g(z) := \int_{\mathcal{F}_d} \overline{\theta(z;\tau)}g(\tau)d\mu(\tau) \tag{26}$$

where $L^2_0(X_{\mathcal{O}})$ denotes the space of non-constant square integrable automorphic functions on $X_{\mathcal{O}} := \mathcal{O}^1\backslash H$ and $\mathcal{C}_{d(\mathcal{O})}$ denotes the space of cusp forms for congruence subgroup $\Gamma_0(d(\mathcal{O}))$ whose level is equal to the discriminant of the order $\mathcal{O}$. $\mathcal{F}_{\mathcal{O}^1}$ and $\mathcal{F}_d$ are the corresponding fundamental domains and $\theta(z;\tau)$ is the Siegel theta function.

In [8] it is proved that by choosing a suitable reference point z_0, for a basis $\{\varphi_k \mid k \in \mathbb{N}_0\}$ for $L^2(X_{\mathcal{O}})$ which consists of all eigenfunctions of the hyperbolic Laplacian on $X_{\mathcal{O}}$, $\Theta(\varphi_k)$ is not identically zero, for all k. Thus we have the following theorem whose proof one can see in [8]

Theorem 5. *All eigenvalues of the hyperbolic Laplacian on $L^2(X_{\mathcal{O}})$ also occur as eigenvalues of the hyperbolic Laplacian on $L^2(X_d)$ where $d = d(\mathcal{O})$.*

In [7] for a maximal order $\mathcal{O}_{max}$, it is shown that the right hand side of Selberg trace formula for $\mathcal{O}^1_{max}$ is equal to the right hand side of Selberg trace formula for new forms of a congruence subgroup (see the formula 41) of the level equal to the discriminant of $\mathcal{O}_{max}$. Therefore one gets the equality of the left hand sides of these trace formulas which is the the following identity

$$\sum_{\varphi_k \in L^2(X_{\mathcal{O}_{max}})} h(r_k) = \sum_{g_k \in \mathbb{C}\oplus\mathcal{C}^{new}_d} h(r_k) \tag{27}$$

where $\mathcal{C}^{new}_d$ denotes the space of new Maass cusp forms for the congruence subgroup of level $d = d(\mathcal{O}_{max})$. This together with the last theorem leads to

Theorem 6. *For a maximal order $\mathcal{O}_{max}$, the eigenvalues of the hyperbolic Laplacian, including multiplicities, on $X_{\mathcal{O}_{max}}$ coincides with the Laplace spectrum on the new Maass forms for the congruence subgroup $\Gamma_0(d)$, where d is the discriminant of the maximal order $\mathcal{O}_{max}$.*

3 Selberg trace formula

In this section we introduce shortly the Selberg trace formula for the general case of a cofinite Fuchsian group and then for the special cases of congruence subgroups we define the Selberg trace formula for new forms.

The trace formula is a general identity connecting geometrical and spectral terms

$$\sum \{\text{spectral terms}\} = \sum \{\text{geometric terms}\} \tag{28}$$

In the Selberg trace formula the spectral terms come from discrete and continuous spectrum of the automorphic hyperbolic Laplacian $A(\Gamma)$ for a cofinite Fuchsian group Γ and the geometrical terms are integral operators depending on the conjugacy classes of Γ. As in [6] one can calculate the integrals explicitly and achieve the final form of the Selberg trace formula. We give it in the form of a theorem [13]

Theorem 7. *Let $h(r^2+\frac{1}{4})$ be a function of a complex variable r which satisfies the following assertions:*

(1) As a function of r, $h(r^2+\frac{1}{4})$ is holomorphic in the strip $\left\{r \in \mathbb{C} : |Im(r)| < \frac{1}{2}+\varepsilon\right\}$ for some $\varepsilon > 0$.

(2) In that strip, $h(r^2+\frac{1}{4}) = O((1+|r^2|)^{-1-\varepsilon})$ and all of the series and integrals appearing below converge absolutely.

Then the following identity hold

$$\sum_{k=0}^{\infty} h(\lambda_k) + C = I + H + E + P \tag{29}$$

where $\{\lambda_n \mid 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$ are the discrete eigenvalues of $A(\Gamma)$. C corresponds to the continuous part of the spectrum given by

$$C = -\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\varphi'}{\varphi}(\frac{1}{2}+ir)h(r^2+\frac{1}{4})dr - \frac{K_0}{4}h(\frac{1}{4}) \tag{30}$$

where φ denotes determinant of the scattering matrix $\Phi(s)$ and $K_0 = tr(\Phi(\frac{1}{2}))$. On the right hand side of (29) we have I which corresponds to the identity element of the group and given by

$$I = \frac{vol(\Gamma\backslash H)}{4\pi}\int_{-\infty}^{\infty} r\tanh(\pi r)\ h(r^2+\frac{1}{4})dr \tag{31}$$

The term H denotes the contribution of hyperbolic conjugacy classes and is given by

$$\sum_{\{P\}_\Gamma}\sum_{m=1}^{\infty} \frac{logN(P)}{N(P)^{\frac{m}{2}} - N(P)^{-\frac{m}{2}}} g(m\ logN(P)) \tag{32}$$

where $\{P\}_\Gamma$ denotes the primitive hyperbolic conjugacy classes and the function g appears in the process of the Selberg transformation:

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iru}h(r^2+\frac{1}{4})dr \tag{33}$$

The next term E refers to the contribution of elliptic elements given by a summation over primitive elliptic conjugacy classes $\{R\}_\Gamma$ with order ν

$$E = \frac{1}{2}\sum_{\{R\}_\Gamma}\sum_{m=1}^{\nu-1} \frac{1}{\nu\sin\pi m/\nu}\int_{-\infty}^{\infty} \frac{exp(-2\pi rm/\nu)}{1+exp(-2\pi r)}h(r^2+\frac{1}{4})dr \tag{34}$$

Finally the last term comes from the parabolic conjugacy classes given by

$$P = -K\ g(0)\ log2 + \frac{K}{4}h(\frac{1}{4}) - \frac{K}{2\pi}\int_{-\infty}^{\infty} \psi(1+ir)h(r^2+\frac{1}{4})dr \tag{35}$$

where K is the number of cusps and ψ is the di-gamma function.

In the case of cocompact groups including the unit group of quaternion algebras there is no continuous spectrum and no parabolic element, thus the trace formula (29) reduces to

$$\sum_{k=0}^{\infty} h(\lambda_k) = I + H + E \tag{36}$$

Now consider a congruence subgroup $\Gamma_0(n) \subset SL(2,\mathbb{Z})$. As in [7] the space $\mathcal{C}_n(\lambda)$ of Maass forms with eigenvalue λ, can be decomposed into two subspaces of new and old forms $\mathcal{C}_n(\lambda) = \mathcal{C}_n^{old}(\lambda) \oplus \mathcal{C}_n^{new}(\lambda)$. The space $\mathcal{C}_n^{old}(\lambda)$ is the linear span of all forms with the same eigenvalue λ coming from all overgroups $\Gamma_0(m) \supset \Gamma_0(n)$ with $m|n$ and $\mathcal{C}_n^{new}(\lambda)$ is defined to be the orthogonal complement of $\mathcal{C}_n^{old}(\lambda)$. Let us denote the dimension of $\mathcal{C}_n(\lambda)$ and $\mathcal{C}_n^{new}(\lambda)$ by $\delta(n,\lambda)$ and $\delta^{new}(n,\lambda)$ respectively. Then the following identity holds [9]

$$\delta^{new}(n,\lambda) = \sum_{m|n} \beta(\frac{n}{m})\delta(m,\lambda) \tag{37}$$

with

$$\beta(a) = \sum_{l|a} \mu(l)\mu(\frac{a}{l}), \tag{38}$$

where $\mu(a)$ is the Moebius function. Identity (37) leads to the following formula [7]

$$\sum_{u_k \in \mathcal{C}_n^{new}} h(\lambda_k) = \sum_{m|n} \beta(\frac{n}{m}) \sum_{u_k \in \mathcal{C}_m} h(\lambda_k) \tag{39}$$

This suggests us to take the sum

$$\sum_{u_k \in \mathcal{C}_n^{new}} h(\lambda_k) + \sum_{m|n} \beta(\frac{n}{m}) h(\lambda_0) = \sum_{m|n} \beta(\frac{n}{m}) \sum_{u_k \in \mathbb{C} \oplus \mathcal{C}_m} h(\lambda_k) \tag{40}$$

for defining the left hand side of the Selberg trace formula for new forms. Note that if n be equal to the discriminant of some maximal order $\mathcal{O}_{max}$ in an indefinite quaternion division algebra over $\mathbb{Q}$ then as has been explained in [7], the sum $\sum_{m|n} \beta(\frac{n}{m})$ is equal to one and therefore (40) is reduced to

$$\sum_{u_k \in \mathcal{C}_n^{new}} h(\lambda_k) + h(\lambda_0) = \sum_{u_k \in \mathbb{C} \oplus \mathcal{C}_n^{new}} h(\lambda_k) = \sum_{m|n} \beta(\frac{n}{m}) \sum_{u_k \in \mathbb{C} \oplus \mathcal{C}_m} h(\lambda_k) \tag{41}$$

Thus we achieve the following definition of trace formula for new forms for the congruence subgroup $\Gamma_0(n)$

$$\sum_{u_k \in \mathcal{C}_n^{new}} h(\lambda_k) + \sum_{m|n} \beta(\frac{n}{m}) h(\lambda_0) = I_n^{new} + H_n^{new} + E_n^{new} + CP_n^{new} \tag{42}$$

The terms I_n^{new}, H_n^{new}, E_n^{new} and CP_n^{new} are given by

$$I_n^{new} = \sum_{m|n} \beta(\frac{n}{m}) I_m \tag{43}$$

$$H_n^{new} = \sum_{m|n} \beta(\frac{n}{m}) H_m \tag{44}$$

$$E_n^{new} = \sum_{m|n} \beta(\frac{n}{m}) E_m \tag{45}$$

$$CP_n^{new} = \sum_{m|n} \beta(\frac{n}{m})(P_m - C_m) \tag{46}$$

where I_m, H_m, E_m and P_m denotes the identity, hyperbolic, elliptic and parabolic conjugacy classes for the congruence subgroup $\Gamma_0(m)$. The term C_m refers to the contribution of the continuous spectrum for the corresponding group $\Gamma_0(m)$. Now we give the following theorem which is proved in [7].

Theorem 8. *Let $\mathcal{O}_{max}$ be a maximal order with discriminant $d(\mathcal{O}_{max})$ in an indefinite quaternion division algebra over the field of rationals. Then the right hand side of the Selberg-new trace formula for congruence subgroup $\Gamma_0(n)$ with $n = d(\mathcal{O}_{max})$ coincides with the right hand side of the Selberg trace formula for the unit group of $\mathcal{O}_{max}$ that is*

$$I_{\mathcal{O}^1_{max}} = I_n^{new}, \qquad E_{\mathcal{O}^1_{max}} = E_n^{new}, \qquad CP_n^{new} = 0 \tag{47}$$

$$H_{\mathcal{O}^1_{max}} = H_n^{new} = \sum_{m|n} \beta(\frac{n}{m}) H_m \tag{48}$$

where $I_{\mathcal{O}^1_{max}}$, $E_{\mathcal{O}^1_{max}}$, $H_{\mathcal{O}^1_{max}}$ denotes the contribution of identity, elliptic and hyperbolic elements in the right hand side of Selberg trace formula for the unit group of quaternions $\mathcal{O}^1_{max}$.

4 Selberg zeta function and its determinant expression

In this section we define Selberg zeta function for new forms for congruence subgroup $\Gamma_0(n)$ and using the result of previous section we show that for some levels n it is equal to the Selberg zeta function of the unit group of quaternions $\mathcal{O}^1_{max}$. The Selberg zeta function has the following property [13]

$$\frac{d}{ds} H(s) = \frac{d}{ds} \frac{1}{2s-1} \frac{d}{ds} log\ Z(s) \tag{49}$$

where the $H(s)$ is the contribution of hyperbolic conjugacy classes in the Selberg trace formula with the test function depending on the complex parameter s given by

$$h(r^2 + \frac{1}{4}) = \frac{1}{r^2 + \frac{1}{4} + s(s-1)} - \frac{1}{r^2 + \beta^2} \qquad \beta > \frac{1}{2}, \quad s \in \mathbb{C} \tag{50}$$

Now we define the Selberg-new zeta function for a congruence subgroup $\Gamma_0(n)$ such that it satisfies the property (49) and has the asymptotic behavior

$$\lim_{Re(s) \to +\infty} Z_n^{new} = 1 \tag{51}$$

that is

$$Z_n^{new}(s) := \Pi_{m|n} Z_m^{\beta(\frac{n}{m})}(s) \tag{52}$$

where $Z_m(s)$ denotes the Selberg zeta function for congruence subgroup $\Gamma_0(m)$. By simple calculation one can see the following identity which is in accordance with (49)

$$\frac{d}{ds}\sum_{m|n}\beta(\frac{n}{m})H_m(s) = \frac{d}{ds}H_n^{new}(s) = \frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}log\ Z^{new}(s) \qquad (53)$$

On the other hand theorem (8) and formula (53) together with the fact that both $Z_n^{new}(s)$ and $Z_{\mathcal{O}^1}(s)$ have limit equal to one at infinity leads to the following theorem

Theorem 9. *The following formula holds between different Selberg zeta functions*

$$Z_{\mathcal{O}^1_{max}}(s) = Z_n^{new}(s) = \Pi_{m|n}Z_m^{\beta(\frac{n}{m})}(s) \qquad (54)$$

where $n = d(\mathcal{O}_{max})$.

Finally we can connect transfer operators for different groups through their Fredholm determinants. Indeed by replacing formula (4) in (54) we get the following theorem

Theorem 10. *The following identity holds*

$$Det(I - L_{\mathcal{O}^1_{max}}(s)) = \frac{\Xi^2_{\mathcal{O}^1_{max}}(s)}{\Pi_{m|n}\Xi_m^{2\beta(\frac{n}{m})}(s)}\Pi_{m|n}Det(I - L_{\Gamma_0(m)}(s))^{\beta(\frac{n}{m})} \qquad (55)$$

where $n = d(\mathcal{O}_{max})$

Remark 1. The right hand side of (55) can be represented in a different form. That is

$$\Xi^2_{\mathcal{O}^1_{max}}(s)\Pi_{m|n}Det(I - \mathcal{L}_s^{\Gamma_0(m)})^{\beta(\frac{n}{m})} \qquad (56)$$

Here the operator $\mathcal{L}_s^{\Gamma_0(m)}$ was introduced and studied by Mayer (see for example [18]). The operator acts on a certain Banach space of vector valued functions $f : \Omega \to \mathbb{C}^\mu$, where Ω is a domain in $\mathbb{C}$ and μ is the index of $\Gamma_0(m)$ in $SL(2,\mathbb{Z})$.

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References

[1] D.Mayer, *The thermodynamic formalism approach to Selberg's zeta function for* $PSL(2,\mathbb{Z})$, Bull. Amer. Math. Soc, **25** (1991), 55–60.

[2] T.Morita, *Markov systems and transfer operators associated with cofinite Fuchsian groups*, Ergod. Th. and Dynam. Sys. **17** (1997), 1147–1181.

[3] D.Mayer,*Thermodynamics formalism and quantum mechanics on the modular surface*, Topics in Modern Statistical Physics (1992).

[4] R.Bowen and C.Series, *Markov maps associated with Fuchsian groups*, Publ. IHES **50** (1979), 401–418.

[5] D.Mayer, Continued fractions and related transformations. In *Ergodic theory, Symbolic Dynamics and Hyperbolic Spaces*, chapter 7, Oxford University Press, Oxford, (1991), 175–222.

[6] A.B.Venkov, *Spectral theory of automorphic functions and its applications*, Kluwer Academic Publishers, 1990.

[7] J.Bolte, S.Johansson, *A Spectral correspondence for Maass waveforms*, GAFA, Geom. funct. anal. **9** (1999), 1128–1155.

[8] J.Bolte, S.Johansson, *Theta-lifts of Maass waveforms*, in "Emerging applications of number theory" (D.A. Hejhal, F. Chung, J. Friedman, M. C. Gutzwiller, A. Odlyzko, eds.), IMA **109** , Springer-Verlag, New York (1999), 39–72.

[9] A.O.L.Atkin, J.Lehner, *Hecke Operators on* $\Gamma_0(m)$, Math. Ann. **185** (1970), 134–160.

[10] D. A. Hejhal, *A classical approach to a well-known spectral correspondence on quaternion groups* in Number theory, New York 1983-84, D. Chudnovsky, G. Chudnovsky, H. Cohen, and M. Nathanson, eds., Lecture Notes in Mathematics 1135, Berlin-Heidelberg-New York, 1985, Springer-Verlag.

[11] T.Miyake, *Modular forms* , Springer, 1989.

[12] D.W.Lewis, *Quaternion Algebras and the Algebraic Legacy of Hamilton's Quaternions*, Irish Math. Soc. Bulletin **57** (2006), 41–64.

[13] S.Y.Koyama, *Determinant Expression of Selberg Zeta functions.***I**, Transactions of the American mathematical Society, **324**, (1991), no.1, 149–168.

[14] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with application to Dirichlet series*. J.Indian Math. Soc. **20** (1956), 47–87.

[15] A. Weil, *Basic Number Theory*. Springer, Berlin Heidelberg New York, 1967.

[16] E. Balslev, A. Venkov, *The Weyl law for subgroups of the modular group*, Geom. Funct. Anal.(GAFA),**8** (1998), 437–465.

[17] A. Stroembergsson, *Studies in the analytic and spectral theory of automorphic forms*. Phd Thesis, Uppsala University 2001, 226 pp.

[18] C.-H. Chang, D. Mayer, *Thermodynamic Formalism and Selberg's zeta function for modular groups*, Regul Chaotic Dyn, **5** (2000), 281-312.

[19] M.-F. Vigneras, *Arithmetique des Algebres de Quaternions*, Lecture Notes in Math. **800**, Springer-Verlag, Berlin, 1980.