# An isomorphism between polynomial eigenfunctions of the transfer operator and the Eichler cohomology for modular groups

D.Mayer and J. Neunhäuserer
Institut für Theoretische Physik,
Technische Universität Clausthal. [1]
Arnold-Sommerfeld-Str. 6,
38678 Clausthal-Zellerfeld, Germany
e-mail: dieter.mayer@tu-clausthal.de
joerg.neunhaeuserer@tu-clausthal.de

**Abstract**

For the group $PSL(2,\mathbb{Z})$ it is known that there is an isomorphism between polynomial eigenfunctions of the transfer operator for the geodesic flow and the Eichler cohomology in the theory of modular forms, see [3], [8], [9]. In [3] it is indicated that such an isomorphism exists as well for the subgroups $\Gamma(2)$ and $\Gamma_0(2)$ of $PSL(2,\mathbb{Z})$. We will prove this and provide some evidence by computer aided algebraic calculations that such an isomorphism exists for all principal congruence subgroups $\Gamma(N)$ and all congruence subgroups of Hecke type $\Gamma_0(N)$.

**MSC 2000: 11F11, 30F35, 37C30**

[1]This work is supported by the DFG research group 'Zeta functions and locally symmetric spaces'.

# 1 Introduction

The transfer operator $L_s$ for the geodesic flow on the surface corresponding to a modular group is an generalization of the classical Perron Frobenius operator in ergodic theory for the Poincare map of this dynamical system. The operator is of special interest because it provides a new approach to Selberg's zeta function which encodes information about the length spectrum of the geodesic flow. In fact it was shown that Selberg's zeta function can be expressed using the Fredholm determinant of this operator and that the zeros of Selberg's zeta function correspond to values of $s$ for which the transfer operator $L_s$ has eigenvalue $+1$ or $-1$ (see [10], [1], [2] and [3]). In this paper we are interested in polynomial eigenfunctions of the transfer operator to this eigenvalues. These eigenfunctions are determined by the so called Lewis equation found in [2] and [9]. Our general conjecture is that using the Lewis equation one can directly find an isomorphism between the spaces of polynomial eigenfunctions of the transfer operator and the spaces of Eichler's cohomology classes which are well known in number theory [4]. In fact these cohomology classes correspond in a special way to automorphic forms of modular groups. In the case of $PSL(2,\mathbb{Z})$ the existence of such an isomorphism is known and in [3] we presented numerical indications that such an isomorphism exists for the simplest subgroups of $PSL(2,\mathbb{Z})$. We will present here rigorous arguments proving the existence of such an isomorphism in the case of the groups $\Gamma(2)$ (see Proposition 7.1) and $\Gamma_0(2)$ (see Proposition 8.1 and Proposition 8.2). Furthermore we will provide strong numerical evidence that such an isomorphism exists for all principal congruence subgroups $\Gamma(N)$ (see Table 1) and all congruence subgroups of Hecke type $\Gamma_0(N)$ (see Table 2). The main problem to prove this fact rigorously is the complicated combinatorial interplay between the action of the generators on the coset sets of these groups and the structure of the Lewis equation.

The rest of this paper is organized as follows. In section 2 and 3 we give a brief overview about modular groups, modular forms and the Eichler cohomology emphasizing those facts about congruence subgroups that we need. In section 4 we introduce the transfer operator. In section 5 we state the Lewis equation for modular groups which is then exploited in section 6 to describe the spaces of polynomial eigenfunctions of the transfer operator. In section 7 we present our results for principal congruence modular groups and in section 8 we present our results for congruence subgroups of Hecke type.

# 2 Modular groups and modular surfaces

We will give here a brief introduction to modular groups and modular surfaces. Our reference for this material are the books of Miyake [11] and Shimura [12]. Consider the upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$ with the Poincare metric given by

$$ds^2(z) = \frac{dz\ d\bar{z}}{\Im(z)^2}.$$

The special linear group

$$SL(2,\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in \mathbb{R} \text{ and } ad - bc = 1\}$$

acts on the upper half plane $\mathbb{H}$ by the Möbius transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

The group

$$PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\pm I,$$

is known to be the isometry group of $\mathbb{H}^2$. Discrete subgroups of $PSL(2,\mathbb{R})$ like

$$\Gamma(1) = PSL(2,\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R}) | a,b,c,d \in \mathbb{Z}\}$$

are called **Fuchsian groups** and the subgroups of $\Gamma(1)$ of finite index are called **modular groups**. Given a modular group $\Gamma$ we denote by $\tilde{\Gamma}$ the set $\Gamma\backslash\Gamma(1)$ of right cosets.

In this paper we are especially interested in **principal congruence subgroups**

$$\Gamma(N) := \{A \in \Gamma(1) | A = I \bmod N\}$$

and **congruence subgroups of Hecke type**

$$\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) | c = 0 \bmod N\}$$

where $N \geq 2$. These groups are modular and their index is given by

$$|\Gamma(1) : \Gamma(N)| = \{ \begin{array}{cl} 6 & \text{if } N = 2 \\ 1/2N^3 \prod_{p|N}(1 - 1/p^2) & \text{if } N > 2 \end{array}$$

$$|\Gamma(1) : \Gamma_0(N)| = N \prod_{p|N} (1 + 1/p)$$

where the product runs over all prime divisors $p$ of $N$.

Later on we will need the generators for the groups $\Gamma(1)$, $\Gamma(2)$ and $\Gamma_0(2)$. The group $\Gamma(1)$ is generated by the elements

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

fulfilling the relations

$$Q^2 = (QT)^3 = I.$$

[2] $I$ denotes the identity matrix through out this paper

The group $\Gamma(2)$ is freely generated by

$$A_1 = T^2 \quad \text{and} \quad A_2 = QT^2Q$$

and $\Gamma_0(2)$ is generated by

$$B_1 = T \quad , \quad B_2 = QT^{-2}Q \quad \text{and} \quad B_3 = T^{-1}B_2$$

which fulfill the relations

$$B_1B_2B_3 = I \quad \text{and} \quad B_3^2 = I.$$

Now let us mention a few facts about the coset sets. Consider the map

$$Mod_N : PSL(2, \mathbb{Z}) \longmapsto PSL(2, \mathbb{Z}/N\mathbb{Z}) \text{ with } Mod_N(A) = A \bmod N$$

This is a surjective homomorphism with kernel $\Gamma(N)$. Hence $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$ and the group $\tilde{\Gamma}(N) = \Gamma(N)\backslash\Gamma(1)$ is isomorphic to $PSL(2, \mathbb{Z}/N\mathbb{Z})$.
Now consider $\Gamma_0(N)$. Obviously we have $\Gamma(N) \subseteq \Gamma_0(N) \subseteq \Gamma(1)$ and $Mod_N(\Gamma_0(N)) = \Theta(N)$ where $\Theta(N)$ is the group

$$\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid b \in \mathbb{Z}/N\mathbb{Z}, \quad a \in (Z/N\mathbb{Z})^\star\}.$$

$\Gamma_0(N)$ is not normal in $\Gamma(1)$ but there is a natural bijection from $\tilde{\Gamma}_0(N) = \Gamma_0(N)\backslash\Gamma(1)$ to $\Theta(N)\backslash\tilde{\Gamma}(1)$. For $N = p$ prime a system of representatives of this coset set is given by

$$\{QT^i | i = 0 \ldots p-1\} \cup \{I\}.$$

To each modular group $\Gamma$ there corresponds a modular surface given by the quotient space $\Gamma\backslash\mathbb{H}$ which consists of the equivalence classes of $\Gamma$-equivalent points of $\mathbb{H}$. This surface is topologically a sphere with finitely many handles and finitely many cusps. The cusps are located at the rationals on the real axis and at $\infty$. The compactification of this surface is a Riemannian surface with the Riemannian metric coming from the Poincare metric on $\mathbb{H}$. Let $g$ be the genus of this surface, $v_\infty$ the number of cusps, $v_2$ be the number of elliptic elements of order 2 in $\Gamma$ and $v_3$ be the number of elliptic elements of order 3 in $\Gamma$[3]. With these notations we have [11]

$$g = 1 + \frac{|\Gamma(1) : \Gamma|}{12} - \frac{v_2}{4} - \frac{v_3}{3} - \frac{v_\infty}{2}.$$

In the case of the principal congruence subgroups and congruence subgroups of Hecke type all these quantities are explicitly know. For $\Gamma(N)$ we have $v_2 = v_3 = 0$ and $v_\infty = |\Gamma(1) : \Gamma(N))|/N$ and hence

$$g(\Gamma(N)) = 1 + \frac{|\Gamma(1) : \Gamma(N)|}{12} - \frac{|\Gamma(1) : \Gamma(N)|}{2N}.$$

[3]An element in $\Gamma$ is called elliptic if it has the fixed points $z$ and $\bar{z}$ with $z \in \mathbb{H}$

For $\Gamma_0(N)$ we have

$$v_2(\Gamma_0(N)) = \left\{ \begin{array}{cc} 0 & \text{if } 4|N \\ \prod_{p|N}(1+\{\frac{-1}{p}\}) & \text{if } 4 \not| N \end{array} \right.$$

$$v_3(\Gamma_0(N)) = \left\{ \begin{array}{cc} 0 & \text{if } 9|N \\ \prod_{p|N}(1+\{\frac{-3}{p}\}) & \text{if } 9 \not| N \end{array} \right.$$

and

$$v_\infty(\Gamma_0(N)) = \sum_{0<d|N} \phi(d, N/d)$$

where $\{-\}$ denotes here the quadratic residue symbol and $\phi$ denotes the Euler function. Again the genus of the surfaces corresponding to $\Gamma_0(N)$ can be explicitly calculated using these expressions.

# 3 Automorphic forms and the Eichler cohomology for modular groups

Let $\Gamma$ be a modular group. We will first recall the definition of **automorphic forms** and **cusp forms** on $\Gamma$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $f : \mathbb{H} \longmapsto \mathbb{C}$ and $k \in \mathbb{Z}$ we set

$$[A]_k f(z) = (cz+d)^{-k} f(Az).$$

$f$ is called an automorphic form of weight $k$ (or degree $-k$) for $\Gamma$ if $f$ is holomorphic on $\mathbb{H}$ and at the cusps of $\Gamma\backslash\mathbb{H}$ such that $[A]_k f = f$ for all $A \in \Gamma$. If in addition $f$ vanishes at the cusps of $\Gamma$ it is called cusp form. We denote by $A_k(\Gamma)$ the space of automorphic forms and by $C_k(\Gamma)$ the space of cusp forms. It is possible to calculate the dimension of these spaces by relating them to differentials on $\Gamma\backslash\mathbb{H}$ and using the Riemann-Roch Theorem. In fact we have the following Proposition, see [11].

**Proposition 3.1** *For arbitrary modular groups $\Gamma$ we have with the above notations*

$$\dim C_k(\Gamma) = (k-1)(g-1) + (k/2-1)v_\infty + [k/4]v_2 + [k/3]v_3$$

*for all $k > 2$ even and $\dim C_2(\Gamma) = g$. Moreover*

$$\dim A_k(\Gamma) = \dim C_k(\Gamma) + v_\infty(\Gamma)$$

*and the automorphic forms that are not cusps forms are given by Eisenstein series.*

For the principal congruence subgroups $\Gamma(N)$ we get the following corollary.

**Corollary 3.1** *For all $N \geq 2$ and all $k \in \mathbb{N}$ even we have*

$$\dim C_k(\Gamma(N)) = \frac{|\Gamma(1) : \Gamma(N)|}{12}(k-1) - \frac{v_\infty(\Gamma(N))}{2}.$$

Using Proposition 3.1 it is also possible to calculate $\dim C_k(\Gamma_0(N))$. In the appendix of [11] the reader will find a table with the explicit values of $\dim C_k(\Gamma_0(N))$ for a wide range of $N$ and $k$.

Now we introduce the **Eichler cohomology**, see [4]. Let $f \in C_{k+2}(\Gamma)$ be a cusp form of weight $k+2$ (degree $-(k+2)$) with $k \in \mathbb{N}_0$ an even number. We are interested in the $k+1$-th undetermined integral of $f$. Obviously the integral

$$\Theta(\tau) = \frac{1}{k!} \int_{\tau_0}^{\tau} (t-z)^k f(z) dz$$

is path independent for $\tau, \tau_0 \in \mathbb{H}$ and

$$\frac{d^{k+1}}{d\tau^{k+1}} \Theta(\tau) = f(\tau).$$

Let $\mathbb{P}_k$ denote the linear space of all polynomials over $\mathbb{C}$ of degree less or equal $k$. We can add an arbitrary polynomial $\phi \in \mathbb{P}_k$ to the above integral as an "integration constant" to obtain an arbitrary $k+1$-th integral of $f$

$$\Theta(\tau) = \frac{1}{k!} \int_{\tau_0}^{\tau} (t-z)^k f(z) dz + \phi(\tau).$$

Moreover it is easy to show that

$$\Omega_A(\tau) := [A]_{-k} \Theta(\tau) - \Theta(\tau)$$

is a polynomial in $\mathbb{P}_k$ for all $A \in \Gamma$. The map

$$\Omega : A \longmapsto \Omega_A$$

from $\Gamma$ into $\mathbb{P}_k$ is a cocycle since

$$\Omega_{AB} = [B]_{-k} \Omega_A + \Omega_B.$$

We denote the space of all cocycles by $\bar{E}_k(\Gamma)$;

$$\bar{E}_k(\Gamma) = \{\Omega : \Gamma \longmapsto \mathbb{P}_k | \Omega_{AB} = [B]_{-k} \Omega_A + \Omega_B\}$$

Special cocycles are the coboundaries

$$\Omega_A(\tau) = [A]_{-k} \phi(\tau) - \phi(\tau)$$

for $\phi \in \mathbb{P}_k$. The space of all Eichler cohomology classes is now the quotient

$$E_k(\Gamma) := \bar{E}_k(\Gamma) / \mathbb{P}_k$$

where we identify $\mathbb{P}_k$ with the space of coboundaries in the obvious way. Now every cusp form $f \in C_{k+2}(\Gamma)$ corresponds to a cohomology class in $E_k(\Gamma)$. Eichler's work [4] shows that this correspondens is two to one. Moreover [4] implicitly contains the following result.

**Proposition 3.2** *For all $k \geq 2$ even we have*

$$\dim E_k(\Gamma) = 2 \dim C_{k+2}(\Gamma) + v_\infty(\Gamma)$$

*and especially*

$$\dim E_k(\Gamma(N)) = \frac{|\Gamma(1) : \Gamma(N)|}{6}(k+1)$$

For $\Gamma(2)$ and $\Gamma_0(2)$ we want to compute the spaces $E_k$ explicitly. Let $A_1$ and $A_2$ be the generators of $\Gamma(2)$. Since $\Gamma(2)$ is freely generated by these matrices it is obvious that every pair of polynomials $\Omega_{A_1}, \Omega_{A_2} \in I\!P_k$ determines a cocycle and vice versa. Hence

$$E_k(\Gamma(2)) = \bar{E}_k(\Gamma(2))/I\!P_k \cong (I\!P_k \times I\!P_k)/I\!P_k \cong I\!P_k.$$

Let $B_1, B_2, B_3$ be the generators of $\Gamma_0(2)$. Choose arbitrary polynomials $\Omega_{B_1}, \Omega_{B_3} \in P_k$. By the relation $B_1 B_2 B_3 = I$ a polynomial $\Omega_{B_2} \in I\!P_k$ is uniquely determined by the cocycle condition. By the relation $B_3^2 = I$, $\Omega_{B_3}$ fulfills the cocycle relation if and only if $[B_3]_{-k}\Omega_{B_3} + \Omega_{B_3} = 0$ which means

$$\Omega_{B_3}(\frac{z+1}{-2z-1})(-2z-1)^k + \Omega_{B_3}(z) = 0.$$

Let $\Upsilon_k$ be the linear space of all polynomials $\Omega \in I\!P_k$ obeying this functional equation. We now see that a cocycle for $\Gamma_0(2)$ is uniquely determined by an arbitrary polynomial in $P_k$ and an polynomial in $\Upsilon_k$. Hence

$$E_k(\Gamma_0(2)) = \bar{E}_k(\Gamma_0(2))/I\!P_k \cong (I\!P_k \times \Upsilon_k)/I\!P_k \cong \Upsilon_k.$$

By Proposition 3.1 and 3.2 we now get a formula for the dimension of the space $\Upsilon_k$

$$\dim \Upsilon_k = 2[(k-1)/4] + 2.$$

# 4 The Transfer operator for modular groups

Here we consider the transfer operator for modular groups $\Gamma$ introduced by Chang and Mayer in [1], [2] and [3]. In these papers the reader will find a detailed discussion of the properties of the transfer operator that are mentioned here.

For a function $f : \mathbb{C} \times \tilde{\Gamma} \times \mathbb{Z}/2\mathbb{Z} \longmapsto \mathbb{C}$ and $s \in \mathbb{C}$ we define a formal operator $L_s$ by

$$L_s f(z, A, \epsilon) = \sum_{n=1}^{\infty} (\frac{1}{z+n})^{2s} f(\frac{1}{z+n}, AQT^{n\epsilon}, -\epsilon)$$

where $\tilde{\Gamma} = \Gamma\backslash\Gamma(1) = \{A_1, A_2, \ldots, A_\mu\}$ with $\mu = |\Gamma(1) : \Gamma|$. We call this operator the **transfer operator**. There is another way to express this operator using representation theory[4]. We define $\chi^\Gamma$ as the representation of $\Gamma(1)$ which is

[4]We refer to [5] for an introduction to representation theory

induced by the trivial representation of $\Gamma$. That means $\chi^\Gamma : \Gamma(1) \longmapsto \{0,1\}^\mu$ is given by

$$\chi^\Gamma(G) = (\chi(A_i G A_j))_{i,j=1,\ldots,\mu}$$

with

$$\chi(G) = \{ \begin{array}{ll} 1 & G \in \Gamma \\ 0 & G \notin \Gamma \end{array} .$$

We can now write the operator $L_s$ in the form

$$L_s f(z,\epsilon) = \sum_{n=1}^{\infty} (\frac{1}{z+n})^{2s} \chi^\Gamma(QT^{n\epsilon}) f(\frac{1}{z+n}, -\epsilon)$$

which acts on functions $f : \mathbb{C} \times \mathbb{Z}/2\mathbb{Z} \longmapsto \mathbb{C}^\mu$. Now let

$$D = \{z | z \in \mathbb{C}, |z-1| < 3/2\}$$

and consider the Banach space

$$\tilde{B} := B(D \times \mathbb{Z}/2\mathbb{Z})^\mu$$

where $B(D \times \mathbb{Z}/2\mathbb{Z})$ is the space of all complex valued functions holomorphic on $D \times \mathbb{Z}/2\mathbb{Z}$ and continuous on the boundary of this set. The operator $L_s$ is well defined on $\tilde{B}$ and holomorphic if $\Re(\beta) > 1/2$. It can be continued to an operator which is meromorphic on the hole complex $\beta$-plane with possible poles only for $\beta = \beta(\kappa) = (1-\kappa)/2$ with $\kappa \in \mathbb{N}_0$. Moreover it was shown by Chang and Mayer that $L_s$ is a nuclear operator and hence of trace class. The Fredholm determinant of the operator is given by Selberg's zeta function with respect to the representation $\chi^\Gamma$, see [13].
The transfer operator has a well known interpretation within the theory of dynamical systems. Let us consider the geodesic flow on the surface $\Gamma \backslash \mathbb{H}$. This is a Hamiltonian flow and provides a classical example of a "chaotic" dynamical systems, see [6]. It is possible to take a Poincare section for the flow in order to obtain a discrete time system defined by the first return map. In appropriate coordinates this map is given by

$$g(z, A, \epsilon) = (\frac{1}{z} - [\frac{1}{z}], AQT^{n\epsilon}, -\epsilon)$$

on the space $I_2 \times \tilde{\Gamma} \times \mathbb{Z}/2\mathbb{Z}$. Now the Transfer operator defined above is nothing but a generalization of the classical Perron Frobenius operator in ergodic theory for this system, see [6] and [7].

# 5 Eigenfunctions of the transfer operator and the Lewis equation

The key tool for determining eigenfunctions of the transfer operator for a fixed parameter $s \in \mathbb{C}$ to an eigenvalue $\lambda = \lambda(s)$ is a certain functional equation, which is called **Lewis equation**. For $\Gamma(1)$ this equation was found in [8]. For modular groups $\Gamma$ we have the following Proposition contained in [3] which relates eigenfunctions of the transfer operator to solutions of this Lewis equation.

**Proposition 5.1** *$f \in \tilde{B}$ is a eigenfunction of the transfer operator $L_s$ with eigenvalue $\lambda$ i.e. $L_s f = \lambda f$ if and only if*

$$\lambda(f(z,\epsilon) - \chi^\Gamma(QT^\epsilon Q)f(z+1,\epsilon)) = (\frac{1}{z+1})^{2s}\chi^\Gamma(QT^\epsilon)f(\frac{1}{z+1}, -\epsilon)$$

*for $\epsilon \in \mathbb{Z}/2\mathbb{Z}$ and*

$$\lim_{z\longmapsto\infty}(\lambda f(z,\epsilon) - \sum_{l=0}^{\kappa}\sum_{m=1}^{r}(1/r)^{2s+l}\chi^\Gamma(QT^{m\epsilon})\frac{f^{(l)}(0,-\epsilon)}{l!}\zeta(2s+l, \frac{z+m}{r})) = 0$$

*for $\epsilon \in \mathbb{Z}/2\mathbb{Z}$ and $\kappa > 2\Re(s)$ where $\zeta$ is the Hurwitz zeta function.*

**Remark 5.1** In the special case $\chi^\Gamma(T^2) = I$ the two equations in the last Proposition are the same for $\epsilon = 1$ and $\epsilon = -1$ and thus reduce to

$$\lambda(f(z) - \chi^\Gamma(QTQ)f(z+1)) = (\frac{1}{z+1})^{2s}\chi^\Gamma(QT)f(\frac{1}{z+1})$$

and

$$\lim_{z\longmapsto\infty}(\lambda f(z) - \sum_{l=0}^{\kappa}\sum_{m=1}^{r}(1/r)^{2s+l}\chi^\Gamma(QT^m)\frac{f^{(l)}(0)}{l!}\zeta(2s+l, \frac{z+m}{r})) = 0 \ \kappa > 2\Re(s).$$

For the groups $\Gamma(2)$ and $\Gamma_0(2)$ we indeed have $\chi^\Gamma(T^2) = I$ since $T^2 = I$ in $\tilde{\Gamma}(2)$ and $\tilde{\Gamma}_0(2)$. We will use this fact later on.

**Remark 5.2** Consider the following functional equation

$$g(z) - \chi^\Gamma(QTQ)g(z+1) = (\frac{1}{z+2})^{2s}\chi^\Gamma(QT^2)g(\frac{-1}{z+2}).$$

Following [9] we call this equation the **Master equation**. Let $g$ be a solution of the Master equation and set

$$g^+(z,\epsilon) = \{ \begin{array}{cc} g(z) & \text{if } \epsilon = 1 \\ (1/(z+1))^{2s}\chi^\Gamma(T)g(-z/(z+1)) & \text{if } \epsilon = -1 \end{array}$$

and

$$g^-(z,\epsilon) = \{ \begin{array}{cc} g(z) & \text{if } \epsilon = -1 \\ -(1/(z+1))^{2s}\chi^\Gamma(T)g(-z/(z+1)) & \text{if } \epsilon = -1 \end{array}$$

Then $g^+$ is a solution of the Lewis equation for $\lambda = +1$ and $g^-$ is a solution of the Lewis equation for $\lambda = -1$. In fact also the converse is true. If $g^+$ is a solution of the Lewis equation with $\lambda = 1$ then $g(z) := g^+(z,1)$ is a solution of the Master equation. The same is true for $g^-$. See section 3 of [3].

# 6 Polynomial Eigenfunctions of the transfer operator

We are interested in this paper primarily in polynomial eigenfunctions of the transfer operator. By Proposition 5.1 we see that the transfer operator $L_s$ may have eigenfunctions in $I\!P_n \times \{-1,1\}$ if and only if $s = -n/2$ with $n \in I\!N$ even. Moreover for all polynomial eigenfunctions the asymptotic condition in Proposition 5.1 is trivially satisfied.
By well known properties of Selberg's zeta function one knows that $L_{-n/2}$ has the eigenvalue $+1$ and $-1$, see [1]. We restrict our attention from now on to these eigenvalues. By $\wp_n^+(\Gamma)$ resp. $\wp_n^-(\Gamma)$ we denote the space of all polynomial eigenfunctions of degree $n$ of the transfer operator $L_{-n/2}$ for the modular group $\Gamma$ to the eigenvalue $+1$ resp. $-1$. More precisely

$$\wp_n^+(\Gamma) = \{p \in (I\!P_n)^\mu | L_{-n/2}p^+ = p^+\}$$

$$\wp_n^-(\Gamma) = \{p \in (I\!P_n)^\mu | L_{-n/2}p^- = -p^-\}$$

where $\mu = |\Gamma(1) : \Gamma|$. Furthermore let

$$\wp_n(\Gamma) = \wp_n^+(\Gamma) \bigoplus \wp_n^-(\Gamma).$$

In the next sections we will study this space for principal congruence subgroups and congruence subgroups of Hecke type.

# 7 The case of principal congruence subgroups

In this section we consider the principal congruence group $\Gamma(N)$ defined in section 2 for $N \geq 2$. We are interested in the dimension of the space $\wp_n(\Gamma(N))$ of all polynomial eigenfunctions of degree $n$ of the transfer operator $L_{-n/2}$ where $n$ is an even number (see section 6) and in the relation of this space to the space of Eichler's cohomology classes $E_n(\Gamma(N))$ (see section 4). We have the following conjecture.

**Conjecture 7.1** *For all $N \geq 2$ and $n \geq 2$ even we have*

$$\dim \wp_n(\Gamma(N)) = \frac{|\Gamma(1) : \Gamma(N)|}{6}(n+1)$$

*and hence*

$$\wp_n(\Gamma(N)) \cong E_n(\Gamma(N)).$$

**The case $N = 2$**

In the case $N = 2$ we can prove this conjecture. Since $|\Gamma(1) : \Gamma(2)| = 6$ it is in fact an immediate consequence of the following Proposition

**Proposition 7.1** *We have*

$$\dim \wp_n^-(\Gamma(2)) = n/2+2 \quad \textit{and} \quad \dim \wp_n^+(\Gamma(2)) = n/2-1.$$

**Proof** Using the fact that $\tilde{\Gamma}(2)$ is isomorphic to $PSL(2,\mathbb{Z}/2\mathbb{Z})$ it is easy to calculate the induced representation $\chi^{\Gamma(2)}$. We get

$$\chi^{\Gamma(2)}(QTQ) = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix} \text{ and } \chi^{\Gamma(2)}(QT) = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{pmatrix}$$

If we set $f(z) = \phi(z+1)$ the Lewis equation from remark 5.1 is given by the following system of functional equations

$$\begin{aligned} \lambda(\phi_1(z) - \phi_6(z+1)) - z^n\phi_5(\tfrac{z+1}{z}) &= 0\\ \lambda(\phi_2(z) - \phi_5(z+1)) - z^n\phi_6(\tfrac{z+1}{z}) &= 0\\ \lambda(\phi_3(z) - \phi_4(z+1)) - z^n\phi_2(\tfrac{z+1}{z}) &= 0\\ \lambda(\phi_4(z) - \phi_3(z+1)) - z^n\phi_1(\tfrac{z+1}{z}) &= 0\\ \lambda(\phi_5(z) - \phi_2(z+1)) - z^n\phi_4(\tfrac{z+1}{z}) &= 0\\ \lambda(\phi_6(z) - \phi_1(z+1)) - z^n\phi_3(\tfrac{z+1}{z}) &= 0. \end{aligned}$$

First note that that $\phi_1$ and $\phi_2$ are uniquely determined by the first two equations if $\phi_5$ and $\phi_6$ are given. By substituting $1/z$ for $z$ and multiplying with $-\lambda z^n$ the last two equation are equivalent to

$$\begin{aligned} z^n(-\phi_5(\tfrac{1}{z}) + \phi_2(\tfrac{z+1}{z})) + \lambda\phi_4(z+1) &= 0\\ z^n(-\phi_6(\tfrac{1}{z}) + \phi_1(\tfrac{z+1}{z})) + \lambda\phi_3(z+1) &= 0. \end{aligned}$$

Now adding the third and the fourth equation to this equation we see that

$$\begin{aligned} \phi_3(z) &= \lambda z^n\phi_5(\tfrac{1}{z})\\ \phi_4(z) &= \lambda z^n\phi_6(\tfrac{1}{z}). \end{aligned}$$

Thus $\phi_3$ and $\phi_4$ are uniquely determined if $\phi_5$ and $\phi_6$ are given. Now inserting the expressions for $\phi_1, \phi_2, \phi_3$ and $\phi_4$ in the last two equations we get

$$\begin{aligned} \phi_5(z) - \phi_5(z+2) &= (z+1)^n(\lambda\phi_6(\tfrac{z+2}{z+1}) + \phi_6(\tfrac{z}{z+1}))\\ \phi_6(z) - \phi_6(z+2) &= (z+1)^n(\lambda\phi_5(\tfrac{z+2}{z+1}) + \phi_5(\tfrac{z}{z+1})) \end{aligned}$$

and substituting $z-1$ for $z$ this gives

$$\begin{aligned} \phi_5(z-1) - \phi_5(z+1) &= z^n(\lambda\phi_6(\tfrac{z+1}{z}) + \phi_6(\tfrac{z-1}{z}))\\ \phi_6(z-1) - \phi_6(z+1) &= z^n(\lambda\phi_5(\tfrac{z+1}{z}) + \phi_5(\tfrac{z-1}{z})) \end{aligned}$$

With

$$\bar{T}\phi(z) := \phi(z-1) - \phi(z+1) \quad \text{and} \quad G\phi(z) := z^n(\lambda\phi(\frac{z+1}{z}) + \phi(\frac{z-1}{z})).$$

we can write this system in the form

$$\bar{T}\phi_5 = G\phi_6 \quad \text{and} \quad \bar{T}\phi_6 = G\phi_5$$

We have to determine the solutions $(\phi_5, \phi_6) \in I\!P_n^2$ of this system of functional equations.
Obviously $G$ is a linear map of $I\!P_n$ into $I\!P_n$ and $\bar{T}$ is a linear map $I\!P_n$ onto $I\!P_{n-1}$. Let $I\!P_n^e = \{\phi \in I\!P_n | \phi \text{ is even}\}$ and $I\!P_n^o = \{\phi \in I\!P_n | \phi \text{ is odd}\}$. If $\phi \in I\!P_n^e$ we have

$$\bar{T}\phi(-z) = \phi(-z-1) - \phi(-z+1) = \phi(z+1) - \phi(z-1) = -\bar{T}\phi(z)$$

and hence $\bar{T}\phi \in I\!P_{n-1}^o$ and if $\phi \in I\!P_n^o$ we have

$$\bar{T}\phi(-z) = \phi(-z-1) - \phi(-z+1) = -\phi(z+1) + \phi(z-1) = \bar{T}\phi(z)$$

and hence $\bar{T}\phi \in I\!P_{n-1}^e$.
Now consider the case $\lambda = -1$. In this case $G\phi \in I\!P_{n-1}$ and since

$$G\phi(-z) = (-z)^n(-\phi(\frac{-z+1}{-z}) + \phi(\frac{-z-1}{-z}))$$

$$= z^n(\phi(\frac{z+1}{z}) - \phi(\frac{z-1}{z})) = -G\phi(z)$$

the map $G$ is onto $I\!P_{n-1}^o$.
Let $(\phi_5, \phi_6) \in I\!P_n^2$ be solutions of $\bar{T}\phi_5 = G\phi_6$ and $\bar{T}\phi_6 = G\phi_5$. We know that $G\phi_6$ and hence $\bar{T}\phi_5$ is odd. Now assume that $\phi_5 = \phi_5^e + \phi_5^o$ where $\phi_5^e$ is an even polynomial and $\phi_5^o$ is an odd polynomial and not zero. We have $\bar{T}\phi_5 = \bar{T}\phi_5^e + \bar{T}\phi_5^o$ where $\bar{T}\phi_5^e$ is odd but $\bar{T}\phi_5^o$ is even and not zero. Hence $\bar{T}\phi_5$ would not be odd. This is a contradiction and hence $\phi_5$ has to be even. By the same argument we can show that $\phi_6^e$ is even.
Now let $\phi_5$ be an arbitrary polynomial in $I\!P_n^e$. We have $\bar{T}\phi_5 \in I\!P_{n-1}^o$ and the map $G : I\!P_n^e \longmapsto P_{n-1}^o$ is onto and has a kernel consisting of all constant polynomials. Hence we see that all solutions $\phi_6 \in I\!P_n$ of $\bar{T}\phi_5 = G\phi_6$ are given by $\phi_6 = \bar{\phi}_6 + d$ where $\bar{\phi}_6 \in I\!P_n^e$ is uniquely determined and $d$ is an arbitrary constant. Furthermore if $\bar{T}\phi_5 = G\phi_6$ we have

$$\phi_5(z-1) - \phi_5(z+1) = z^n(-\phi_6(\frac{z+1}{z}) + \phi_6(\frac{z-1}{z})).$$

Substituting $1/z$ for $z$ and using the fact that $\phi_5$ and $\phi_6$ are even this implies

$$\phi_6(z-1) - \phi_6(z+1) = z^n(\phi_5(-\frac{z+1}{z}) + \phi_5(\frac{z-1}{z})).$$

This means that the functional equation $\bar{T}\phi_6 = G\phi_5$ holds as well. Hence the space of solutions in $I\!P_n$ of our system of functional equations is isomorphic to $I\!P_n^e \times I\!P_o$ and hence $\wp_n(\Gamma(2)) \cong I\!P_n^e \times I\!P_o$. Since $\dim I\!P_n^e = n/2+1$ for $n$ even this completes the proof of our result in the case $\lambda = -1$.
In the case $\lambda = +1$ an argument along the same lines just exchanging the role

of odd and even polynomials gives the desired result. □

**The case $N > 2$**

In the case $N > 2$ we have some numerical evidence for our general conjecture. Table 1 below contains the dimension of $\wp_n(\Gamma(N))$ for some values of $N > 2$ and $n$ even. By the formula for $|\Gamma(1) : \Gamma(N)|$ given in section 2 we see from table 1 that conjecture 7.1 is really true for all values of $N$ and $n$ we have checked.

| | $n=2$ | $n=4$ | $n=6$ | $n=8$ | $n=10$ | $n=12$ |
|---|---|---|---|---|---|---|
| N=3 | 6 | 10 | 14 | 18 | 22 | 24 |
| N=4 | 12 | 20 | 28 | 36 | 44 | 52 |
| N=5 | 30 | 50 | 70 | 90 | 110 | 130 |
| N=6 | 36 | 60 | 84 | 108 | 132 | 156 |
| N=7 | 84 | 140 | 196 | 242 | | |
| N=8 | 96 | 160 | | | | |

**Table 1:** $\dim \wp_n(\Gamma(N))$

We have calculated these dimensions algebraically using Matematica. Let us make a few comments on this computation. It is easy to compute the finite group $PSL(2, \mathbb{Z}/N\mathbb{Z})$ and the action of $Q$ and $T$ on these group. This gives us the regular representation of $PSL(2, \mathbb{Z}/N\mathbb{Z})$ which can be identified with the induced representation $\chi^\Gamma$. Now we can explicitly determine the Lewis equation for $\Gamma(N)$. Inserting polynomials into this equation we get a system of $C(N,n) = |\Gamma(1) : \Gamma(N)|n \approx N^3 n$ linear equation. This system can be solved algebraically in at most a day on a PC if $C(N,n)$ is less then 2000.

# 8 The case of congruence subgroups of Hecke type

In this section we consider the congruence groups of Hecke type $\Gamma_0(N)$ defined in section 2 for $N \geq 2$. We are interested in the relation of the space $\wp_n(\Gamma_0(N))$ of all polynomial eigenfunctions of degree $n$ of the transfer operator $L_{-n/2}$ where $n$ is an even number to the space of Eichler's cohomology classes $E_n(\Gamma_0(N))$, see sections 4 and 6. We have the following conjecture.

**Conjecture 8.1** *For all $N \geq 2$ and $n \geq 2$ even we have*

$$\wp_n(\Gamma_0(N)) \cong E_n(\Gamma_0(N)).$$

**The case $N = 2$**

Let

$$\tilde{\wp}_n^+ = \{p \in (\mathbb{P}_{n+1})^\mu | L_{-n/2} p^+ = p^+\}$$
$$\tilde{\wp}_n^- = \{p \in (\mathbb{P}_{n+1})^\mu | L_{-n/2} p^- = -p^-\}$$

where $L$ is the transfer operator with respect to $\Gamma_0(2)$ and $p^+$ and $p^-$ are defined in remark 5.2 . We consider here polynomials in $\mathbb{P}_{n+1}$ instead of polynomials in $\mathbb{P}_n$ for some technical reasons. In fact there exists such eigenfunctions of $L_{-n/2}$ and we can explicitly relate the space $\tilde{\wp}_n(N) = \tilde{\wp}_n^+(N) \oplus \tilde{\wp}_n^-(N)$ to the space $\Upsilon_n$ which is isomorphic to Eichler's cohomology classes for $\Gamma_0(2)$, see section 3.

**Proposition 8.1** *Let $n \in \mathbb{N}$ be an even number and let $\tilde{T}$ be the linear operator given by $\tilde{T}\phi(z) = \phi(z) - \phi(z+1)$ from $\mathbb{P}_{n+1}$ to $\mathbb{P}_n$. Then*

$$\Upsilon_n \cong \tilde{T}\tilde{\wp}_n.$$

**Proof** Using the fact that a system of representatives of $\tilde{\Gamma}_0(2)$ is given by $\{I, Q, QT\}$ it is easy to calculate the induced representation $\chi^{\Gamma(2)}$. We get

$$\chi^{\Gamma(2)}(QTQ) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \chi^{\Gamma(2)}(QT) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

If we set $f(z) = \phi(z+1)$ the Lewis equation from remark 5.1 is given by the following system of functional equations

$$\begin{aligned} \lambda(\phi_1(z) - \phi_3(z+1)) - z^n\phi_3(\tfrac{z+1}{z}) &= 0 \\ \lambda(\phi_2(z) - \phi_2(z+1)) - z^n\phi_1(\tfrac{z+1}{z}) &= 0 \\ \lambda(\phi_3(z) - \phi_1(z+1)) - z^n\phi_2(\tfrac{z+1}{z}) &= 0. \end{aligned}$$

Expressing $\phi_3$ through $\phi_2$ and $\phi_1$ using the third equation and inserting $\phi_3$ into the second equation yields

$$\phi_3(z) = \lambda z^n \phi_2(1/z).$$

Thus $\phi_3$ can be expressed through $\phi_2$. Moreover by the second equation $\phi_1$ can be expressed through $\phi_2$,

$$\phi_1(z) = \lambda(z-1)^n(\phi_2(1/(z-1)) - \phi_2(z/(z-1)))$$

Inserting these expression into the first equation yields a functional equation for $\phi_2$

$$\phi_2(z) - \phi_2(z+1) = (2z+1)^n(\lambda\phi_2(\frac{z+1}{2z+1}) + \phi_2(\frac{z}{2z+1}))$$

This shows

$$\tilde{\wp}_n^+ \cong \{\phi \in \mathbb{P}_{n+1} | \phi(z) - \phi(z+1) = (2z+1)^n(\phi(\frac{z+1}{2z+1}) + \phi(\frac{z}{2z+1}))\}$$

and

$$\tilde{\wp}_n^- \cong \{\phi \in \mathbb{P}_{n+1} | \phi(z) - \phi(z+1) = (2z+1)^n(-\phi(\frac{z+1}{2z+1}) + \phi(\frac{z}{2z+1}))\}$$

We will identify these isomorphic spaces in the following. We want to show that

$$\tilde{\wp}_n^+ \subseteq I\!P_{n+1}^o \text{ and } \tilde{\wp}_n^- \subseteq I\!P_{n+1}^e$$

Let $\phi \in \tilde{\wp}_n^+$. Substituting $-z-1$ in the functional equation determining $\phi$ and using the fact that $n$ is even we get

$$\phi(-z-1) - \phi(-z) = (2z+1)^n(\phi(\frac{z+1}{2z+1}) + \phi(\frac{z}{2z+1}))$$

and hence

$$\phi(-z-1) - \phi(-z) = \phi(z) - \phi(z+1) = -(\phi(z+1) - \phi(z)).$$

Decomposing $\phi$ in an even and an odd polynomial we see that this implies $\phi = \phi^o + c$ where $\phi^o$ is odd and $c$ is a constant. Inserting this into the functional equitation we get $2(2z+1)^n c = 0$. Hence $c = 0$ and $\phi \in I\!P_{n+1}^o$.
Let $\phi \in \tilde{\wp}_n^-$. Substituting $-z-1$ in the functional equation determining $\phi$ and using the fact that $n$ is even we get

$$\phi(-z-1) - \phi(-z) = (2z+1)^n(\phi(\frac{z+1}{2z+1}) - \phi(\frac{z}{2z+1}))$$

and hence

$$\phi(-z-1) - \phi(-z) = -(\phi(z) - \phi(z+1) = \phi(z+1) - \phi(z)$$

Again decomposing $\phi$ in an even and an odd part we see that this implies $\phi \in I\!P_{n+1}^e$. Now note that

$$\tilde{T}^{-1}\Upsilon_n = \{\phi \in I\!P_{n+1} | \tilde{T}\phi(z) \in \Upsilon_n\}$$

$$= \{\phi \in I\!P_{n+1} | \phi(z) - \phi(z+1) = -(-2z-1)^n(\phi(-\frac{z+1}{2z+1}) - \phi(\frac{z}{2z+1}))\}$$

$$= \{\phi \in I\!P_{n+1} | \phi(z) - \phi(z+1) = (2z+1)^n(-\phi(-\frac{z+1}{2z+1}) + \phi(\frac{z}{2z+1}))\}$$

From this equation and the fact that polynomials in $\tilde{\wp}_n^+$ are odd and polynomials in $\tilde{\wp}_n^-$ are even we get

$$\tilde{\wp}_n = \tilde{\wp}_n^+ \oplus \tilde{\wp}_n^- \subseteq \tilde{T}^{-1}\Upsilon_n$$

It remains to show that

$$\tilde{T}^{-1}\Upsilon_n \subseteq \tilde{\wp}_n^+ \oplus \tilde{\wp}_n^-$$

Assume that there exists $\phi \in \tilde{T}^{-1}\Upsilon_n \backslash (\tilde{\wp}_n^+ \oplus \tilde{\wp}_n^-)$. Decompose $\phi = \phi^e + \phi^o$ with $\phi^e \in I\!P_{n+1}^e$ and $\phi^o \in I\!P_{n+1}^o$. Define $\tilde{\phi}$ by

$$\tilde{\phi}(z) := \phi^e(z) - \phi^e(z+1) - (2z+1)^n(-\phi^e(\frac{z+1}{2z+1}) + \phi^e(\frac{z}{2z+1})).$$

Since we assumed $\phi \in \tilde{T}^{-1}\Upsilon_n$ we have

$$-\tilde{\phi}(z) = \phi^o(z) - \phi^o(z+1) - (2z+1)^n(\phi^o(\frac{z+1}{2z+1}) + \phi^o(\frac{z}{2z+1}))$$

Substituting $-z-1$ for $z$ in these equations yields

$$\tilde{\phi}(z) = -\tilde{\phi}(-z-1) \qquad \text{and} \qquad \tilde{\phi}(z) = \tilde{\phi}(-z-1).$$

But this obviously implies $\tilde{\phi} = 0$. On the other hand since $\phi \notin (\tilde{\wp}_n^+ \oplus \tilde{\wp}_n^-)$ the polynomial $\tilde{\phi}$ can not be zero. This is a contradiction and our proof is complete. $\square$

Since the kernel of $\tilde{T}$ consists of all constant polynomials which are contained as well in $\tilde{\wp}_n^-$ Proposition 8.1 and section 3 has the following corollary.

**Corollary 8.1** *For all $n \geq 2$ even we have*

$$\dim \tilde{\wp}_n = \dim \Upsilon_n + 1 = 2[(k-1)/4] + 3.$$

Combining the following Proposition with this Corollary we see that Conjecture 8.1 is really true in the case $N = 2$.

**Proposition 8.2** *For all $n \geq 2$ even we have*

$$\dim \tilde{\wp}_n = \dim \wp_n(\Gamma_0(2)) + 1$$

**Proof** Looking again at the proof of Proposition 8.1 we see that

$$\wp_n(\Gamma_0(2)) \cong \{\phi \in I\!P_n | \phi(z) - \phi(z+1) = (2z+1)^n(-\phi(-\frac{z+1}{2z+1}) + \phi(\frac{z}{2z+1}))\}$$

and we will identify these isomorphic spaces in the following.
We can write every $\tilde{\phi} \in \tilde{\wp}_n$ as $\tilde{\phi} = \phi + cz^{n+1}$ where $\phi \in I\!P_n$ and $c$ is a constant such that

$$\phi(z) - \phi(z+1) + (2z+1)^n(\phi(-\frac{z+1}{2z+1}) - \phi(\frac{z}{2z+1}))$$

$$= c(z^{n+1} - (z+1)^{n+1} - \frac{(z+1)^{n+1}}{2z+1} - \frac{z^{n+1}}{2z+1})$$

Let

$$\check{\phi}(z) = z^{n+1} - (z+1)^{n+1} - \frac{(z+1)^{n+1}}{2z+1} - \frac{z^{n+1}}{2z+1}$$

Since $(z^{n+1} + (z+1)^{n+1})$ has a zero at $z = -1/2$ if $n$ is even we have $\check{\phi} \in I\!P_n$. Now there exists oa $\phi_1 \in I\!P_n$ such that

$$\phi_1(z) - \phi_1(z+1) + (2z+1)^n(\phi_1(-\frac{z+1}{2z+1}) - \phi_1(\frac{z}{2z+1})) = \check{\phi}(z)$$

Hence we have

$$\tilde{\wp}_n \cong \{\phi + cz^{n+1} | \phi \in I\!P_n, \quad c \in I\!R, \quad \phi \in \wp_n(\Gamma_0(2)) + c\phi_1\}$$

$$\cong \wp_n(\Gamma_0(2)) \oplus < \phi_1 + z^{n+1} >$$

which proves our Proposition. □

**The case $N > 2$**

In the case $N > 2$ prime we have some numerical evidence for our general conjecture. Table 2 below contains the dimension of $\wp_n(\Gamma_0(N))$ for some values $N > 2$ prime and $n \geq 2$ even. If we compare Table 2 with $\dim E_k(\Gamma_0(N))$ using the table in the appendix of [11] for $\dim C_n(\Gamma_0(N)$ and Proposition 3.2 we see that Conjecture 8.1 is true for all values of $N$ and $n$ we have checked.

| | $N=3$ | $N=5$ | $N=7$ | $N=11$ | $N=13$ | $N=17$ |
|---|---|---|---|---|---|---|
| n=2 | 2 | 4 | 4 | 6 | 8 | 10 |
| n=4 | 4 | 4 | 8 | 10 | 12 | 14 |
| n=6 | 4 | 8 | 8 | 14 | 16 | 22 |
| n=8 | 6 | 8 | 12 | 18 | 20 | 26 |
| n=10 | 8 | 12 | 16 | 22 | 28 | 34 |
| n=12 | 8 | 12 | 16 | 26 | 28 | 38 |
| n=14 | 10 | 16 | 20 | 30 | 36 | 46 |
| n=16 | 12 | 16 | 24 | 34 | 40 | 50 |
| n=18 | 12 | 20 | 24 | 38 | 44 | 58 |
| n=20 | 14 | 20 | 28 | 42 | 48 | 62 |
| n=22 | 16 | 24 | 32 | 46 | 56 | 70 |
| n=24 | 16 | 24 | 32 | 50 | 56 | 74 |

**Table 2:** $\dim \wp_n(\Gamma_0(N))$

We have calculated these dimensions algebraically using Matematica. If $N$ is prime we have the simple system of representatives of the coset set $\tilde{\Gamma}_0(N)$ given in section 2. This allows us to calculate the action of $Q$ and $T$ on the coset set and thus the induced representation $\chi^\Gamma$. Now we proceed in exactly the same way as in the calculations for $\Gamma(N)$. Here we have to solve a system of only $(N+1)n$ linear equations.

# References


[1] C.-H. Chang and D. Mayer, *The transfer operator approach to Selberg's zeta function and modular and Maass wave forms for $PSL(2,\mathbb{Z})$*, in D. Hejhal and M. Gutzwiller et al, editors, IMA Volumes **109** 'Emerging applications of number theory', Springer Verlag 72-143 (1999).

[2] C.-H. Chang and D. Mayer, *Thermodynamic formalism and Selberg's zeta function for modular groups*, Regular and Chaotic Dynamics **5**, 281-312 (2000).

[3] C.-H. Chang and D. Mayer, *Eigenfunctions of the transfer operators and the period functions for modular groups*, in 'Proceedings of an AMS workshop on arithmetic, spectral and dynamical zeta functions', San Antionio (1999); to appear in: Contemporary. Math (2001).

[4] M. Eichler, *Eine Verallgemeinerung der abelschen Integrale* , Math. Zeitschrift, **67**, 267-298 (1957).

[5] W. Fulton and J. Harris, *Representation Theory - A first Cours*, Springer Verlag (1991).

[6] A. Katok and B. Hasselblatt, *Modern Theory of Dynamical Systems*, Cambridge University press (1995).

[7] A. Lasota and M. Mackey, *Probabilistic properties of deterministic systems*, Cambridge University Press, (1985).

[8] J. Lewis, *Spaces of holomorphic functions equivalent to Maass cusp forms*, Invent. Math., **127(2)**, 271-306 (1997).

[9] J. Lewis and D. Zagier, *Period functions and the Selberg's zeta function for modular groups*, in The Mathematical Beauty of Physics, Adv. Series in Math. Physics 24, World Scientific, Singapour, 83-97 (1997).

[10] D. Mayer, *The thermodynamic formalism approach to Selberg's zeta function for $PSL(2, \mathbb{Z})$*, Bull. Am. Math. Soc., **25**, 55-60 (1991).

[11] T. Miyake, *Modular Forms*, Springer Verlag (1989).

[12] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press (1971).

[13] A.B. Venkov, *Spectral Theory of Automorphic Functions and Its Applications*, Kluwer Academic Publishers (1990).