Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

THE TRANSFER OPERATOR FOR THE HECKE TRIANGLE GROUPS

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Abstract. In this paper we extend the transfer operator approach to Selberg's zeta function for cofinite Fuchsian groups to the Hecke triangle groups G_q, $q = 3, 4, \ldots$, which are non-arithmetic for $q \neq 3, 4, 6$. For this we make use of a Poincaré map for the geodesic flow on the corresponding Hecke surfaces which has been constructed in [13] and which is closely related to the natural extension of the generating map for the so called Hurwitz-Nakada continued fractions. We derive simple functional equations for the eigenfunctions of the transfer operator which for eigenvalues $\rho = 1$ are expected to be closely related to the period functions of Lewis and Zagier for these Hecke triangle groups.

1. **Introduction.** This paper continues our study of the transfer operator for cofinite Fuchsian groups and their Selberg zeta functions [3],[4]. For the modular groups, i.e. finite index subgroups $\Gamma \subset SL(2,\mathbb{Z})$, the transfer operator approach to Selberg's zeta function [3] has led to interesting new developments in number theory, like the theory of period functions for Maass wave forms by Lewis and Zagier [9]. Obviously, it is necessary to extend this theory to more general Fuchsian groups, especially the nonarithmetic ones. One possibility for such a generalization is via a cohomological approach [1], which has been worked out for the case $G_3 = SL(2,\mathbb{Z})$ recently in [2]. We concentrate on the transfer operator approach to this circle of problems and started to work out this approach in [12],[13] for the Hecke triangle groups which, contrary to modular groups studied up to now, are mostly non-arithmetic.

The transfer operator has been introduced by D. Ruelle [18] among other reasons primarily to investigate analytic properties of dynamical zeta functions. A typical

2000 *Mathematics Subject Classification.* Primary: 11M36, 37C30; Secondary: 37B10, 37D35, 37D40, 37D20.

Key words and phrases. Hecke triangle groups, λ_q-continued fractions, transfer operator, Ruelle and Selberg zeta function.

The second author was supported by German Research Foundation (DFG) grant Ma 633/16-1: Transfer operators and non arithmetic quantum chaos.

example of such a function is the Selberg zeta function, $Z_S(s)$, for the geodesic flow on surfaces of constant negative curvature, which connects the length spectrum of this flow with spectral properties of the corresponding Laplacian. It is defined by

$$Z_S(s) = \prod_\gamma \prod_{k=0}^\infty \left(1 - e^{-(s+k)l(\gamma)}\right), \tag{1}$$

where the outer product is taken over all prime periodic orbits γ of period $l(\gamma)$ of the geodesic flow $\Phi_t : S\mathcal{M} \to S\mathcal{M}$ on the unit tangent bundle of the surface $\mathcal{M}$. The period $l(\gamma)$ coincides in this case with the length of the corresponding closed geodesic. If $\mathcal{P} : \Sigma \to \Sigma$ is the Poincaré map on a section Σ of the flow Φ_t Ruelle showed that $Z_S(s)$ can be rewritten as

$$Z_S(s) = \prod_{k=0}^\infty \frac{1}{\zeta_R(s+k)},$$

where ζ_R denotes the Ruelle zeta function for the Poincaré map $\mathcal{P}$ defined as

$$\zeta_R(s) = \exp\left(\sum_{n=1}^\infty \frac{1}{n} Z_n(s)\right)$$

with

$$Z_n(s) = \sum_{x \in \operatorname{Fix} \mathcal{P}^n} \exp\left(-s \sum_{k=0}^{n-1} r\left(\mathcal{P}^k(x)\right)\right), n \geq 1,$$

the so called dynamical partition functions and $r : \Sigma \to \mathbb{R}_+$ the recurrence time function with respect to the map $\mathcal{P}$, defined through

$$\Phi_{r(x)}(x) \in \Sigma \text{ for } x \in \Sigma \quad \text{and} \quad \Phi_t(x) \notin \Sigma \text{ for } 0 < t < r(x).$$

In the transfer operator approach for the modular groups the Selberg zeta function gets expressed in terms of the Fredholm determinant of an operator $\mathcal{L}_s$ as $Z_S(s) = \det(1 - \mathcal{L}_s)$. From this relation it is clear that the zeros of $Z_S(s)$ are directly related to the values of s for which $\mathcal{L}_s$ has eigenvalue one. Furthermore, the corresponding eigenfunctions in a certain Banach space of holomorphic functions can be directly related to the automorphic functions of these modular groups [4]. It is expected that this approach can be extended to all cofinite Fuchsian groups. In this paper we continue to work it out for the Hecke triangle groups and their corresponding surfaces. A Poincaré map $\mathcal{P} : \Sigma \to \Sigma$ for the geodesic flow on the Hecke surfaces $\mathcal{M}_q = G_q \backslash \mathbb{H}$, $q = 3, 4, 5, \ldots$ was constructed in [13]. Thereby $\mathbb{H}$ denotes the hyperbolic upper half-plane and G_q the Hecke triangle group generated by the isometries

$$S : z \mapsto -1/z \quad \text{and} \quad T : z \mapsto z + \lambda_q$$

with $\lambda_q = 2\cos\left(\frac{\pi}{q}\right)$. In [13] it was shown, that the map $\mathcal{P}$ is closely related to the natural extension $F_q : \Omega_q \to \Omega_q$ of the generating map $f_q : I_q \to I_q$, $I_q = [-\frac{\lambda_q}{2}, \frac{\lambda_q}{2}]$ of the Hurwitz-Nakada continued fractions [14],[17], also called λ_q-continued fractions or shortly λ_q-CF's. These are closely related to the Rosen λ-continued fractions [16, 17, 19]. For a precise description of this relationship see e.g. [13] Remark 15. We recall the necessary facts about the λ_q-continued fractions in §2. Contrary to the case of the modular surfaces, where a Poincaré map $\mathcal{P} : \Sigma \to \Sigma$ has been constructed through the natural extension of the Gauss map $T_G : [0, 1] \to [0, 1]$, $T_g(x) = \frac{1}{x} \mod 1$, $x \neq 0$, in the present case of the Hecke surfaces $\mathcal{M}_q$ there is not a one-to-one correspondence between the periodic orbits of the map f_q generating the λ_q-CF 's and the periodic orbits of the geodesic flow $\Phi_t : S\mathcal{M}_q \to S\mathcal{M}_q$. Indeed there exist for every G_q exactly two periodic points $r_q, -r_q \in I_q$ which correspond to the same periodic orbit $\mathcal{O}$ of the flow Φ_t. In the case $q = 3$ Hurwitz already showed in [8] that there exist exactly two closed f_3-orbits which are equivalent under the action of the group $G_3 = SL(2,\mathbb{Z})$ and hence lead to the same orbit of the flow Φ_t. As a consequence the Fredholm determinant $\det(1-\mathcal{L}_s)$ of the Ruelle transfer operator $\mathcal{L}_s$ of the the Hurwitz-Nakada map f_q does not correctly describe the Selberg zeta function (1) for the Hecke triangle groups G_q, since it contains the contribution of the closed orbit $\mathcal{O}$ twice. To correct this overcounting we introduce another transfer operator, $\mathcal{K}_s$, whose Fredholm determinant describes exactly the contribution of this orbit $\mathcal{O}$ to the Selberg function. The form of this operator can be directly deduced from the λ_q-CF expansion of the point $r_q \in I_q$. The spectrum of the operator $\mathcal{K}_s$ can be determined explicitly, leading to regularly spaced zeros of its Fredholm determinant, $\det(1 - \mathcal{K}_s)$, in the complex s-plane. In in Section 6.2 we will use the operator $\mathcal{K}_s$ to show the following formula for the Selberg zeta function for Hecke triangle groups:

$$Z_S(s) = \frac{\det(1-\mathcal{L}_s)}{\det(1-\mathcal{K}_s)}. \tag{2}$$

As in the case of the modular surfaces and the Gauss map T_G, the holomorphic eigenfunctions $\overrightarrow{g}$ of the transfer operator $\mathcal{L}_s$ fulfil simple functional equations. In the case $q = 3$ it was recently shown [2] that if $\Re s = \frac{1}{2}$ then there is a one-to-one correspondence between eigenfunctions of $\mathcal{L}_s$ with eigenvalue 1 and Maass waveforms, i.e. square-integrable eigenfunctions of the Laplace-Beltrami operator, on the modular surface $\mathcal{M}_3$. We therefore expect that the holomorphic eigenfunctions $\overrightarrow{g}$ of the operator $\mathcal{L}_s$ with eigenvalue $\rho(s) = 1$ can be related for all G_q and general s to the automorphic functions of these Hecke triangle groups which almost all are non arithmetic. This will extend the transfer operator approach to the theory of period functions of Lewis and Zagier [9] to a whole class of non-arithmetic Fuchsian groups. We hope to come back to this question soon.

The structure of this article is as follows: In Section 2 we briefly introduce the Hecke triangle groups G_q and recall the λ_q-continued fractions as discussed in [12]. Section 3 recalls the geodesic flow on the Hecke surface $\mathcal{M}_q$, the Selberg zeta function and the construction of the Poincaré section Σ and the Poincaré map $\mathcal{P} : \Sigma \to \Sigma$ in [13]. The transfer operator $\mathcal{L}_s$ for the H-N map $f_q : I_q \to I_q$ is discussed in Section 4. We show that it is a nuclear operator when acting in a certain Banach space B of vector-valued holomorphic functions whose dimension is determined by the Markov partitions for f_q and has a meromorphic extension to the entire complex s-plane. In Section 5 we define a symmetry operator, $P : B \to B$, commuting with the transfer operator. This allows us to restrict the operator $\mathcal{L}_s$ to the two eigenspaces B_ϵ, $\epsilon = \pm 1$ of P. From this we derive the scalar functional equations which the eigenfunctions of the restricted transfer operators $\mathcal{L}_{s,\epsilon}$ are shown to fulfil. In Section 6 we discuss the Ruelle zeta function for the H-N map f_q and show that it can be expressed in terms of Fredholm determinants of the operator $\mathcal{L}_s$ respectively the reduced operators $\mathcal{L}_{s,\epsilon}$. Finally, in section 6.2 we introduce the transfer operator, $\mathcal{K}_s$, whose Fredholm determinant describes the contribution of the orbit, $\mathcal{O}_+$ corresponding to the point r_q and which is needed to obtain (2) above.

2. λ_q-continued fractions and their generating maps.

2.1. The Hecke triangle groups.

Let

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \mod \{\pm\mathbf{1}\}$$

denote the *projective linear group*, where $\mathrm{SL}(2,\mathbb{R})$ denotes the group of 2×2 matrices with real entries and determinant 1 and $\pm\mathbf{1} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$. We usually identify the elements of $\mathrm{PSL}(2,\mathbb{R})$ with one of its matrix representatives in $\mathrm{SL}(2,\mathbb{R})$.

For an integer $q \geq 3$, the *Hecke triangle group* $G_q \subset \mathrm{PSL}(2,\mathbb{R})$ is the group generated by the elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T_q = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \tag{3}$$

where

$$\lambda_q = 2\cos\left(\frac{\pi}{q}\right) \in [1,2).$$

The elements S and T_q satisfy the relations

$$S^2 = (ST_q)^q = \mathbf{1}.$$

Later on we also need the element

$$T_q' = \begin{bmatrix} 1 & 0 \\ \lambda_q & 1 \end{bmatrix} = ST_q^{-1}S \in G_q.$$

The action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}$ is given by *Möbius transformations*:

$$g\,z := \frac{az+b}{cz+d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}).$$

One can easily verify that $g\,z \in \mathbb{H}$ for $z \in \mathbb{H}$ and $g\,x \in \mathbb{P}_{\mathbb{R}}$ for $x \in \mathbb{P}_{\mathbb{R}}$ where $\mathbb{H} = \{x+iy \mid y > 0\}$ denotes the upper half-plane and $\mathbb{P}_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ denotes its boundary, the projective real line.

We say that two points $x, y \in \mathbb{H} \cup \mathbb{P}_{\mathbb{R}}$ are G_q*-equivalent* if there exists a $g \in G_q$ such that $x = g\,y$.

An element $g \in \mathrm{PSL}(2,\mathbb{R})$ is called *elliptic*, *hyperbolic* or *parabolic* depending on whether $|\mathrm{Tr}\,(g)| := |a+d| < 2$, > 2 or $= 2$. The same notation applies for the fixed points of the corresponding Möbius transformation.

In the following we identify the element $g \in \mathrm{PSL}(2,\mathbb{R})$ with the induced map $z \mapsto g\,z$ on $\mathbb{H}$. Note that the type of fixed point is preserved under conjugation, $g \mapsto AgA^{-1}$, by $A \in \mathrm{PSL}(2,\mathbb{R})$. A parabolic fixed point is a degenerate fixed point, belongs to $\mathbb{P}_{\mathbb{R}}$, and is usually called a *cusp*. Elliptic fixed points appear in pairs, z and $\overline{z}$ with $z \in \mathbb{H}$, and $G_q(z)$, the stabilizer subgroup of z in G_q, is cyclic of finite order. Hyperbolic fixed points also appear in pairs $x, x^\star \in \mathbb{P}_{\mathbb{R}}$, where $x^\star$ is said to be the repelling conjugate point of the attractive fixed point x.

2.2. λ_q-continued fractions and their duals.

Consider finite or infinite sequences $[a_i]_i$ with $a_i \in \mathbb{Z}$ for all i. We denote a periodic subsequence within an infinite sequence by overlining the periodic part and a finitely often repeated pattern is denoted by a power, where the power ^{0} means absence of the pattern, hence

$$[a_1, \overline{a_2, a_3}] = [a_1,\ a_2, a_3,\ a_2, a_3,\ a_2, a_3,\ \ldots],$$
$$[a_1, (a_2, a_3)^i, a_4, \ldots] = [a_1, \underbrace{a_2, a_3, a_2, a_3, \ldots, a_2, a_3}_{i \text{ times } a_2, a_3}, a_4, \ldots] \quad \text{and}$$
$$[a_1, (a_2)^0, a_3, \ldots] = [a_1, a_3, \ldots].$$

Furthermore, by the negative of a sequence we mean the following:

$$-[a_1, a_2, \ldots] = [-a_1, -a_2, \ldots].$$

Put

$$h_q := \begin{cases} \frac{q-2}{2} & \text{for even } q \text{ and} \\ \frac{q-3}{2} & \text{for odd } q. \end{cases}$$

Next we define the set $\mathcal{B}_q$ of *forbidden blocks* as

$$\mathcal{B}_q := \begin{cases} \{[\pm 1]\} \cup \bigcup_{m=1}^{\infty} \{[\pm 2, \pm m]\} & \text{for } q = 3, \\ \{[(\pm 1)^{h_q+1}]\} \cup \bigcup_{m=1}^{\infty} \{[(\pm 1)^{h_q}, \pm m]\} & \text{for even } q \text{ and} \\ \{[(\pm 1)^{h_q+1}]\} & \\ \ \cup \bigcup_{m=1}^{\infty} \{[(\pm 1)^{h_q}, \pm 2, (\pm 1)^{h_q}, \pm m]\} & \text{for odd } q \geq 5. \end{cases}$$

The choice of the sign is the same within each block and $m \geq 1$. For example $[2, 3]$, $[-2, -3] \in \mathcal{B}$ and $[2, -3] \notin \mathcal{B}$ for $q = 3$.

We call a sequence $[a_1, a_2, a_3, \ldots]$ *q-regular* if $[a_k, a_{k+1}, \ldots, a_l] \notin \mathcal{B}_q$ for all $1 \leq k < l$ and *dual q-regular* if $[a_l, a_{l-1}, \ldots, a_k] \notin \mathcal{B}_q$ for all $1 \leq k < l$. Denote by $\mathcal{A}_q^{\text{reg}}$ respectively by $\mathcal{A}_q^{\text{dreg}}$ the set of infinite q-regular respectively dual q-regular sequences $(a_i)_{i \in \mathbb{N}}$.

A *nearest λ_q-multiple continued fraction*, or λ_q-CF, is a formal expansion

$$[a_0; a_1, a_2, a_3, \ldots] := a_0 \lambda_q + \cfrac{-1}{a_1 \lambda_q + \cfrac{-1}{a_2 \lambda_q + \cfrac{-1}{a_3 \lambda_q + \cdots}}}$$

with $a_i \in \mathbb{Z}_{\neq 0}$, $i \geq 1$ and $a_0 \in \mathbb{Z}$.

A λ_q-CF$[a_0; a_1, a_2, a_3, \ldots]$ is said to *converge* if either $[a_0; a_1, a_2, a_3, \ldots, a_l]$ has finite length or $\lim_{l \to \infty} [a_0; a_1, a_2, a_3, \ldots, a_l]$ exists in $\mathbb{R}$. The notations for sequences, as introduced above, are also used for λ_q-CF's.

We say that a λ_q-CF is *regular* respectively *dual regular* if the sequence $[a_1, a_2, a_3, \ldots]$ is q-regular respectively dual q-regular. Regular and dual regular λ_q-CF 's are denoted by $[\![a_0; a_1, \ldots]\!]$ respectively $[\![a_0; a_1, \ldots]\!]^\star$.

It follows from [13] Lemmas 16 and 34 that regular and dual regular λ_q-CF 's converge. Moreover, it is known [13] that x has a regular expansion $x = [\![0; a_1, a_2, \ldots]\!]$ with leading $a_0 = 0$ if and only if $x \in I_q := \left[-\frac{\lambda_q}{2}, \frac{\lambda_q}{2}\right]$.

Convergent λ_q-CF 's can be rewritten in terms of the generators of the Hecke triangle group G_q: if the expansion (2.2) is finite it can be written as follows

$$\begin{aligned} [a_0; a_1, a_2, a_3, \ldots, a_l] &= a_0 \lambda_q + \cfrac{-1}{a_1 \lambda_q + \cfrac{-1}{a_2 \lambda_q + \cfrac{-1}{a_3 \lambda_q + \cdots \cfrac{-1}{a_l \lambda_q}}}} \\ &= T^{a_0}\, ST^{a_1}\, ST^{a_2}\, ST^{a_3} \, \cdots \, ST^{a_l}\, 0, \end{aligned}$$

since $\frac{-1}{a\lambda_q+x} = ST^a\ x$. For infinite converging λ_q-CF's the expansion has to be interpreted as

$$\begin{aligned}[a_0; a_1, a_2, a_3, \ldots] &= \lim_{l\to\infty} [a_0; a_1, a_2, a_3, \ldots, a_l] \\ &= \lim_{l\to\infty} T^{a_0}\, ST^{a_1}\, ST^{a_2}\, ST^{a_3} \cdots ST^{a_l}\, 0 \\ &= T^{a_0}\, ST^{a_1}\, ST^{a_2}\, ST^{a_3} \cdots 0.\end{aligned}$$

An immediate consequence of this is [12, Lemma 2.2.2]:

Lemma 2.1. *For a finite regular λ_q-CF one finds for $q = 2h_q + 2$*

$$[\![a_0; a_1, \ldots, a_n, (1)^{h_q}]\!] = [\![a_0; a_1, \ldots, a_n - 1, (-1)^{h_q}]\!]$$

respectively for $q = 2h_q + 3$

$$[\![a_0; \ldots, a_n, (1)^{h_q}, 2, (1)^{h_q}]\!] = [\![a_0; \ldots, a_n - 1, (-1)^{h_q}, -2, (-1)^{h_q}]\!].$$

2.3. **Special values and their expansions.** The following results are well-known (see [13] and [12, §2.3]). The point $x = \mp\frac{\lambda}{2}$ has the regular λ_q-CF

$$\mp\frac{\lambda}{2} = \begin{cases} [\![0; (\pm1)^{h_q}]\!] & \text{for even } q \text{ and} \\ [\![0; (\pm1)^{h_q}, \pm2, (\pm1)^{h_q}]\!] & \text{for odd } q. \end{cases}$$

Define

$$R_q := \lambda_q + r_q \quad \text{with} \tag{4}$$

$$r_q := \begin{cases} [\![0; \overline{3}]\!] & \text{for } q = 3, \\ [\![0; \overline{(1)^{h_q-1}, 2}]\!] & \text{for even } q \text{ and} \\ [\![0; \overline{(1)^{h_q}, 2, (1)^{h_q-1}, 2}]\!] & \text{for odd } q \geq 5, \end{cases} \tag{5}$$

whose expansion hence is periodic with period κ_q, where

$$\kappa_q := \begin{cases} h_q = \frac{q-2}{2} & \text{for even } q \text{ and} \\ 2h_q + 1 = q - 2 & \text{for odd } q. \end{cases}$$

The regular respectively dual regular λ-CF of the point $x = R_q$ has the form

$$\begin{aligned} R_q &= \begin{cases} [\![1; \overline{(1)^{h_q-1}, 2}]\!] & \text{for even } q, \\ [\![1; \overline{(1)^{h_q}, 2, (1)^{h_q-1}, 2}]\!] & \text{for odd } q \geq 5, \text{ and} \\ [\![1; \overline{3}]\!] & \text{for } q = 3. \end{cases} \\ &= \begin{cases} [\![0; (-1)^h_q, \overline{-2, (-1)^{h_q-1}}]\!]^\star & \text{for even } q, \\ [\![0; (-1)^h_q, \overline{-2, (-1)^{h_q}, -2, (-1)^{h_q-1}}]\!]^\star & \text{for odd } q \geq 5, \text{ and} \\ [\![0; -2, \overline{-3}]\!]^\star & \text{for } q = 3. \end{cases} \end{aligned}$$

Moreover,

$$\begin{aligned} &R_q = 1 \quad \text{and} \quad -R_q = S\, R_q && \text{for even } q \text{ and} \\ &R_q^2 + (2-\lambda)R_q = 1 \quad \text{and} \quad -R_q = \left(T_q S\right)^{h_q+1} R_q && \text{for odd } q, \end{aligned}$$

and R_q satisfies the inequality

$$\frac{\lambda}{2} < R_q \leq 1.$$

2.4. **A lexicographic order.** Let $x, y \in I_{R_q} := [-R_q, R_q]$ have the regular λ_q-CF's $x = [\![a_0; a_1, \ldots]\!]$ and $y = [\![b_0; b_1, \ldots]\!]$. Denote by $l(x) \le \infty$ respectively $l(y) \le \infty$ the number of entries in the above λ_q-CF's. We introduce a *lexicographic order* "$\prec$" for λ_q-CF's as follows: If $a_i = b_i$ for all $0 \le i \le n$ and $l(x), l(y) \ge n$, we define

$$x \prec y :\iff \begin{cases} a_0 < b_0 & \text{if } n = 0, \\ a_n > 0 > b_n & \text{if } n > 0, \text{ both } l(x), l(y) \ge n+1 \text{ and } a_n b_n < 0, \\ a_n < b_n & \text{if } n > 0, \text{ both } l(x), l(y) \ge n+1 \text{ and } a_n b_n > 0, \\ b_n < 0 & \text{if } n > 0 \text{ and } l(x) = n \text{ or} \\ a_n > 0 & \text{if } n > 0 \text{ and } l(y) = n. \end{cases}$$

We also write $x \preceq y$ for $x \prec y$ or $x = y$.

This is indeed an order on regular λ_q-CF's, since Lemmas 22 and 23 in [13] imply:

Lemma 2.2. *Let x and y have regular λ_q-CF's. Then $x \prec y \iff x < y$.*

2.5. **The generating interval maps f_q and $f_q^\star$.** Denote by I_q and I_{R_q} the intervals

$$I_q = \left[-\frac{\lambda_q}{2}, \frac{\lambda_q}{2}\right] \quad \text{and} \quad I_{R_q} = [-R_q, R_q]$$

with λ_q and R_q given in (2.3) and (5). The *nearest λ_q-multiple map* $\langle\cdot\rangle_q$ is defined as

$$\langle\cdot\rangle_q : \mathbb{R} \to \mathbb{Z}; \quad x \mapsto \langle x \rangle_q := \left\lfloor \frac{x}{\lambda_q} + \frac{1}{2} \right\rfloor$$

where $\lfloor\cdot\rfloor$ is the floor function

$$\lfloor x \rfloor = n \iff \begin{cases} n < x \le n+1 & \text{if } x > 0 \text{ and} \\ n \le x < n+1 & \text{if } x \le 0. \end{cases}$$

We also need the map $\langle\cdot\rangle_q^\star$ given by

$$\langle\cdot\rangle_q^\star : \mathbb{R} \to \mathbb{Z}; \quad x \mapsto \langle x \rangle_q^\star := \begin{cases} \left\lfloor \frac{x}{\lambda_q} + 1 - \frac{R_q}{\lambda_q} \right\rfloor & \text{if } x \ge 0 \text{ and} \\ \left\lfloor \frac{x}{\lambda_q} + \frac{R_q}{\lambda_q} \right\rfloor & \text{if } x < 0. \end{cases}$$

The interval maps $f_q \colon I_q \to I_q$ and $f_q^\star : R_q \to R_q$ are then defined as follows:

$$f_q(x) = \begin{cases} -\frac{1}{x} - \left\langle \frac{-1}{x} \right\rangle_q \lambda_q & \text{if } x \in I_q \backslash \{0\}, \\ 0 & \text{if } x = 0 \end{cases} \tag{6}$$

respectively

$$f_q^\star(y) = \begin{cases} -\frac{1}{y} - \left\langle \frac{-1}{y} \right\rangle_q^\star \lambda_q & \text{if } y \in I_{R_q} \backslash \{0\}, \\ 0 & \text{if } y = 0. \end{cases} \tag{7}$$

These maps generate the regular respectively dual regular λ_q-CF's in the following sense:

For given $x, y \in \mathbb{R}$ the entries a_i and b_i, $i \in \mathbb{Z}_{\ge 0}$, in their λ_q-CF are determined by the algorithms:

(0) $a_0 = \langle x \rangle_q$ and $x_1 := x - a_0 \lambda_q \in I_q$,

(1) $a_1 = \left\langle \frac{-1}{x_1} \right\rangle_q$ and $x_2 := \frac{-1}{x_1} - a_1 \lambda_q = f_q(x_1) \in I_q$,

(i) $a_i = \left\langle \frac{-1}{x_i} \right\rangle_q$ and $x_{i+1} := \frac{-1}{x_i} - a_i\lambda_q = f_q(x_i) \in I_q$, $i = 2, 3, \dots$,

($\star$) the algorithm terminates if $x_{i+1} = 0$,

respectively

(0) $b_0 = \langle x \rangle_q^\star$ and $y_1 := y - b_0\lambda_q \in I_{R_q}$,

(1) $b_1 = \left\langle \frac{-1}{y_1} \right\rangle_q^\star$ and $y_2 := \frac{-1}{y_1} - b_1\lambda_q = f_q^\star(y_1) \in I_{R_q}$,

(i) $b_i = \left\langle \frac{-1}{y_i} \right\rangle_q^\star$ and $y_{i+1} := \frac{-1}{y_i} - b_i\lambda_q = f_q^\star(y_i) \in I_{R_q}$, $i = 2, 3, \dots$,

($\star$) the algorithm terminates if $y_{i+1} = 0$.

In ([12], Lemmas 17 and 33) it is shown that these algorithms lead to regular respectively dual regular λ_q-CF's in the sense of section (2.2):

$$x = [\![a_0; a_1, a_2, \dots]\!] \quad \text{and} \quad y = [\![b_0; b_1, b_2, \dots]\!]^\star.$$

2.6. **Markov partitions and transition matrices for f_q.** As shown in [12], both maps f_q and $f_q^\star$ have the Markov property. This means that there exist partitions of the intervals I_q and I_{R_q} into subintervals whose boundary points are mapped by f_q respectively $f_q^\star$ onto boundary points. In particular the following Markov partitions for f_q have been constructed in ([12], 3.3).

Define the orbit $\mathcal{O}(x)$ of x under the map f_q as

$$\begin{aligned} \mathcal{O}(x) &= \left\{x, f_q(x), f_q^2(x) := f_q(f_q(x)), f_q^3(x), \dots\right\} \\ &= \left\{f_q^n(x);\ n = 0, 1, 2, \dots\right\}. \end{aligned}$$

The orbit $\mathcal{O}(-\frac{\lambda_q}{2})$ is finite; indeed if $\#\{S\}$ denotes the cardinality of the set S then

$$\#\{\mathcal{O}(-\frac{\lambda_q}{2})\} = \kappa_q + 1.$$

We denote the elements of $\mathcal{O}(-\frac{\lambda_q}{2})$ for $q = 2h_q + 2$ by

$$\phi_i = f_q^i\left(-\frac{\lambda_q}{2}\right) = [\![0; 1^{h_q - i}]\!], \qquad 0 \le i \le h_q = \kappa_q,$$

respectively for $q = 2h_q + 3$ by

$$\begin{aligned} \phi_{2i} &= f_q^i\left(-\frac{\lambda_q}{2}\right) = [\![0; 1^{h_q - i}, 2, 1^{h_q}]\!] \quad \text{and} \\ \phi_{2i+1} &= f_q^{h_q + i + 1}\left(-\frac{\lambda_q}{2}\right) = [\![0; 1^{h_q - i}]\!], \qquad 0 \le i \le h_q = \frac{\kappa_q - 1}{2}. \end{aligned}$$

The ϕ_i's then satisfy the ordering [13]

$$-\frac{\lambda_q}{2} = \phi_0 < \phi_1 < \phi_2 < \dots < \phi_{\kappa_q - 2} < \phi_{\kappa_q - 1} = -\frac{1}{\lambda_q} < \phi_{\kappa_q} = 0.$$

Define next $\phi_{-i} := -\phi_i$, $0 \le i \le \kappa_q$. The intervals

$$\Phi_i := \left[\phi_{i-1}, \phi_i\right] \quad \text{and} \quad \Phi_{-i} := \left[\phi_{-i}, \phi_{-(i-1)}\right], \qquad 1 \le i \le \kappa_q$$

define a Markov partition of the interval I_q for the map f_q. This means that

$$\bigcup_{\epsilon = \pm} \bigcup_{i=1}^{\kappa_q} \Phi_{\epsilon i} = I_q \quad \text{and} \quad \Phi_{\epsilon i}^\circ \cap \Phi_{\delta j}^\circ = \emptyset \quad \text{for} \quad \epsilon i \ne \delta j,\ \epsilon, \delta = \pm 1,$$

holds, where S° denotes the interior of the set S.

As in [12] we introduce next a finer partition which is compatible with the intervals of monotonicity for f_q.

In the case $q = 3$ where $\lambda_3 = 1$ define for $m = 2, 3, 4, \ldots$ the intervals J_m as

$$J_2 = \left[-\frac{1}{2}, -\frac{2}{5} \right] \quad \text{and} \quad J_m = \left[-\frac{2}{2m-1}, -\frac{2}{2m+1} \right], \quad m = 3, 4, \ldots$$

and set $J_{-m} := -J_m$ for $m = 2, 3, 4, \ldots$. This partition of I_3, which we denote by $\mathcal{M}(f_3)$, is Markovian. The maps $f_3|_{J_m}$ are monotone with $f_3|_{J_m}(x) = -\frac{1}{x} - m$ and locally invertible with $(f_3|_{J_m})^{-1}(y) = -\frac{1}{y+m}$ for $y \in f_3(J_m)$, $m = 2, 3, \ldots$. For $q \geq 4$ consider the intervals J_m, $m = 1, 2, \ldots$, with

$$J_1 = \left[-\frac{\lambda_q}{2}, -\frac{2}{3\lambda_q} \right] \quad \text{and}$$
$$J_m = \left[-\frac{2}{(2m-1)\lambda_q}, -\frac{2}{(2m+1)\lambda_q} \right], \quad m = 2, 3, \ldots$$

and set $J_{-m} := -J_m$ for $m \in \mathbb{N}$. The intervals J_m are intervals of monotonicity for f_q, i.e the maps $f_q|_{J_m}$ are monotone increasing with $f_q|_{J_m}(x) = -\frac{1}{x} - m\lambda_q$ and $(f_q|_{J_m})^{-1}(y) = -\frac{1}{y+m\lambda_q}$ for $m = \pm 1, \pm 2, \pm 3, \ldots$. Since some points in $\mathcal{O}(-\frac{\lambda_q}{2})$ do not fall onto a boundary point of any of the intervals J_m, $m \in \mathbb{N}$ the partition given by the intervals J_m has to be refined.

For even q define the intervals $J_{\pm 1_i}$ as

$$J_{\epsilon 1_i} := J_{\epsilon 1} \cap \Phi_{\epsilon i} \quad \text{for } \epsilon = \pm, \quad 1 \leq i \leq \kappa_q$$

and therefore $J_{\epsilon 1_i} = \Phi_{\epsilon i}$ for $1 \leq i \leq \kappa_q - 1$. This way one arrives at the partition $\mathcal{M}(f_q)$, defined as

$$I_q = \bigcup_{\epsilon=\pm} \left(\bigcup_{i=1}^{\kappa_q} J_{\epsilon 1_i} \cup \bigcup_{m=2}^{\infty} J_{\epsilon m} \right),$$

which is clearly again Markovian.

Consider next the case of odd $q \geq 5$. Here one has $\phi_{\epsilon i} \in J_{\epsilon 1}$ for $1 \leq i \leq \kappa_q - 2$ and $\phi_{\epsilon(\kappa_q - 1)} \in J_{\epsilon 2}$ for $\epsilon = \pm$. For $\epsilon = \pm$ define the intervals

$$J_{\epsilon 1_i} := J_{\epsilon 1} \cap \Phi_{\epsilon i} \quad 1 \leq i \leq \kappa_q - 1 \,\text{and hence}\, J_{\epsilon 1_i} = \Phi_{\epsilon i},\ 1 \leq i \leq \kappa_q - 2 \text{and}$$
$$J_{\epsilon 2, i} := J_{\epsilon 2} \cap \Phi_{\epsilon i},\ i = \kappa_q - 1, \kappa_q.$$

Then it is easy to see that the partition $\mathcal{M}(f_q)$ defined by

$$I_q = \bigcup_{\epsilon=\pm} \left(\bigcup_{i=1}^{\kappa_q - 1} J_{\epsilon 1_i} \cup \bigcup_{i=\kappa_q - 1}^{\kappa_q} J_{\epsilon 2_i} \cup \bigcup_{m=3}^{\infty} J_{\epsilon m} \right)$$

is again Markovian.

A useful tool for understanding the dynamics of the map f_q are the transition matrices $\mathbb{A} = \left(\mathbb{A}_{i,j} \right)_{i,j \in F}$, where F is the alphabet given by the Markov partition $\mathcal{M}(f_q)$:

$$F = \begin{cases} \{\pm 2, \pm 3, \ldots\} & \text{for } q = 3, \\ \{\pm 1_1, \ldots, \pm 1_{\kappa_q - 1}, \pm 2, \pm 3, \ldots\} & \text{for even } q \text{ and} \\ \{\pm 1_1, \ldots, \pm 1_{\kappa_q - 1}, \pm 2_{\kappa_q}, \pm 2_{\kappa_q + 1}, \pm 3, \ldots\} & \text{for odd } q \geq 5. \end{cases}$$

Each entry $\mathbb{A}_{i,j}$, $i,j \in F$ is given by

$$\mathbb{A}_{i,j} = \begin{cases} 0 & \text{if } J_j \cap f_q(J_i) = \emptyset \text{ or} \\ 1 & \text{if } J_j \subset f_q(J_i). \end{cases}$$

Tables 1, 2 and 3 show the transition matrices $\mathbb{A}$ for $q = 3$, even q respectively odd $q \geq 5$, as determined in [12].

$$\begin{aligned} \mathbb{A}_{\pm 2,\pm m} &= 0, \quad m \geq 2, \\ \mathbb{A}_{\pm 2,\mp m} &= 1, \quad m \geq 2, \\ \mathbb{A}_{\pm k,m} &= 1, \quad k \geq 3, \quad \text{and all } m \in F. \end{aligned}$$

TABLE 1. The matrix elements of the transition matrix $\mathbb{A} = (\mathbb{A}_{i,j})_{i,j\in F}$ for $q = 3$.

$$\begin{aligned} \mathbb{A}_{\pm 1_l,\pm 1_{l+1}} &= 1, \quad 1 \leq l \leq \kappa_q - 1, \\ \mathbb{A}_{\pm 1_{\kappa_q - 1},\pm m} &= 1, \quad m = 2, 3, \ldots, \\ \mathbb{A}_{\pm 1_{\kappa_q},\mp 1_l} &= 1, \quad 1 \leq l \leq \kappa_q, \\ \mathbb{A}_{\pm 1_{\kappa_q},-m} &= 1, \quad m = 2, 3, \ldots, \\ \mathbb{A}_{m,n} &= 1, \quad m \in \mathbb{Z} \setminus \{0, \pm 1\}, \quad n \in F, \end{aligned}$$

TABLE 2. The nonvanishing matrix elements of the transition matrix $\mathbb{A} = (\mathbb{A}_{i,j})_{i,j\in F}$ for even q.

2.7. **The local inverses of f_q.** Consider the local inverses $\vartheta_{\pm n} : J_{\pm n} \to \mathbb{R}$ in (2.6) respectively (2.6) of the interval map f_q on the monotonicity intervals $J_{\pm n}$, $1 \leq n \leq \infty$, respectively $2 \leq n \leq \infty$ for $q = 3$. They are given by

$$\vartheta_{\pm n}(x) := \left(f_q\big|_{J_{\pm n}} \right)^{-1}(x) = \frac{-1}{x \pm n\lambda_q} = ST^{\pm n}\, x.$$

Lemma 2.3. *The maps ϑ_n, $n \in \mathbb{Z}_{\neq 0}$, respectively $n \in \mathbb{Z}_{\neq 0,\pm 1}$ for $q = 3$, satisfy:*

1. *ϑ_n extends to a holomorphic function $\mathbb{C} \setminus \{-n\lambda_q\} \to \mathbb{C}$.*
2. *ϑ_n is strictly increasing on $(-\lambda_q, \lambda_q)$.*
3. *For $x \in (-\lambda_q, \lambda_q)$ we have $\vartheta_n(x) < \vartheta_m(x)$ for either $0 < n < m$ or $n < m < 0$ or $m < 0 < n$.*

Proof. 1. That ϑ_n extends holomorphically to $\mathbb{C} \setminus \{-n\lambda_q\}$ is obvious.

2. Since $\vartheta_n'(x) = (n\lambda_q + x)^{-2}$ the derivative ϑ_n' restricted to $(-\lambda_q, \lambda_q)$ is positive and hence ϑ_n is strictly increasing on $(-\lambda_q, \lambda_q)$.
3. Consider the three cases $0 < n < m$, $n < 0 < m$ and $n < m < 0$: since $x \in (-\lambda_q, \lambda_q)$ we find that

$$\begin{aligned} 0 < n < m &\iff 0 < n\lambda_q + x < m\lambda_q + x \\ &\iff \frac{-1}{n\lambda_q + x} < \frac{-1}{m\lambda_q + x} \\ &\iff \vartheta_n(x) < \vartheta_m(x), \end{aligned}$$

$$
\begin{aligned}
\mathbb{A}_{\pm 1_{2i},\pm 1_{2i+1}} &= 1 \quad 1 \le i \le h_q - 2,\\
\mathbb{A}_{\pm 1_{2h_q-2},\pm 1_{2h_q}} &= 1,\\
\mathbb{A}_{\pm 1_{2h_q-2},\pm 2_{\kappa_q}} &= 1,\\
\mathbb{A}_{\pm 1_{2h_q},\mp 1_i} &= 1 \quad 1 \le i \le \kappa_q - 1,\\
\mathbb{A}_{\pm 1_{2h_q},\mp 2_i} &= 1 \quad \kappa_q \le i \le \kappa_q + 1,\\
\mathbb{A}_{\pm 1_{2h_q},\mp m} &= 1 \quad m \ge 3,\\
\mathbb{A}_{\pm 1_{2i-1},\pm 1_{2i+1}} &= 1, \quad 1 \le i \le h_q - 1,\\
\mathbb{A}_{\pm 1_{2h_q-1},\pm 2_{\kappa_q+1}} &= 1,\\
\mathbb{A}_{\pm 1_{2h_q-1},\mp m} &= 1, \quad m \ge 3,\\
\mathbb{A}_{\pm 2_{\kappa_q},\mp 1_1} &= 1,\\
\mathbb{A}_{\pm 2_{\kappa_q+1},\delta 1_i} &= 1, \quad \delta = \pm, \quad 2 \le i \le \kappa_q - 1,\\
\mathbb{A}_{\pm 2_{\kappa_q+1},\delta 2_i} &= 1, \quad \delta = \pm, \quad 2 \le i \le \kappa_q - 1,\\
\mathbb{A}_{\pm 2_{\kappa_q+1},\delta n} &= 1, \quad \delta = \pm, \quad n \ge 3,\\
\mathbb{A}_{\pm 2_{\kappa_q+1},\mp 1_1} &= 1,\\
\mathbb{A}_{m,n} &= 1, \quad m \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}, \quad n \in F.
\end{aligned}
$$

TABLE 3. The nonvanishing matrix elements of the transition matrix $\mathbb{A} = (\mathbb{A}_{i,j})_{i,j \in F}$ for odd $q \ge 5$.

$$
\begin{aligned}
n < m < 0 &\iff n\lambda_q + x < m\lambda_q + x < 0\\
&\iff \frac{-1}{n\lambda_q + x} < \frac{-1}{m\lambda_q + x}\\
&\iff \vartheta_n(x) < \vartheta_m(x) \qquad \text{and}\\
m < 0 < n &\iff m\lambda_q + x < 0 < n\lambda_q + x\\
&\iff \frac{-1}{m\lambda_q + x} > \frac{-1}{n\lambda_q + x}\\
&\iff \vartheta_m(x) > \vartheta_n(x).
\end{aligned}
$$

□

3. The geodesic flow on Hecke surfaces.

Let $\mathbb{H} = \{z = x + iy \mid y > 0, x \in \mathbb{R}\}$ denote the upper half-plane equipped with the hyperbolic metric $\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$. The boundary of $\mathbb{H}$ is $\partial\mathbb{H} = \mathbb{P}_\mathbb{R} = \mathbb{R} \cup \{\infty\}$.

A geodesic γ on $\mathbb{H}$ is either a half-circle based on $\mathbb{R}$ or a line parallel to the imaginary axis. The pair of base points of γ are denoted by $\gamma_\pm \in \mathbb{P}_\mathbb{R}$ such that the geodesic flow $\Phi_t : S\mathbb{H} \to S\mathbb{H}$ along the oriented geodesic γ satisfies $\lim_{t\to\pm\infty} \phi_t = \gamma_\pm$. We identify an oriented geodesic γ on $\mathbb{H}$ with the pair of its base points (γ_-, γ_+).

3.1. A Poincaré map for $\Phi_t : S\mathcal{M}_q \to S\mathcal{M}_q$ and its Ruelle zeta function.

The *Hecke surfaces* $\mathcal{M}_q$, hyperbolic surfaces of constant negative curvature -1, are given for $q = 3, 4, \ldots$ as the quotient

$$\mathcal{M}_q = G_q \backslash \mathbb{H},$$

which we sometimes identify with the standard fundamental domain of G_q,

$$\mathcal{F}_q = \left\{ z \in \mathbb{H} \mid |z| \geq 1, |\mathrm{Re}\,(z)| \leq \tfrac{1}{2} \right\}$$

with sides pairwise identified by the generators in (3).

If $\pi : \mathbb{H} \to \mathcal{M}_q$ denotes the natural projection map $z \mapsto G_q\, z$, then the geodesic flow Φ_t on $\mathbb{H}$ projects to the geodesic flow on $\mathcal{M}_q$ which we denote by the same symbol Φ_t. The geodesic $\gamma^\star = \pi\gamma$ is a closed geodesic on $\mathcal{M}_q$ if and only if γ_+ and γ_- are hyperbolic fixed points of a hyperbolic element $g_{\gamma^*} \in G_q$. In [13] a Poincaré section Σ and a Poincaré map $\mathcal{P} : \Sigma \to \Sigma$ have been constructed for the geodesic flow on the Hecke surfaces $\mathcal{M}_q$ using λ_q-CF expansions. For this the authors construct a map $\tilde{\mathcal{P}} : \tilde{\Sigma} \to \tilde{\Sigma}$ with $\tilde{\Sigma} \subset \partial\mathcal{F}_q \times S_1$ where $\partial\mathcal{F}_q$ denotes the boundary of the standard fundamental domain (3.1) and S_1 denotes the unit circle. The induced map $\mathcal{P} : \Sigma \to \Sigma$ on the projection $\Sigma := \pi_1^\star\left(\tilde{\Sigma}\right) \subset S\mathcal{M}_q$ defines a Poincaré map for the geodesic flow on the Hecke surface $\mathcal{M}_q$.

To be more precise, let γ be a geodesic corresponding to an element $\tilde{z} \in \tilde{\Sigma}$ such that its base points $\gamma_\pm \in \mathbb{R}$ have the regular and dual regular λ_q-CF expansions

$$\gamma_- = [\![a_0; (\pm 1)^{k-1}, a_k, a_{k+1}, \ldots]\!] \quad \text{and} \quad \gamma_+ = [\![0; b_1, b_2, \ldots]\!]^\star.$$

Then $\tilde{\mathcal{P}}(\tilde{z})$ corresponds to the geodesic $g\,(\gamma_-, \gamma_+) = (g\,\gamma_-, g\,\gamma_+)$ for $g \in G_q$ such that its base points have the regular and dual regular expansions

$$g\,\gamma_- = [\![a_k; a_{k+1} \ldots]\!] \quad \text{and} \quad g\,\gamma_+ = [\![0; (\pm 1)^{k-1}, a_0, b_1, b_2, \ldots]\!]^\star$$

corresponding to a $k-$fold shift of the symbol sequences determined by the entries in the λ_q-CF 's of γ_- and γ_+. But these shifts also correspond to the action of the map f_q^k, that is

$$S\,(g\,\gamma_-) = f_q^k(S\,\gamma_-) \qquad \text{with} \qquad S\,\gamma_- = \frac{-1}{\gamma_-} \in I_q.$$

From these relations we deduce that the periodic orbits of the map $\tilde{\mathcal{P}}$ are determined by the periodic orbits of the map f_q which, in turn, are determined by the points $x \in I_q$ with a periodic regular λ_q-CFexpansion $x = [\![0; \overline{a_1, \ldots, a_n}]\!]$. The base points $\gamma_\pm$ of the corresponding closed geodesic γ are given by

$$\gamma_- = [\![a_1; \overline{a_2, \ldots, a_n, a_1}]\!] \quad \text{and} \quad \gamma_+ = [\![0; \overline{a_n, \ldots, a_1}]\!]^\star.$$

Hence it follows from [13] that there is a one-to-one relation between the orbits of the points $x \in \mathbb{Q}$ with periodic regular λ_q-CF expansion, i.e. the periodic orbits of the map $f_q : I_q \to I_q$, and the periodic orbits of the map $\tilde{\mathcal{P}}$. For the Poincaré map $\mathcal{P} : \Sigma \to \Sigma$ this relation is also bijective except for the periodic orbits under f_q of the two points $\pm r_q$, which correspond to the same periodic orbit of $\mathcal{P} : \Sigma \to \Sigma$ as shown in [12, Theorem 2.5.1]. Because of (3.1) the period of a periodic orbit $\mathcal{O}^\star$ of $\mathcal{P}$ is smaller than the one of the corresponding periodic orbit of f_q if the λ_q-CF expansion of a point $x = [\![0; \overline{a_1, \ldots, a_n}]\!]$ in the periodic orbit of f_q contains the block $[(\pm 1)^k]$ for some $k > 0$.

A *prime* periodic orbit is a periodic orbit which is not obtained by traversing a shorter orbit several times. Analogously a periodic point $x = [\![0; \overline{a_1, \ldots, a_n}]\!]$ is said to be prime if n is the shortest period length of the sequence $\overline{a_1, \ldots, a_n}$. Consider now a prime periodic orbit $\gamma^\star = (\gamma_-, \gamma_+)$ of the geodesic flow determined by the prime periodic point $x^\star = S\gamma_- = [\![0;\overline{a_1,\ldots,a_n}]\!] \in I_q$. The period $l(\gamma^\star)$ of $\gamma^\star$ is given by the well-known formula

$$l(\gamma^*) = 2\ln\lambda$$

where λ is the larger one among the two real positive eigenvalues of the (appropriately chosen) hyperbolic element $g^\star \in G_q$ whose attracting fixed point is $x^\star \in I_q$.

But $f_q^n(x^*) = x^*$ and hence f_q^n defines a hyperbolic element $g^* \in G_q$. A straightforward calculation then shows that

$$\begin{aligned} l(\gamma^*) &= \ln\left|\frac{\mathrm{d}}{\mathrm{d}x} f_q^n(x^*)\right| = \sum_{l=0}^{n-1} \ln\left|\frac{\mathrm{d}}{\mathrm{d}x} f_q(f_q^l(x^*))\right| \\ &= \sum_{l=0}^{n-1} r\left(f_q^l(x^*)\right), \end{aligned}$$

where $r(x) = \ln f_q'(x)$. Since $\tilde{\mathcal{P}}^k(\tilde{z}^*) = \tilde{z}^*$ for some $k \le n$ for $\tilde{z}^* \in \tilde{\Sigma}$ corresponding to x^* , the period $l(\gamma^*)$ can also be written as

$$l(\gamma^*) = \sum_{l=0}^{k-1} r\left(\tilde{\mathcal{P}}^l(\tilde{z}^*)\right).$$

Observe here, that $r(z^*) = \ln f_q'(x^*)$ is exactly the recurrence time function for the Poincaré map $\tilde{\mathcal{P}}$. The Ruelle zeta function $\zeta_R(s)$ for the generating map f_q of the λ_q-CF expansion is defined as

$$\zeta_R(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} Z_n(s)\right)$$

with

$$Z_n(s) = \sum_{x\in\mathrm{Fix}\, f_q^n} \exp\left(-s\sum_{k=0}^{n-1} \ln\, f_q'\big(f_q^k(x)\big)\right)$$

where we used the positivity of f_q' for real arguments. It is well-known that for $\Re s$ large enough $\zeta_R(s)$ is a holomorphic function. A prime periodic orbit

$$\mathcal{O} = \left(x, f_q(x), \ldots, f_q^{n-1}(x)\right)$$

of period n contributes to all partition functions $Z_{ln}(s)$, $l \in \mathbb{N}$.

If $Z_{\mathcal{O}}(s) = \sum_{l=1}^{\infty} \frac{1}{l}\exp(-slr_{\mathcal{O}})$ denotes the contribution of a prime orbit $\mathcal{O}$ to the Ruelle zeta function then one finds by using the Taylor expansion for $\ln(1-x)$

$$\zeta_R(s) = \exp\left(\sum_{\mathcal{O}} Z_{\mathcal{O}}(s)\right) = \exp\left(-\sum_{\mathcal{O}} \ln\left(1 - e^{-s\, r_{\mathcal{O}}}\right)\right).$$

where $r_{\mathcal{O}} = \ln\left(f_q^n\right)'(x)$ depends only on the orbit $\mathcal{O}$ and not on the specific point $x \in \mathcal{O}$. Summing over all the prime orbits $\mathcal{O}$ of f_q leads to the well-known formula [18]

$$\zeta_R(s) = \prod_{\mathcal{O}} \left(1 - e^{-sr_{\mathcal{O}}}\right)^{-1}.$$

Consider on the other hand the Ruelle zeta function for the map $\tilde{\mathcal{P}} : \tilde{\Sigma} \to \tilde{\Sigma}$. We know that the prime periodic orbits $\tilde{\mathcal{O}}$ of this map are in a $1-1$ correspondence with those of the map $f_q : I_q \to I_q$. Let $x = [\![0;\overline{a_1,\ldots,a_n}]\!]$ determine such a periodic orbit $\mathcal{O}$. Then the corresponding periodic orbit $\tilde{\mathcal{O}}$ of $\tilde{\mathcal{P}}$ is determined by the point $\tilde{z}_{\mathcal{O}} \in \tilde{\Sigma}$ where $\tilde{z}_{\mathcal{O}}$ is the intersection of $\tilde{\Sigma}$ with the geodesic γ with base points $(S\,x, -y)$, $y = [\![0; \overline{a_n, \ldots, a_1}]\!]^\star$, such that $\tilde{\mathcal{P}}^k(\tilde{z}_{\mathcal{O}}) = \tilde{z}_{\mathcal{O}}$ for some $1 \le k \le n$. Observe that

$$r_{\tilde{\mathcal{O}}} = \sum_{m=0}^{k-1} r\left(\tilde{\mathcal{P}}^m(\tilde{z}_{\mathcal{O}})\right) = \sum_{m=0}^{n-1} \ln f_q'\left(f_q^m(x)\right) = r_{\mathcal{O}}.$$

Hence the contribution $Z_{\tilde{\mathcal{O}}}(s)$ of the orbit $\tilde{\mathcal{O}}$ to the Ruelle zeta function $\zeta_R(s)$ of the map $\tilde{\mathcal{P}}$ coincides with the contribution $Z_{\mathcal{O}}(s)$ of the orbit $\mathcal{O}$ to $\zeta_R(s)$ of the map f_q. Therefore the Ruelle zeta functions for the two maps are identical.

In [13] it was shown that there is a $1-1$ correspondence between the prime periodic orbits $\tilde{\mathcal{O}}$ of the map $\tilde{\mathcal{P}} : \tilde{\Sigma} \to \tilde{\Sigma}$ and the prime periodic orbits γ of the geodesic flow on $S\,\mathcal{M}_q$ up to the two orbits $\tilde{\mathcal{O}}_\pm$ determined by the endpoints $\big(S\,(\pm r_q), \mp r_q\big)$. These two orbits coincide under the projection $\pi_q^\star : S\mathbb{H} \to S\mathcal{M}_q$. However, the contributions of both of these two orbits are contained in the Ruelle zeta function ζ_R for the map $\tilde{\mathcal{P}}$ respectively f_q. The period of the orbit $\mathcal{O}_+$ of the point r_q under the map f_q is given by κ_q defined in (2.3). Define therefore the partition function $Z_n^{\mathcal{O}_+}(s)$, $n \in \mathbb{N}$ as follows:

$$Z_n^{\mathcal{O}_+}(s) = 0 \qquad \text{for all } n \text{ with } \kappa_q \nmid n \text{ and}$$
$$Z_n^{\mathcal{O}_+}(s) = \kappa_q \exp\left(-sl \ln\left(f_q^{\kappa_q}\right)'(r_q)\right) \qquad n = \kappa_q l,\ l = 1, 2, \ldots.$$

Then one gets

$$\exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} Z_n^{\mathcal{O}_+}(s)\right) = \exp\left(-\sum_{l=1}^{\infty} \frac{1}{l} e^{-sl\,r_{\mathcal{O}_+}}\right) = 1 - e^{-s\,r_{\mathcal{O}_+}}.$$

Hence the Ruelle zeta function $\zeta_R^{\mathcal{P}}(s)$ for the Poincaré map $\mathcal{P} : \Sigma \to \Sigma$ of the geodesic flow $\Phi_t : S\mathcal{M}_q \to S\mathcal{M}_q$ has the form

$$\zeta_R^{\mathcal{P}}(s) = \prod_{\mathcal{O} \ne \mathcal{O}_+} \left(1 - e^{-s\,r_{\mathcal{O}}}\right)^{-1}.$$

3.2. **The Selberg zeta function.** The *Selberg zeta function* $Z_S(s) = Z_S^{G_q}(s)$ for the Hecke triangle group G_q is defined as

$$Z_S(s) = \prod_{k=0}^{\infty} \prod_{\gamma^\star \text{ prime}} \left(1 - e^{-(s+k)l(\gamma^\star)}\right).$$

The inner product is taken over all prime periodic orbits $\gamma^\star$ of the geodesic flow on $\mathcal{M}_q$ and $l(\gamma^\star)$ denotes the period of $\gamma^\star$ (and hence the length of the corresponding closed geodesic). It is now clear that we can write $Z_S(s)$ as

$$Z_S(s) = \prod_{k=0}^{\infty} \prod_{\tilde{\mathcal{O}} \ne \tilde{\mathcal{O}}_+} \left(1 - e^{-(s+k)r_{\mathcal{O}}}\right),$$

where the inner product is over all prime periodic orbits $\tilde{\mathcal{O}} \ne \tilde{\mathcal{O}}_+$ of the Poincaré map $\tilde{P} : \tilde{\Sigma} \to \tilde{\Sigma}$. For $\Re s > 1$ it can also be written in the form

$$Z_S(s) = \frac{\prod_{k=0}^{\infty} \prod_{\mathcal{O}} \left(1 - e^{-(s+k)r_{\mathcal{O}}}\right)}{\prod_{k=0}^{\infty} \left(1 - e^{-(s+k)r_{\mathcal{O}_+}}\right)},$$

where the product $\prod_{\mathcal{O}}$ is over all prime periodic orbits $\mathcal{O}$ of the map f_q. If the period of $\mathcal{O}$ is $\#\mathcal{O} = l$ then $r_{\mathcal{O}} = \ln\left(f_q^l\right)'(x)$ for $x \in \mathcal{O}$. Hence $r_{\mathcal{O}_+} = \ln\left(f_q^{\kappa_q}\right)'(r_q)$. Note that the zeros of the denominator all lie in the left half s-plane. We will show next how this function can be expressed in terms of the Fredholm determinant of the transfer operator for the map $f_q : I_q \to I_q$.

4. Ruelle's transfer operator for the Hurwitz-Nakada map $f_q : I_q \to I_q$.

For $g : I_q \to \mathbb{C}$ a function on the interval I_q Ruelle's transfer operator $\mathcal{L}_s$ for f_q acts on it as follows:

$$\mathcal{L}_s g(x) = \sum_{y \in f_q^{-1}(x)} e^{-s\, r(y)}\, g(y)$$

where $r(y) = \ln f_q'(y)$ and and $\mathrm{Re}\,(s) > 1$ to ensure convergence of the series. To get a more explicit form for $\mathcal{L}_s$ one has to determine the preimages y of any point $x \in I_q$. For this recall the Markov partition $I_q = \bigcup_{i \in A_{\kappa_q}} \Phi_i$, $A_{\kappa_q} = \{\pm 1, \ldots, \pm\kappa_q\}$ in (2.6), determined by the intervals Φ_i defined in (2.6) and the local inverses ϑ_n of f_q in (2.7). The Markov property of the map f_q shows that the preimages of points in Φ_i^o can be characterized by the following lemma

Lemma 4.1. *For $x \in \Phi_i^o \subset I_q$ for some $i \in A_{\kappa_q}$ the preimage $f_q^{-1}(x)$ is given by the set $f_q^{-1}(x) = \{y \in I_q : y = \vartheta_n(x),\, n \in \mathcal{N}_i\}$, where $\mathcal{N}_i = \bigcup_{j \in A_{\kappa_q}} \mathcal{N}_{i,j}$ with $\mathcal{N}_{i,j} = \{n \in \mathbb{Z} : \vartheta_n(\Phi_i) \subset \Phi_j\}$*

Proof. The preimages of a point x in the open interval Φ_i can be determined by looking at its λ_q-CF expansion $x = [\![0; a_1, a_2, \ldots]\!]$. The boundary points of the interval Φ_i are members of the orbit $\mathcal{O}(-\frac{\lambda_q}{2})$ such that for $q = 2h_q + 2$ one finds that $\Phi_i = [[\![0; 1^{h_q+1-i}]\!], [\![0; 1^{h_q-i}]\!]]$, $1 \le i \le h_q$, respectively for $q = 2h_q + 3$ one finds that $\Phi_{2i+1} = [[\![0; 1^{h_q-i}, 2, 1^{h_q}]\!], [\![0; 1^{h_q-i}]\!]]$, $1 \le i \le h_q$, and $\Phi_{2i} = [[\![0; 1^{h_q+1-i}]\!], [\![0; 1^{h_q-i}, 2, 1^{h_q}]\!]]$, $1 \le i \le h_q$. For the intervals Φ_{-i} one gets analogous expressions with negative entries in the λ_q-CF expansions. If for $q = 2h_q + 2$ one has $x \in \Phi_i^o$ then its λ_q-CF expansion must be either of the form $x = [\![0; 1^{h_q+1-i}, -m, \ldots]\!]$ for some $m \ge 1$ or of the form $x = [\![0; 1^{h_q-i}, m, \ldots]\!]$ for some $m \ge 2$. It is easy to see that the set of $n \in \mathbb{Z}$ such that $\vartheta_n(x) = [\![0; n, x]\!] \in I_q$ does not depend on x but only on the interval Φ_i. Thereby $[\![0; n, x]\!]$ denotes the concatenation of the corresponding sequences. If furthermore $\vartheta_n(x) \in \Phi_j$ for $x \in \Phi_i$ then $\vartheta_n(\Phi_i) \in \Phi_j$ and hence $\mathcal{N}_i = \bigcup_{j \in A_{\kappa_q}} \mathcal{N}_{i,j}$. The same reasoning applies to the case $q = 2h_q + 3$. □

Remark 1. We will define the operator $\mathcal{L}_s$ on a space of piecewise continuous functions, hence it is enough to determine the preimages of points in the interior of the intervals Φ_i. In general, points on the boundary of an interval Φ_i can have more preimages than those in the interior.

We are now able to determine the sets $\mathcal{N}_{i,j}$ explicitly. For this we denote by $\mathbb{Z}_{\ge n}$ respectively by $\mathbb{Z}_{\le -n}$ for $n = 1, 2, \ldots$ the sets $\mathbb{Z}_{\ge n} := \{l \in \mathbb{N} : l \ge n\}$ respectively $\mathbb{Z}_{\le -n} := \{l \in \mathbb{Z} : l \le -n\}$.

Lemma 4.2. *For $q = 2h_q + 2$ the sets $\mathcal{N}_{i,j}$ are given as follows:*

$$\begin{aligned} \mathcal{N}_{1,h_q} &= \mathbb{Z}_{\geq 2},\, \mathcal{N}_{1,-h_q} = \mathbb{Z}_{\leq -1} \\ \mathcal{N}_{i,i-1} &= \{1\},\, \mathcal{N}_{i,h_q} = \mathbb{Z}_{\geq 2},\, \mathcal{N}_{i,-h_q} = \mathbb{Z}_{\leq -1},\, 2 \leq i \leq h_q, \\ \mathcal{N}_{i,j} &= \emptyset \quad \text{for all other} \quad 1 \leq i \leq h_q, \quad \text{and all} \quad j \in A_{\kappa_q} \\ \mathcal{N}_{-i,j} &= -\mathcal{N}_{i,-j} \quad \text{for all} \quad i,j \in A_{\kappa_q}. \end{aligned}$$

For $q = 3$ the sets $\mathcal{N}_{i,j}$ are given by

$$\mathcal{N}_{1,1} = \mathbb{Z}_{\geq 3}, \mathcal{N}_{1,-1} = \mathbb{Z}_{\leq -2},$$
$$\mathcal{N}_{-1,j} = -\mathcal{N}_{1,-j}, j = \pm 1.$$

For $q = 2h_q + 3$ one has

$$\begin{aligned} \mathcal{N}_{1,2h_q} &= \{2\},\, \mathcal{N}_{1,-2h_q} = \{-1\},\, \mathcal{N}_{1,2h_q+1} = \mathbb{Z}_{\geq 3},\, \mathcal{N}_{1,-(2h_q+1)} = \mathbb{Z}_{\leq -2} \\ \mathcal{N}_{2,-2h_q} &= \{-1\},\, \mathcal{N}_{2,2h_q+1} = \mathbb{Z}_{\geq 2},\, \mathcal{N}_{2,-(2h_q+1)} = \mathbb{Z}_{\leq -2} \\ \mathcal{N}_{i,i-2} &= \{1\},\, 3 \leq i \leq \kappa_q,\, \mathcal{N}_{i,-2h_q} = \{-1\},\, 1 \leq i \leq \kappa_q, \\ \mathcal{N}_{i,2h_q+1} &= \mathbb{Z}_{\geq 2},\, \mathcal{N}_{i,-(2h_q+1)} = \mathbb{Z}_{\leq -2}. \end{aligned}$$

$\mathcal{N}_{i,j} = \emptyset$ for all other $1 \leq i \leq \kappa_q, j \in A_{\kappa_q}$ and again $\mathcal{N}_{-i,j} = -\mathcal{N}_{i,-j}$ for all $1 \leq i \leq \kappa_q$, $j \in A_{\kappa_q}$.

Proof. We will prove the case $q = 2h_q + 2$, the case of odd q is similar. If $x \in \Phi_1^o$ then either $x = [\![0; 1^{h_q}, -m, \ldots]\!]$ for some $m \geq 1$ or $x = [\![0; 1^{h_q-1}, m, \ldots]\!]$ for some $m \geq 2$. In both cases $y = [\![0; 1, x]\!] \notin I_q$ whereas $y = [\![0; -1, x]\!] \in \Phi_{-h_q}$ respectively $y = [\![0; \pm n, x]\!] \in \Phi_{\pm h_q}$ for $n \geq 2$. For $x \in \Phi_{h_q}^o$ one has $x = [\![0; 1, -m, \ldots]\!]$ for some $m \geq 1$ or $x = [\![0; m, \ldots]\!]$ for some $m \geq 2$. In this case $y = [\![0; 1, x]\!] \in \Phi_{h_q-1}$ and $y = [\![0; \pm n, x]\!] \in \Phi_{\pm h_q}$ whereas $y = [\![0; -1, x]\!] \in \Phi_{-h_q}$. For $x \in \Phi_i$, $2 \leq i \leq h_q - 1 = \kappa_q - 1$ finally one has either $x = [\![0; 1^{h_q+1-i}, -m, \ldots]\!]$ for some $m \geq 1$ or $x = [\![0; 1^{h_q-i}, m, \ldots]\!]$ for some $m \geq 2$. In both cases $y = [\![0; 1, x]\!] \in \Phi_{i-1}$ and $y = [\![0; \pm n, x]\!] \in \Phi_{\pm h_q}$ for all $n \geq 2$ whereas $y = [\![0; -1, x]\!] \in \Phi_{h_q}$. All other sets $\mathcal{N}_{i,j}$ are empty for $1 \leq i \leq h_q$ and $j \in A_{\kappa_q}$. That $\mathcal{N}_{-i,j} = -\mathcal{N}_{i,-j}$ for all $1 \leq i \leq \kappa_q$, $j \in A_{\kappa_q}$ is obvious from $\Phi_{-i} = -\Phi_i$ and the form of the maps ϑ_n. □

This Lemma allows us to derive explicit expressions for the transfer operator $\mathcal{L}_s$ for the map $f_q : I_q \to I_q$. Using the index sets $\mathcal{N}_i = \bigcup_{j \in A_{\kappa_q}} \mathcal{N}_{i,j}$ we can rewrite the transfer operator $\mathcal{L}_s$ in (4) as

$$\mathcal{L}_s g(x) = \sum_{i \in A_{\kappa_q}} \chi_{\Phi_i}(x) \sum_{n \in \mathcal{N}_i} \big(\vartheta_n'(x)\big)^s g\big(\vartheta_n(x)\big),$$

where χ_{Φ_i} is the characteristic function of the set Φ_i. If we now introduce vector valued functions $\underline{g} = (g)_{i \in A_{\kappa_q}}$ with $g_i := g|_{\Phi_i}$ then the operator $\mathcal{L}_s$ can also be written as follows

$$\begin{aligned} (\mathcal{L}_s \underline{g})_i(x) &= \sum_{j \in A_{\kappa_q}} \sum_{n \in \mathcal{N}_{i,j}} \big(\vartheta_n'(x)\big)^s g_j\big(\vartheta_n(x)\big) \\ &= \sum_{j \in A_{\kappa_q}} \sum_{n \in \mathcal{N}_{i,j}} \left(\frac{1}{z + n\lambda_q}\right)^{2s} g_j \left(\frac{-1}{z + n\lambda_q}\right), \quad x \in \Phi_i. \end{aligned}$$

If g_i is continuous on Φ_i for all $i \in A_{\kappa_q}$ then $(\mathcal{L}_s \underline{g})_i$ is also continuous on Φ_i since $\vartheta_n(\Phi_i) \subset \Phi_j$ for $n \in \mathcal{N}_{i,j}$. This implies that $\mathcal{L}_s$ is well defined on the Banach space $B = \oplus_{i \in A_{\kappa_q}} C(\Phi_i)$ of piecewise continuous functions on the intervals Φ_i. To give explicit expressions for this operator on the space B denote for $n \in \mathbb{N}$ by $\mathcal{L}^{\infty}_{\pm n,s}$ the operator

$$\mathcal{L}^{\infty}_{\pm n,s} g(x) = \sum_{l=n}^{\infty} \frac{1}{(x \pm n\lambda_q)^{2s}}\, g\left(\frac{-1}{x \pm n\lambda_q}\right),$$

and by $\mathcal{L}_{\pm n,s}$ the operator

$$\mathcal{L}_{\pm n,s} g(x) = \frac{1}{(x \pm n\lambda_q)^{2s}}\, g\left(\frac{-1}{x \pm n\lambda_q}\right).$$

Then we have

Lemma 4.3. *For $q = 3$ the operator $\mathcal{L}_s$ is given by*

$$(\mathcal{L}_s \underline{g})_1 = \mathcal{L}^{\infty}_{3,s} g_1 + \mathcal{L}^{\infty}_{-2,s} g_{-1}$$
$$(\mathcal{L}_s \underline{g})_{-1} = \mathcal{L}^{\infty}_{2,s} g_1 + \mathcal{L}^{\infty}_{-3,s} g_{-1}.$$

For $q = 2h_q + 2$ one has

$$(\mathcal{L}_s \underline{g})_1 = \mathcal{L}^{\infty}_{2,s} g_{h_q} + \mathcal{L}^{\infty}_{-1,s} g_{-h_q}$$
$$(\mathcal{L}_s \underline{g})_i = \mathcal{L}_{1,s} g_{i-1} + \mathcal{L}^{\infty}_{2,s} g_{h_q} + \mathcal{L}^{\infty}_{-1,s} g_{-h_q},\ 2 \le i \le h_q,$$

respectively

$$(\mathcal{L}_s \underline{g})_{-1} = \mathcal{L}^{\infty}_{1,s}\, g_{h_q} + \mathcal{L}^{\infty}_{-2,s}\, g_{-h_q}$$
$$(\mathcal{L}_s \underline{g})_{-i} = \mathcal{L}_{-1,s}\, g_{-(i-1)} + \mathcal{L}^{\infty}_{1,s}\, g_{h_q} + \mathcal{L}^{\infty}_{-2,s}\, g_{-h_q},\ 2 \le i \le h_q.$$

For $q = 2h_q + 3$ one has

$$(\mathcal{L}_s \underline{g})_1 = \mathcal{L}_{2,s}\, g_{2h_q} + \mathcal{L}^{\infty}_{3,s}\, g_{2h_q+1} + \mathcal{L}^{\infty}_{-2,s}\, g_{-(2h_q+1)} + \mathcal{L}_{-1,s}\, g_{-2h_q},$$
$$(\mathcal{L}_s \underline{g})_2 = \mathcal{L}^{\infty}_{2,s}\, g_{2h_q+1} + \mathcal{L}^{\infty}_{-2,s}\, g_{-(2h_q+1)} + \mathcal{L}_{-1,s}\, g_{-2h_q},$$
$$(\mathcal{L}_s \underline{g})_i = \mathcal{L}_{1,s}\, g_{(i-2)} + \mathcal{L}^{\infty}_{2,s}\, g_{2h_q+1} + \mathcal{L}^{\infty}_{-2,s}\, g_{-(2h_q+1)} + \mathcal{L}_{-1,s}\, g_{-2h_q},\ 1 \le i \le 2h_q + 1,$$

respectively

$$(\mathcal{L}_s \underline{g})_{-1} = \mathcal{L}_{1,s}\, g_{2h_q} + \mathcal{L}^{\infty}_{2,s}\, g_{2h_q+1} + \mathcal{L}^{\infty}_{-3,s}\, g_{-(2h_q+1)} + \mathcal{L}_{-2,s}\, g_{-2h_q},$$
$$(\mathcal{L}_s \underline{g})_{-2} = \mathcal{L}_{1,s}\, g_{2h_q} + \mathcal{L}^{\infty}_{2,s}\, g_{2h_q+1} + \mathcal{L}^{\infty}_{-2,s}\, g_{-(2h_q+1)},$$
$$(\mathcal{L}_s \underline{g})_{-i} = \mathcal{L}_{1,s}\, g_{2h_q} + \mathcal{L}^{\infty}_{2,s}\, g_{2h_q+1} + \mathcal{L}^{\infty}_{-2,s}\, g_{-(2h_q+1)} + \mathcal{L}_{-1,s}\, g_{2-i},\ 1 \le i \le 2h_q + 1.$$

Unfortunately, on the space of piecewise continuous functions the operator $\mathcal{L}_s$ is not of trace class. In fact, it not even compact.

Much better spectral properties however can be achieved by defining $\mathcal{L}_s$ on a Banach space $B = \oplus_{i \in A_{\kappa_q}} B(D_i)$ with $B(D_i)$ the Banach space of holomorphic functions on certain discs $D_i \subset \mathbb{C}$ with $\Phi_i \subset D_i$, $i \in A_{\kappa_q}$, continuous on $\overline{D}_i$ together with the sup norm. This is possible, since all the maps $\vartheta_{\pm m}$, $m \ge 1$ have holomorphic extensions to complex neighborhoods of I_q with the following properties:

Lemma 4.4. *There exist open discs $D_i \subset \mathbb{C}$, $i \in A_{\kappa_q}$, with $\Phi_i \subset D_i$ such that for all $n \in \mathcal{N}_{i,j}$ we have $\vartheta_n(\overline{D}_i) \subset D_j$.*

Here $\overline{D_i}$ denotes the closure of the set D_i. For the proof of the Lemma it suffices to show the existence of open intervals $I_i \subset \mathbb{R}$, $i \in A_{\kappa_q}$, which have the properties

- $\Phi_i \subset I_i$ and
- $\vartheta_n(\overline{I_i}) \subset I_j$ for all $n \in \mathcal{N}_{i,j}$.

Since the maps ϑ_n are conformal it is clear that the discs D_i with center on the real axis and intersection equal to the open intervals I_i satisfy Lemma (4.4).

Using (2.7) the two conditions on I_i can also be written as

$$ST^n \,\overline{I}_i \subset I_j \quad \text{and} \quad \Phi_i \subset I_i \quad \text{for all } n \in \mathcal{N}_{i,j} \text{ and all } \quad i,j \in A_{\kappa_q}$$

In the cases $q = 3$ and $q = 4$ we give explicit intervals fulfilling conditions (4). For the case $q \geq 5$ we first show the existence of intervals I_i satisfying the weaker condition

$$ST^n \, I_i \subset I_j \quad \text{and} \quad \Phi_i \subset I_i \quad \text{for all } n \in \mathcal{N}_{i,j}.$$

The existence of intervals I_i satisfying (4) then follows by a simple perturbation argument.

Lemma 4.5. *The intervals*

$$I_1 := \left(-1, \frac{1}{2}\right) \quad \textit{and} \quad I_{-1} := -I_1 \quad \textit{for} \quad q = 3,$$

respectively

$$I_1 := \left(-1, \frac{\lambda_q}{4}\right) \quad \textit{and} \quad I_{-1} := -I_1 \quad \textit{for} \quad q = 4,$$

satisfy Condition (4).

Proof. The above intervals $I_i, i = \pm 1$ obviously satisfy

$$\Phi_1 = \left[-\frac{\lambda_q}{2}, 0\right] \subset I_1 \qquad \text{and} \qquad \Phi_{-1} = \left[0, \frac{\lambda_q}{2}\right] \subset I_{-1},$$

since $\lambda_3 = 1$ and $\lambda_4 = \sqrt{2}$. For $q = 3$ one has $\mathcal{N}_{1,1} = \mathbb{Z}_{\geq 3}$ and $\mathcal{N}_{1,-1} = \mathbb{Z}_{\leq -2}$ respectively $\mathcal{N}_{-1,-1} = -\mathbb{Z}_{\geq 3}$ and $\mathcal{N}_{-1,1} = -\mathbb{Z}_{\leq -2}$. Hence we have to show that $\theta_n(\overline{I_1}) \subset I_1$ for all $n \geq 3$ and $\theta_n(\overline{I_1}) \subset I_{-1}$ for all $n \leq -2$. Since all the maps are strictly increasing it is enough to show $\theta_n(-1) > -1$ and $\theta_n(\frac{1}{2}) < \frac{1}{2}$ for all $n \geq 3$ respectively $\theta_n(-1) > -\frac{1}{2}$ and $\theta_n(\frac{1}{2}) < 1$ for all $n \leq -2$. But $\theta_n(-1) = \frac{-1}{-1+n} \geq \frac{-1}{2} > -1$ and $\theta_{-n}(\frac{1}{2}) = \frac{-1}{\frac{1}{2}-n} = \frac{1}{n-\frac{1}{2}} \leq \frac{2}{3} < 1$ for all $n \geq 2$. Since $\mathcal{N}_{-i,j} = -\mathcal{N}_{i,-j}$ and $I_{-1} = -I_1$ the result for the interval I_{-1} follows .

Consider now the case $q = 4$. Then one has $\mathcal{N}_{1,1} = \mathbb{Z}_{\geq 2}$ and $\mathcal{N}_{1,-1} = \mathbb{Z}_{\leq -1}$ respectively $\mathcal{N}_{-1,-1} = -\mathbb{Z}_{\geq 2}$ and $\mathcal{N}_{-1,1} = -\mathbb{Z}_{\leq -1}$. Hence one finds that $\theta_n(-1) = \frac{-1}{-1+n\lambda_4} \geq \frac{-1}{-1+2\lambda_4} > -1$ since $2\lambda_4 = 2\sqrt{2} > 2$ and $\theta_n(\frac{\lambda_4}{4}) = \frac{-1}{\frac{\lambda_4}{4}+n\lambda_4} < 0 < \frac{\lambda_4}{4}$ for all $n \geq 2$. Furthermore $\theta_{-n}(-1) = \frac{-1}{-1-n\lambda_4} \geq 0 > -\frac{\lambda_4}{4}$ and $\theta_{-n}(\frac{\lambda_4}{4}) = \frac{-1}{\frac{\lambda_4}{4}-n\lambda_4} \leq \frac{\lambda_4}{4} - \frac{\lambda_4}{4} = \frac{4}{3\lambda_4} < 1$, since $3\sqrt{2} > 4$ for all $n \geq 1$. Since $\mathcal{N}_{-i,j} = -\mathcal{N}_{i,-j}$ and $D_i = -D_{-i}$ the Lemma is proved. □

Proof of Lemma 4.4 for $q = 3, 4$. The maps ϑ_n are all conformal, which means that they preserve angles and generalized circles. It is now easy to see that the discs $D_{\pm 1}$ which have diameters along the intervals $I_{\pm 1}$ satisfy the conditions of the lemma for $q = 3$ and $q = 4$. □

To prove Lemma 4.4 for $q \geq 5$ we need several Lemmas.

Lemma 4.6. *For* $q = 2h_q + 2, h_q \geq 2$ *and* $0 \leq i \leq h_q$

$$(ST)^{h_q - i} \left(-\frac{\lambda_q}{2}\right) = [\![-1; (-1)^i]\!], \qquad i = 0, \dots, h_q$$

and

$$-\lambda_q = [\![-1;]\!] < [\![-1;(-1)^1]\!] < \dots$$
$$< [\![-1;(-1)^{h_q-1}]\!] < [\![-1;(-1)^{h_q}]\!] = -\frac{\lambda_q}{2}.$$

Proof. By (2.1), (2.2) and (2.3) we have

$$\begin{aligned}(ST)^{h_q-i}\left(-\frac{\lambda}{2}\right) &= (ST)^{h_q-i}(ST)^{h_q}\,0 = (ST)^{-i-2}\,0\\ &= (T^{-1}S)^i\,T^{-1}\,ST^{-1}S\,0 = T^{-1}\,(ST^{-1})^i\,0\\ &= [\![-1;(-1)^i]\!].\end{aligned}$$

This shows relation (4.6). By definition it is clear that $-\lambda_q = [\![-1;]\!]$ and then inequalities (4.6) follow immediately from §2.4. □

Next we can prove

Lemma 4.7. *For $q = 2h_q + 2, h_q \geq 2$ define the intervals I_i respectively I_{-i} for $1 \leq i \leq h_q$ by $I_i := \left([\![-1;(-1)^i]\!], \frac{\lambda_q}{4}\right)$ and $I_{-i} := -I_i$. Then*

$$\begin{aligned}&\vartheta_{\pm n}(\overline{I}_{\pm i}) \subset I_{\pm h_q} \quad \text{for all } n \geq 2,\ i = 1, \dots, h_q,\\ &\vartheta_{\pm n}(\overline{I}_{\mp i}) \subset I_{\pm h_q} \quad \text{for all } n \geq 1,\ i = 1, \dots, h_q \quad \text{and}\\ &\vartheta_{\pm 1}(I_{\pm i}) \subset I_{\pm i-1} \quad \text{for all } i = 2, \dots, h_q.\end{aligned}$$

Hence $\vartheta_n(I_i) \subset I_j$ for all $n \in \mathcal{N}_{i,j}$.

Proof. Since $\mathcal{N}_{i,h_q} = \mathbb{Z}_{\geq 2}$ for all $1 \leq i \leq h_q$ we have to show that $\vartheta_n(\overline{I}_i) \subset I_{h_q}$ for all $1 \leq i \leq h_q$ and all $n \geq 2$. But $\vartheta_n([\![-1;(-1)^i]\!]) = [\![0;n-1,(-1)^i]\!] > -\frac{\lambda_q}{2}$ by 2.2.

On the other hand $\vartheta_n(\frac{\lambda_q}{4}) = \frac{-1}{n\lambda_q + \frac{\lambda_q}{4}} < 0 < \frac{\lambda_q}{4}$ and hence $\vartheta_n(\overline{I}_i) \subset I_{h_q}$. Consider next the case $\mathcal{N}_{i,-h_q} = \mathbb{Z}_{\leq -1}$ for $1 \leq i \leq h_q$. There one has $\vartheta_{-n}([\![-1;(-1)^i]\!]) = [\![0;-n-1,(-1)^i]\!] > 0 > -\frac{\lambda_q}{4}$. Furthermore one finds for all $n \geq 1$ that $\vartheta_{-n}(\frac{\lambda_q}{4}) = \frac{-1}{-n\lambda_q + \frac{\lambda_q}{4}} \leq \frac{4}{3\lambda_q} < \frac{\lambda_q}{2}$ since for $q \geq 6$ one has $\lambda_q \geq \sqrt{3}$. Hence $\vartheta_{-n}(\overline{I}_i) \subset I_{-h_q}$ for all $n \geq 1$. Consider finally the case $\mathcal{N}_{i,i-1} = \{1\}$ for $2 \leq i \leq h_q$. In this case one finds $\vartheta_1([\![-1;(-1)^i]\!]) = \frac{-1}{\lambda_q + [\![-1;(-1)^i]\!]} = [\![-1;(-1)^{i-1}]\!]$. Furthermore $\vartheta_1(\frac{\lambda_q}{4}) = \frac{-1}{\lambda_q + \frac{\lambda_q}{4}} < 0 < \frac{\lambda_q}{4}$ and hence $\vartheta_1(I_i) \subset I_{i-1}$ for all $2 \leq i \leq h_q$. The intervals I_{-i} have again analogous properties. □

For odd $q \geq 5$ we need

Lemma 4.8. *For $q = 2h_q + 3, h_q \geq 1$ one has*

$$\begin{aligned}(ST)^{h_q-i}\left(-\frac{\lambda_q}{2}\right) &= [\![-1;(-1)^i,-2,(-1)^{h_q}]\!] \qquad \text{for} \quad 0 \leq i \leq h_q,\\ (ST)^{h_q+1-i}\left([\![0;1^{h_q}]\!]\right) &= [\![-1;(-1)^i]\!] \qquad \text{for} \quad 1 \leq i \leq h_q\end{aligned}$$

and

$$\begin{aligned}-\lambda_q &= [\![-1;]\!] < [\![-1;-2,(-1)^{h_q}]\!] < [\![-1;-1]\!] < [\![-1;(-1)^1,-2,(-1)^{h_q}]\!] \\ &< [\![-1;(-1)^2]\!] < [\![-1;(-1)^2,-2,(-1)^{h_q}]\!] < [\![-1;(-1)^3]\!] < \dots \\ &< [\![-1;(-1)^{h_q-1},-2,(-1)^{h_q}]\!] < [\![-1;(-1)^{h_q}]\!] < [\![-1;(-1)^{h_q},-2,(-1)^{h_q}]\!] \\ &= -\frac{\lambda_q}{2}.\end{aligned}$$

Proof. By (2.1), (2.2) and (2.3) we have

$$\begin{aligned}(ST)^{h_q-i}\left(-\frac{\lambda}{2}\right) &= (ST)^{h_q-i}\,(ST)^{h_q}\,ST^2\,(ST)^{h_q}\,0 \\ &= (ST)^{-i-3}\,ST^2\,(ST)^{h_q}\,0 \\ &= T^{-1}\,(ST^{-1})^i\,ST^{-1}\,ST^{-1}S\,ST^2\,(ST)^{h_q}\,0 \\ &= T^{-1}\,(ST^{-1})^i\,ST^{-1}\,(ST)^{h_q+1}\,0 \\ &= T^{-1}\,(ST^{-1})^i\,ST^{-1}\,(ST)^{-h_q-2}\,0 \\ &= T^{-1}\,(ST^{-1})^i\,ST^{-2}\,(ST^{-1})^{h_q}\,ST^{-1}S\,0 \\ &= T^{-1}\,(ST^{-1})^i\,ST^{-2}\,(ST^{-1})^{h_q}\,0 \\ &= [\![-1;(-1)^i,-2,(-1)^{h_q}]\!].\end{aligned}$$

Similarly, we have

$$\begin{aligned}(ST)^{h_q+1-i}\left([\![0;1^{h_q}]\!]\right) &= (ST)^{h_q+1-i}\,(ST)^{h_q}\,0 \\ &= (ST)^{-i-2}\,0 \\ &= T^{-1}\,(ST^{-1})^i\,ST^{-1}S\,0 \\ &= T^{-1}\,(ST^{-1})^i\,0 \\ &= [\![-1;(-1)^i]\!].\end{aligned}$$

This shows relation (4.8).

The lexicographic ordering in §2.4 implies

$$\begin{aligned}&[\![-1;\,]\!] < [\![-1;-2,(-1)^{h_q}]\!] < [\![-1;-1]\!] < [\![-1;-1,-2,(-1)^{h_q}]\!] < \dots \\ &\qquad\dots < [\![-1;(-1)^{h_q}]\!] < [\![-1;(-1)^{h_q},2,(-1)^{h_q}]\!] = -\frac{\lambda_q}{2}.\end{aligned}$$

Using the identities in (4.8) shows (4.8). □

Now we can prove the following lemma for odd q

Lemma 4.9. *For $q = 2h_q + 3$, $h_q \geq 1$ define the intervals*

$$\begin{aligned}I_{2i+1} &= \left([\![-1;(-1)^i,-2,(-1)^{h_q}]\!],\frac{\lambda_q}{4}\right) \quad \textit{for} \quad 0 \leq i \leq h_q, \\ I_{2i} &= \left([\![-1;(-1)^i]\!],\frac{\lambda_q}{4}\right) \quad \textit{for} \quad 1 \leq i \leq h_q.\end{aligned}$$

respectively $I_{-i} := -I_i$, $1 \leq i \leq \kappa_q = 2h_q + 1$. Then $\Phi_i \subset I_i$ for all $1 \leq i \leq \kappa_q$ and $\vartheta_n(\overline{I}_i) \subset I_j$ for all $n \in \mathcal{N}_{i,j}$, $(i,j) \neq (\pm k, \pm(k-2))$, $3 \leq k \leq 2h_q + 1$, whereas $\vartheta_n(I_{\pm k}) \subset I_{\pm(k-2)}$ for all $n \in \mathcal{N}_{\pm k,\pm(k-2)}$, $3 \leq k \leq 2h_q + 1$.

Proof. Since the proof of this Lemma proceeds along the same lines as in the case of even q, we restrict ourselves to the case where $\vartheta_n(I_i) \subset I_j$ for $n \in \mathcal{N}_{i,j}$. This happens only for the pairs $(i,j) = (\pm k, \pm(k-2))$, where $\mathcal{N}_{i,j} = \{1\}$ respectively $\mathcal{N}_{i,j} = \{-1\}$. In these cases one finds indeed that $\vartheta_1([\![-1;(-1)^i,-2,(-1)^{h_q}]\!]) = [\![-1;(-1)^{i-1},-2,(-1)^{h_q}]\!]$ respectively $\vartheta_1([\![-1;(-1)^i]\!]) = [\![-1;(-1)^{i-1}]\!]$. Hence the left boundary point of these intervals is mapped onto the left boundary point of the image interval. The case of negative indices (i,j) follows again from the symmetry of the intervals and the sets $\mathcal{N}_{i,j}$. □

To prove finally Lemma 4.4 one has to enlarge the intervals I_i a little bit so that $\vartheta_n(\bar{I}_i) \subset I_j$ for all $n \in \mathcal{N}_{i,j}$. In the case $q = 2h_q + 2$, $h_q \geq 2$ one can take the intervals $I_i = -I_{-i} = ([\![-1;(-1)^i, n_i]\!], \frac{\lambda_q}{4})$ with $n_i > n_{i-1}$ for $2 \leq i \leq h_q$ and n_1 large enough. In the case $q = 2h_q + 3$, $h_q \geq 1$ one can choose the intervals

$$I_{2i+1} = -I_{-2i-1} = \big([\![-1;(-1)^i,-2,(-1)^{h_q},n_{2i+1}]\!], \frac{\lambda_q}{4}\big) \quad \text{for} \quad 0 \leq i \leq h_q,$$
$$I_{2i} = -I_{-2i} = \big([\![-1;(-1)^i,n_{2i}]\!], \frac{\lambda_q}{4}\big) \quad \text{for} \quad 1 \leq i \leq h_q.$$

with $n_{2i+1} > n_{2i} > n_{2i-1} > n_{2i-2}$ for all $1 \leq i \leq h_q$ and n_1 large enough.

The existence of the discs $D_i, i \in A_{\kappa_q}$, of Lemma 4.4 shows that the operator $\mathcal{L}_s$ is well defined on the Banach space $B = \oplus_{i \in A_{\kappa_q}} B(D_i)$ with $B(D_i)$ the Banach space of holomorphic functions on the disc D_i with $\Phi_i \subset D_i$, $i \in A_{\kappa_q}$, with the sup norm. In fact we have the following theorem.

Theorem 4.10. *The operator $\mathcal{L}_s : B \to B$ is nuclear of order zero for $\Re s > \frac{1}{2}$ and extends to a meromorphic family of nuclear operators of order zero with poles only at the points $s_k = \frac{1-k}{2}$, $k = 0,1,2\ldots$.*

Proof. It is now easy to verify that the operator $\mathcal{L}_s$ can be written as a $2\kappa_q \times 2\kappa_q$ matrix operator which for even q has the form

$$\mathcal{L}_s = \begin{pmatrix}
0 & 0 & \ldots & 0 & \mathcal{L}^\infty_{2,s} & \mathcal{L}^\infty_{-1,s} & 0 & \ldots & 0 & 0 \\
\mathcal{L}_{1,s} & 0 & \ldots & 0 & \mathcal{L}^\infty_{2,s} & \mathcal{L}^\infty_{-1,s} & 0 & \ldots & 0 & 0 \\
0 & \mathcal{L}_{1,s} & 0 & \ldots & \mathcal{L}^\infty_{2,s} & \mathcal{L}^\infty_{-1,s} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & \mathcal{L}_{1,s} & \mathcal{L}^\infty_{2,s} & \mathcal{L}^\infty_{-1,s} & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \mathcal{L}^\infty_{1,s} & \mathcal{L}^\infty_{-2,s} & \mathcal{L}_{-1,s} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \mathcal{L}^\infty_{1,s} & \mathcal{L}^\infty_{-2,s} & \ldots & 0 & \mathcal{L}_{-1,s} & 0 \\
0 & 0 & \ldots & 0 & \mathcal{L}^\infty_{1,s} & \mathcal{L}^\infty_{-2,s} & 0 & \ldots & 0 & \mathcal{L}_{-1,s} \\
0 & 0 & \ldots & 0 & \mathcal{L}^\infty_{1,s} & \mathcal{L}^\infty_{-2,s} & 0 & \ldots & 0 & 0
\end{pmatrix}.$$

For odd q it has the form

$$\mathcal{L}_s = \begin{pmatrix} 0 & \dots & 0 & \mathcal{L}_{2,s} & \mathcal{L}^\infty_{3,s} & \mathcal{L}^\infty_{-2,s} & \mathcal{L}_{-1,s} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \mathcal{L}^\infty_{2,s} & \mathcal{L}^\infty_{-2,s} & \mathcal{L}_{-1,s} & 0 & \dots & 0 \\ \mathcal{L}_{1,s} & \ddots & 0 & 0 & \mathcal{L}^\infty_{2,s} & \mathcal{L}^\infty_{-2,s} & \mathcal{L}_{-1,s} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \mathcal{L}_{1,s} & 0 & \mathcal{L}^\infty_{2,s} & \mathcal{L}^\infty_{-2,s} & \mathcal{L}_{-1,s} & 0 & \dots & 0 \\ 0 & \dots & 0 & \mathcal{L}_{1,s} & \mathcal{L}^\infty_{2,s} & \mathcal{L}^\infty_{-2,s} & 0 & \mathcal{L}_{-1,s} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mathcal{L}_{1,s} & \mathcal{L}^\infty_{2,s} & \mathcal{L}^\infty_{-2,s} & 0 & 0 & \ddots & \mathcal{L}_{-1,s} \\ 0 & \dots & 0 & \mathcal{L}_{1,s} & \mathcal{L}^\infty_{2,s} & \mathcal{L}^\infty_{-2,s} & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \mathcal{L}_{1,s} & \mathcal{L}^\infty_{2,s} & \mathcal{L}^\infty_{-3,s} & \mathcal{L}_{-2,s} & 0 & \dots & 0 \end{pmatrix}$$

with

$$\mathcal{L}^\infty_{\pm l,s} f(z) = \sum_{n=l}^{\infty} (\frac{1}{z \pm n\lambda_q})^{2s} f(\frac{-1}{z \pm n\lambda_q}) \text{ for } l = 1,2,3,$$

respectively

$$\mathcal{L}_{l,s} f(z) = (\frac{1}{z + l\lambda_q})^{2s} f(\frac{-1}{z + l\lambda_q}) \text{ for } l = \pm 1, \pm 2.$$

As for the transfer operator of the Gauss map (cf. [10]) one shows that the operators $\mathcal{L}^\infty_{l,s}$, $l = \pm 1, \pm 2, \pm 3$ define meromorphic families of nuclear operators $\mathcal{L}^\infty_{l,s} : B(D_i) \to B(D_j)$ in the Banach spaces of holomorphic functions on the discs for which $\mathcal{N}_{i,j} = \mathbb{Z}_{\geq l}$ respectively $\mathcal{N}_{i,j} = \mathbb{Z}_{\leq -l}$ for $l = 1,2,3$. These operators have poles at the points $s = s_k = \frac{1-k}{2}$ for $k = 0, 1, \cdots$. On the other hand the operators $\mathcal{L}_{l,s}$, $l = \pm 1, \pm 2$ with $\mathcal{L}_{l,s} : B(D_i) \to B(D_j)$ are holomorphic nuclear operators in the entire s-plane in the corresponding Banach spaces of holomorphic functions on the discs for which $\mathcal{N}_{i,j} = \{\pm l\}$, $l = 1, 2$. Hence the operator $\mathcal{L}_s$ has these properties in the Banach space $B = \oplus_{i \in A_{\kappa_q}} B(D_i)$. □

5. The reduced transfer operators $\mathcal{L}_{s,\epsilon}$, $\epsilon = \pm 1$ and functional equations for their eigenfunctions.

5.1. The symmetry operator $P : B \to B$.

From the above matrix representation of the transfer operator $\mathcal{L}_s$ it can be seen that this operator has a certain symmetry which we will discuss next. For this purpose, define the operator $P : B \to B$ as

$$(P\underline{f})_i(z) := f_{-i}(-z) \quad \text{for} \quad \underline{f} = (f_i)_{i \in A_{\kappa_q}}.$$

This operator is well-defined, since $D_{-i} = -D_i$ for all $i \in A_{\kappa_q}$ and $P^2 = id_B$. That P is indeed a symmetry follows from the following lemma.

Lemma 5.1. *The operators $P : B \to B$ and $\mathcal{L}_s : B \to B$ commute for all $s \in \mathbb{C}, s \neq s_k$, $k = 0, 1, 2, \cdots$.*

Proof. Let $\Re s > \frac{1}{2}$ and suppose that $\underline{f} \in B$. To extend ϑ'_n to the complex discs D_i. we use the convention $(n+z)^{2s} := ((n+z)^2)^s$ It is then easy to see that $\mathcal{L}_{l,s}(Pf)_i(z) = \sum_{n \geq l} \left(\frac{1}{z+n\lambda_q}\right)^{2s} f_{-i}\left(\frac{1}{z+n\lambda_q}\right) = \sum_{n \geq l} \left(\frac{1}{-z-n\lambda_q}\right)^{2s} f_{-i}\left(\frac{-1}{-z-n\lambda_q}\right) = \mathcal{L}_{-l,s}(f_{-i})(-z)$. for any positive integer l. From the form of the matrices in the proof of Theorem 4.10 it follows that the matrix elements of $\mathcal{L}_s$ satisfy the identities: $(\mathcal{L}_s)_{i,j} = \mathcal{L}_{l,s}$ if and only if $(\mathcal{L}_s)_{-i,-j} = \mathcal{L}_{-l,s}$ respectively $(\mathcal{L}_s)_{i,j} = \mathcal{L}^\infty_{l,s}$

if and only if $(\mathcal{L}_s)_{-i,-j} = \mathcal{L}^{\infty}_{-l,s}$. Combining these two observations, the fact that $\mathcal{L}_s P \underline{f}(z) = P\mathcal{L}_s \underline{f}(z)$ follows immediately. Since both the operators $P\mathcal{L}_s$ and $\mathcal{L}_s P$ are meromorphic in the entire s-plane with only poles at the points $s = s_k$, $k = 0, 1, \dots$ they coincide there. □

This allows us to restrict the operator $\mathcal{L}_s$ to the eigenspaces of the operator P which is an involution and therefore has the eigenvalues ± 1. Denote these eigenspaces by $B_\pm$. Then $\underline{f} = (f_i)_{i\in A_{\kappa_q}} \in B_\pm$ if and only if $f_{-i}(z) = \pm f_i(-z)$ for $i \in A_{\kappa_q}$. Let B_{κ_q} denote the Banach space $B_{\kappa_q} = \oplus_{1\le i\le \kappa_q} B(D_i)$ with the discs as defined earlier in Lemma 4.4. Then the transfer operator $\mathcal{L}_s$ restricted to the spaces $B_\pm$ induces the following operators $\mathcal{L}_{s,\pm}$ in the Banach space B_{κ_q}. For $q = 2h_q + 2$, $\kappa_q = h_q$ and $\overrightarrow{g} = (g_i)_{1\le i\le \kappa_q}$ we get:

$$\begin{aligned}
&(\mathcal{L}_{s,\pm}\overrightarrow{g})_1(z) = L^{\infty}_{2,s}\, g_{h_q}(z) \pm L^{\infty}_{-1,s}\, g_{h_q}(z)\\
&(\mathcal{L}_{s,\pm}\overrightarrow{g})_i\,(z) = L_{1,s}\, g_{i-1}(z) + L^{\infty}_{2,s}\, g_{h_q}(z) \pm L^{\infty}_{-1,s}\, g_{h_q}(z),\ 2 \le i \le h_q.
\end{aligned} \tag{8}$$

For $q = 3$ on the other hand we get

$$(\mathcal{L}_{s,\pm}\overrightarrow{g})_1(z) = L^{\infty}_{3,s}\, g_1(z) \pm L^{\infty}_{-2,s}\, g_1(z),$$

respectively for $q = 2h_q + 3 > 5,\ \kappa_q = 2h_q + 1$

$$\begin{aligned}
(\mathcal{L}_{s,\pm}\overrightarrow{g})_1(z) =& L_{2,s}\, g_{2h_q}(z) + L^{\infty}_{3,s}\, g_{\kappa_q}(z) \pm L_{-1,s}\, g_{2h_q}(z) \pm L^{\infty}_{-2,s}\, g_{\kappa_q}(z)\\
(\mathcal{L}_{s,\pm}\overrightarrow{g})_2(z) =& L^{\infty}_{2,s}\, g_{\kappa_q}(z) \pm L_{-1,s}\, g_{2h_q}(z) \pm L^{\infty}_{-2,s}\, g_{\kappa_q}(z)\\
(\mathcal{L}_{s,\pm}\overrightarrow{g})_i(z) =& L_{1,s}\, g_{i-2}(z) + L^{\infty}_{2,s}\, g_{\kappa_q}(z) \pm L_{-1,s}\, g_{2h_q}(z) \pm L^{\infty}_{-2,s}\, g_{\kappa_q}(z),\\
& 3 \le i \le \kappa_q.
\end{aligned} \tag{9}$$

For $i > 0$ the operators $L^{\infty}_{i,s}$ and $L_{i,s}$ coincide with the operators $\mathcal{L}^{\infty}_{i,s}$ and $\mathcal{L}_{i,s}$, whereas $L^{\infty}_{-i,s} g(z) = \sum_{n=i}^{\infty} \frac{1}{(z-n\lambda_q)^{2s}} g(\frac{1}{z-n\lambda_q})$ and $L_{-i,s} g(z) = \frac{1}{(z-i\lambda_q)^{2s}} g(\frac{1}{z-i\lambda_q})$.

5.2. **Functional equations.** It is well-known that for modular groups, i.e. finite index subgroups of $G_3 = PSL(2,\mathbb{Z})$, the eigenfunctions of the transfer operator $\mathcal{L}_s$ with eigenvalue $\rho = 1$ fulfill simple finite term functional equations [7], so called Lewis equations, which are closely related to the period functions of Lewis and Zagier [9] for these groups. In the present case we can also derive such functional equations, but it is not clear how their holomorphic solutions are related to the period functions of the Hecke triangle groups G_q for arbitrary q. In the case $q = 3$ it was shown in [2] that the solutions of the functional equation derived from our transfer operator $\mathcal{L}_s$ for $\Re s = \frac{1}{2}$ are indeed in one to one correspondence with the Maass cusp forms for G_3. Since the spectrum of the operator $\mathcal{L}_s$ is the union of the spectra of the two operators $\mathcal{L}_{s,\epsilon}$, $\epsilon = \pm 1$, we use these two operators to derive the corresponding functional equations. In the case $q = 3$ their eigenfunctions $\overrightarrow{g} = (g_1)$ with eigenvalue $\rho = 1$ obey the equation

$$g_1 = g_1|(\mathbb{N}_3 + \epsilon\mathbb{N}_{-2}) \tag{10}$$

where we used the so-called slash-action defined by

$$g_1|\mathbb{N}_k(z) = g_1|\sum_{l=k}^{\infty} ST^l := \sum_{l=k}^{\infty}\left(\frac{1}{z+l\lambda_q}\right)^{2s} g_1\left(\frac{-1}{z+l\lambda_q}\right),\ k \ge 1,$$

respectively

$$g_1|\mathbb{N}_{-k}(z) = g_1|\sum_{l=k}^{\infty} \tilde{S}T^{-l} := \sum_{l=k}^{\infty} \left(\frac{1}{z-l\lambda_q}\right)^{2s} g_1\left(\frac{1}{z-l\lambda_q}\right),\ k\geq 1,$$

where $Tz = z+\lambda_q$, $Sz = \frac{-1}{z}$ and $\tilde{S}z = \frac{1}{z}$. This action is similar to the usual slash-action of weight s but we extended it in the natural way to the group algebra of G_q over $\mathbb{C}$. One now sees that $g_1|\mathbb{N}_3(1-T) = g_1|ST^3$ and $g_1|\mathbb{N}_{-2}(1-T) = -g_1|\tilde{S}T^{-1}$, which leads to the following four term functional equation

$$g_1|(1-T) = g_1|(ST^3 - \epsilon\tilde{S}T^{-1}),$$

or explicitly,

$$g_1(z) = g_1(z+1) + \left(\frac{1}{z+3}\right)^{2s} g_1\left(\frac{-1}{z+3}\right) - \epsilon\left(\frac{1}{z-1}\right)^{2s} g_1\left(\frac{1}{z-1}\right). \tag{11}$$

An easy calculation shows that every solution of eq. (10) satisfies the equation $g_1(z) = \epsilon g_1(-z-1)$. Therefore only solutions g_1 of eq.(11) with this property lead to eigenfunctions of the transfer operator. But then g_1 obeys also the four term functional equation studied in [2]

$$g_1(z) = g_1(z+1) + \left(\frac{1}{z+3}\right)^{2s} g_1\left(\frac{-1}{z+3}\right) - \left(\frac{1}{z-1}\right)^{2s} g_1\left(\frac{-z}{z-1}\right). \tag{12}$$

On the other hand, every solution g_1 of eq. (12) with $g_1(z) = \epsilon g_1(-z-1)$ is also a solution of eq. (11).

For $q = 2h_q+2$, $h_q \geq 1$, on the other hand we find that for an eigenfunction $\overrightarrow{g} = (g_i)_{1\leq i\leq h_q}$ $g_1 = g_{h_q}|(\mathbb{N}_2 + \epsilon\mathbb{N}_{-1})$ and by induction on i:

$$g_i = g_1|P_{i-1}(ST),\ 2\leq i\leq h_q$$

where $g|P_i(g)$ for $g\in G_q$ is an abbreviation for $g|P_i(g) = g|\sum_{l=0}^{i} g^l$. Hence the function g_1 fulfills the equation

$$g_1 = g_1|P_{h_q-1}(ST)(\mathbb{N}_2 + \epsilon\mathbb{N}_{-1}).$$

But $|\mathbb{N}_2(1-T) = |ST^2$ and $|\mathbb{N}_{-1}(1-T) = |-\tilde{S}$ leading to the q-term functional equation

$$g_1|(1-T) = g_1|P_{h_q-1}(ST)(ST^2 - \epsilon\tilde{S}),$$

which for $q=4$ reads explicitly as

$$g_1(z) = g_1(z+\lambda_4) + \left(\frac{1}{z+2\lambda_4}\right)^{2s} g_1\left(\frac{-1}{z+2\lambda_4}\right) - \epsilon\left(\frac{1}{z}\right)^{2s} g_1\left(\frac{1}{z}\right).$$

For $q=6$ corresponding to $h_q = 2$ one finds that

$$\begin{aligned} g_1(z) &= g_1(z+\lambda_6) + \left(\frac{1}{z+2\lambda_6}\right)^{2s} g_1\left(\frac{-1}{z+2\lambda_6}\right) - \epsilon\left(\frac{1}{z}\right)^{2s} g_1\left(\frac{1}{z}\right) \\ &+ \left(\frac{1}{-\lambda_6 z+1-2\lambda_6^2}\right)^{2s} g_1\left(\frac{z+2\lambda_6}{-\lambda_6 z+1-2\lambda_6^2}\right) - \epsilon\left(\frac{1}{1+\lambda_6 z}\right)^{2s} g_1\left(\frac{-z}{1+\lambda_6 z}\right). \end{aligned}$$

For $q = 2h_q+3$, $h_q\geq 1$ one finds that

$$g_1 = g_{2h_q}|ST^2 + g_{2h_q+1}|\mathbb{N}_3 + \epsilon g_{2h_q}|\tilde{S}T^{-1} + \epsilon g_{2h_q+1}|\mathbb{N}_{-2}$$

respectively

$$g_2 = g_{2h_q+1}|\mathbb{N}_2 + \epsilon g_{2h_q}|\tilde{S}T^{-1} + \epsilon g_{2h_q+1}|\mathbb{N}_{-2}$$

and hence

$$g_1 = g_2 + g_{2h_q}|ST^2 - g_{2h_q+1}|ST^2. \tag{13}$$

Induction on i shows furthermore

$$g_{2i} = g_2|P_{i-1}(ST),\ 1 \le i \le h_q,$$

respectively

$$g_{2i+1} = g_1|(ST)^i + g_2|P_{i-1}(ST),\ 1 \le i \le h_q.$$

Therefore

$$g_{2h_q} = g_2|P_{h_q-1}(ST) \quad \text{and} \quad g_{2h_q+1} = g_1|(ST)^{h_q} + g_2|P_{h_q-1}(ST).$$

Inserting this into equation (13) shows that

$$g_2 = g_1 + g_1|(ST)^{h_q+1}T.$$

This allows to express both g_{2h_q} and g_{2h_q+1} in terms of g_1:

$$g_{2h_q} = g_1|\left(\mathbf{1} + (ST)^{h_q+1}T\right) P_{h_q-1}(ST)$$

respectively

$$g_{2h_q+1} = g_1|(ST)^{h_q} + \left(g_1 + g_1|(ST)^{h_q+1}T\right) P_{h_q-1}(ST).$$

Inserting these expressions into equation (13) gives finally the following functional equation for g_1:

$$\begin{aligned} g_1 &= g_1|P_{h_q-1}(ST)ST^2 + g_1|(ST)^{h_q+1}TP_{h_q-1}(ST)ST^2 + g_1|(ST)^{h_q}\mathbb{N}_3 \\ &\quad + g_1|P_{h_q-1}(ST)\mathbb{N}_3 + g_1|(ST)^{h_q+1}TP_{h_q-1}(ST)\mathbb{N}_3 \\ &\quad + \epsilon(g_1|P_{h_q-1}(ST)\tilde{S}T^{-1} + g_1|(ST)^{h_q+1}TP_{h_q-1}(ST)\tilde{S}T^{-1} + g_1|(ST)^{h_q}\mathbb{N}_{-2} \\ &\quad + g_1|(ST)^{h_q+1}TP_{h_q-1}(ST)\mathbb{N}_{-2} + g_1|P_{h_q-1}(ST)\mathbb{N}_{-2}).. \end{aligned}$$

Since $|\mathbb{N}_3(1-T) = |ST^3$ and $|\mathbb{N}_{-2} = |-\tilde{S}T^{-1}$ the "Lewis" equation for the Hecke triangle group G_q, $q = 2h_q + 3$ has the form:

$$\begin{aligned} g_1|(1-T) &= g_1|(P_{h_q-1}(ST)ST^2 + (ST)^{h_q+1}TP_{h_q-1}(ST)ST^2 + (ST)^{h_q}ST^3) \\ &\quad - \epsilon\, g_1|(P_{h_q-1}(ST)\tilde{S} + (ST)^{h_q+1}TP_{h_q-1}(ST)\tilde{S} + (ST)^{h_q}\tilde{S}T^{-1}). \end{aligned}$$

For $q = 5$ corresponding to $h_q = 1$ this reads

$$\begin{aligned} g_1|(1-T) &= g_1|(ST^2 + (ST)^2TST^2 + (ST)^2T^2) \\ &\quad - \epsilon\, g_1|(\tilde{S} + (ST)^2T\tilde{S} + ST\tilde{S}T^{-1}). \end{aligned}$$

6. **The Selberg zeta function for Hecke triangle groups G_q.** We want to express the Selberg zeta function for the Hecke triangle groups G_q in terms of Fredholm determinants of the transfer operator $\mathcal{L}_s$ for the map f_q. Our construction is analogous to that for modular groups and the Gauss map [3]. We start with a discussion of the Ruelle zeta function for the H-N map f_q.

6.1. **The Ruelle zeta function and the transfer operator for the Hurwitz-Nakada map f_q.** We have seen that the transfer operator $\mathcal{L}_s$ for the map $f_q : I_q \to I_q$ can be written as

$$(\mathcal{L}_s f)(x) \,=\, \sum_{n\in\mathbb{Z}:[\![0;n,x]\!]\in\mathcal{A}_q} (\vartheta_n'(x))^s f(\vartheta_n(x)), \tag{14}$$

where $\mathcal{A}_q$ denotes the set of regular λ_q-continued fraction expansions of all points $x \in I_q$ and if $x = [\![0; a_1, a_2, \ldots]\!] \in I_q$ then $[\![0; n, x]\!]$ denotes the continued fraction $[\![0; n, a_1, a_2, \ldots]\!]$. The iterates $\mathcal{L}_s^k$, $k = 1, 2, \ldots$ of this operator then have the form

$$(\mathcal{L}_s^k f)(x) \,=\, \sum_{(n_1,\ldots,n_k)\in\mathbb{Z}^k:[\![0;n_1,\ldots,n_k,x]\!]\in\mathcal{A}_q} (\vartheta_{n_1,\ldots,n_k}'(x))^s f(\vartheta_{n_1,\ldots,n_k}(x)),$$

where $\vartheta_{n_1,\ldots,n_k}$ denotes the map $\vartheta_{n_1} \circ \ldots \circ \vartheta_{n_k}$.

We have seen that the set of k-tuples $(n_1, \ldots, n_k) \in \mathbb{Z}^k$ with the property that $[\![0; n_1, \ldots, n_k, x]\!] \in \mathcal{A}_q$ depends only on the interval I_i for $x \in I_i^\circ$, the interior of the interval I_i. Denote by $\mathcal{F}_i^k$, $1 \le i \le \kappa_q$ the set

$$\mathcal{F}_i^k = \{(n_1, \ldots, n_k) \in \mathbb{Z}^k : [\![0; n_1, \ldots, n_k, x]\!] \in \mathcal{A}_q \quad \text{for all} \quad x \in I_i^\circ\}.$$

It follows that if $x \in I_i$ then $\mathcal{L}_s^k$ can be written as

$$(\mathcal{L}_s^k f)(x) \,=\, \sum_{(n_1,\ldots,n_k)\in\mathcal{F}_i^k} (\vartheta_{n_1,\ldots,n_k}')^s(x) f(\vartheta_{n_1,\ldots,n_k}(x)).$$

If f_j, $j \in \mathcal{A}_{\kappa_q}$ denotes the restriction $f|I_j$ and $\underline{n}_k = (n_1, \ldots, n_k) \in \mathbb{Z}^k$ we get

$$(\mathcal{L}_s^k f)_i(x) \,=\, \sum_{j\in\mathcal{A}_{\kappa_q}} \sum_{\underline{n}_k\in\mathcal{F}_i^k} (\vartheta_{\underline{n}_k}')^s(x)\chi_{I_j}(\vartheta_{\underline{n}_k}(x)) f_j(\vartheta_{\underline{n}_k}(x)).$$

On the Banach space $B = \oplus_{i\in\mathcal{A}_{\kappa_q}} B(D_i)$ we get

$$(\mathcal{L}_s^k \underline{f})_i(z) \,=\, \sum_{j\in\mathcal{A}_{\kappa_q}} \sum_{\underline{n}_k\in\mathcal{F}_i^k} (\vartheta_{\underline{n}_k}')^s(z)\chi_{D_j}(\vartheta_{\underline{n}_k}(z)) f_j(\vartheta_{\underline{n}_k}(z)).$$

The trace of this operator on this Banach space is then given by the well-known formula for such composition operators [11]

$$\operatorname{trace}\mathcal{L}_s^k = \sum_{i\in\mathcal{A}_{\kappa_q}} \sum_{\underline{n}_k\in\mathcal{F}_i^k} (\vartheta_{\underline{n}_k}')^s(z_{\underline{n}_k}^\star)\frac{1}{1-\vartheta_{\underline{n}_k}'(z_{\underline{n}_k}^\star)},$$

where $z_{\underline{n}_k}^\star = [\![0; \overline{n_1, \ldots, n_k}]\!]$ is the unique fixed point of the map $\vartheta_{n_1,\ldots,n_k} : D_i \to D_i$ which defines a hyperbolic element in the group G_q. These points are however in one-to-one correspondence with the periodic points of period k of the map $f_q : I_q \to I_q$. Hence also the following identity holds

$$\begin{aligned}
\operatorname{trace}\mathcal{L}_s^k - \operatorname{trace}\mathcal{L}_{s+1}^k &= \sum_{i\in\mathcal{A}_{\kappa_q}} \sum_{\underline{n}_k\in\mathcal{F}_i^k} ((\vartheta_{n_1} \circ \ldots \circ \vartheta_{n_k})')^s(z_{\underline{n}_k}^\star) \\
&= \sum_{i\in\mathcal{A}_{\kappa_q}} \sum_{\underline{n}_k\in\mathcal{F}_i^k} \prod_{l=1}^{k} \Big(\vartheta_{n_l}'(\vartheta_{n_{l+1}} \circ \ldots \circ \vartheta_{n_k}(z_{\underline{n}_k}^\star))\Big)^s \\
&= \sum_{z^\star\in\operatorname{Fix} f_q^k} \prod_{l=0}^{k-1} \left(f_q'(f_q^l(z^\star))\right)^{-s}.
\end{aligned} \tag{15}$$

Therefore we get

Proposition 1. *The Ruelle zeta function* $\zeta_R(s) = \exp\sum_{n=1}^{\infty}\frac{1}{n}Z_n(s)$ *for the H-N map* $f_q : I_q \to I_q$ *can be written as* $\zeta_R(s) = \frac{\det(1-\mathcal{L}_{s+1})}{\det(1-\mathcal{L}_s)}$ *with* $\mathcal{L}_s : B \to B$ *as defined in Theorem 14 and* $s \in \mathbb{C}$.

Proof. For $\Re s$ large enough we have, comparing equations (3.1) and (15) $Z_n(s) = \operatorname{trace}\mathcal{L}_s^n - \operatorname{trace}\mathcal{L}_{s+1}^n$ and therefore $\zeta_R(s) = \frac{\det(1-\mathcal{L}_{s+1})}{\det(1-\mathcal{L}_s)}Z_n(s)$. Since the operators $\mathcal{L}_s$ are meromorphic and nuclear in the entire s-plane their Fredholm determinants also allow such a meromorphic continuation, which proves the proposition. □

6.2. **The transfer operator $\mathcal{K}_s$.** As discussed earlier, there is a one to one correspondence between the closed orbits of the map $f_q : I_q \to I_q$ and the closed orbits of the geodesic flow on the Hecke surfaces apart from the closed orbits of the two points $r_q = [\![0; \overline{1^{h_q-1}, 2}]\!]$ for even q , respectively $r_q = [\![0; \overline{1^{h_q}, 2, 1^{h_q-1}, 2}]\!]$ for odd q, and $-r_q$, which are not equivalent under the map f_q, but are equivalent under the group G_q and hence correspond to the same closed orbit of the geodesic flow. In the Ruelle zeta function $\zeta_R(s)$ for the map f_q the contributions of both the orbits of r_q and $-r_q$ are included. To relate this function to the Selberg zeta function $Z_S(s)$ for the geodesic flow on the Hecke surfaces we have to subtract the contribution of one of these two orbits of f_q, say of r_q, to this function. This we can achieve by subtracting the contribution of the orbit $\mathcal{O}_+$ of the point r_q from the partition functions $Z_{l\kappa_q}(s)$ for the map f_q for all $l = 1, 2, \ldots$. Consider therefore the corresponding Ruelle function $\zeta_R^{\mathcal{O}}(s) = \exp(\sum_{n=1}^{\infty}\frac{1}{n}Z_n^{\mathcal{O}_+}(s))$ with

$$Z_n^{\mathcal{O}}(s) = \begin{cases} 0 & \text{if } \kappa_q \nmid \text{n} \\ \sum_{x\in\mathcal{O}_+}\exp\left(-s\sum_{k=0}^{n-1}\ln f_q'(f_q^k(x))\right) & \text{if } \kappa_q \mid n. \end{cases} \tag{16}$$

If $n = l\kappa_q$ we find that $Z_{l\kappa_q}^{\mathcal{O}_+}(s) = \kappa_q\exp(-slr_{\mathcal{O}_+})$ and hence $\zeta_R^{\mathcal{O}_+}(s) = \frac{1}{1-\exp(-sr_{\mathcal{O}_+})}$ where $r_{\mathcal{O}_+} = \ln(f_q^{\kappa_q})'(r_q)$. We now define the transfer operator $\mathcal{L}_s^{\mathcal{O}_+} : B \to B$. Here $B_{\kappa_q} = \oplus_{i=1}^{\kappa_q}B(D_i)$ as defined in 5.1. For $q = 2h_q + 2$ we set

$$\begin{aligned} (\mathcal{L}_s^{\mathcal{O}_+}\overrightarrow{g})_i(z) &= \mathcal{L}_{1,s}g_{i+1}(z),\ 1 \le i \le h_q - 1, \\ (\mathcal{L}_s^{\mathcal{O}_+}\overrightarrow{g})_{h_q}(z) &= \mathcal{L}_{2,s}g_1(z). \end{aligned} \tag{17}$$

For $q = 2h_q + 3$ on the other hand define

$$\begin{aligned} (\mathcal{L}_s^{\mathcal{O}_+}\overrightarrow{g})_i(z) =& \mathcal{L}_{1,s}g_{i+1}(z),\ 1 \le i \le h_q \\ (\mathcal{L}_s^{\mathcal{O}_+}\overrightarrow{g})_{h_q+1}(z) =& \mathcal{L}_{2,s}g_{h_q+2}(z) \\ (\mathcal{L}_s^{\mathcal{O}_+}\overrightarrow{g})_{h_q+i}(z) =& \mathcal{L}_{1,s}g_{h_q+i+1}(z),\ 2 \le i \le h_q \\ (\mathcal{L}_s^{\mathcal{O}_+}\overrightarrow{g})_{2h_q+1}(z) =& \mathcal{L}_{2,s}g_1(z). \end{aligned} \tag{18}$$

In both cases the operator $\mathcal{L}_s^{\mathcal{O}_+}$ then has the form

$$\mathcal{L}_s^{\mathcal{O}_+} = \begin{pmatrix} 0 & \mathcal{L}_{1,s} & 0 & \dots & 0 & 0 \\ 0 & 0 & \mathcal{L}_{1,s} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathcal{L}_{1,s} & 0 \\ 0 & 0 & 0 & \ddots & 0 & \mathcal{L}_{1,s} \\ \mathcal{L}_{2,s} & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

respectively

$$\mathcal{L}_s^{\mathcal{O}_+} = \begin{pmatrix} 0 & \mathcal{L}_{1,s} & 0 & \dots & 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & \mathcal{L}_{1,s} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & \mathcal{L}_{2,s} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & \mathcal{L}_{1,s} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \mathcal{L}_{1,s} & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \ddots & 0 & \mathcal{L}_{1,s} \\ \mathcal{L}_{2,s} & 0 & \dots & \dots & \dots & \dots & \dots & 0 & 0 \end{pmatrix}.$$

Then one has

Lemma 6.1. *The trace of the operator $(\mathcal{L}_s^{\mathcal{O}_+})^n$ for $q = 2h_q + 2$ and $\kappa_q = h_q$ is given by*

$$\mathit{trace}\,(\mathcal{L}_s^{\mathcal{O}_+})^n = \begin{cases} 0 & \mathit{for} \quad \kappa_q \nmid n \\ \kappa_q\, \mathit{trace}\,(\mathcal{L}_{1,s}^{h_q-1}\mathcal{L}_{2,s})^l & \mathit{for} \quad n = l\kappa_q, \end{cases}$$

respectively for $q = 2h_q + 3$ and $\kappa_q = 2h_q + 1$ by

$$\mathit{trace}\,(\mathcal{L}_s^{\mathcal{O}_+})^n = \begin{cases} 0 & \mathit{for} \quad \kappa_q \nmid n \\ \kappa_q\, \mathit{trace}\,(\mathcal{L}_{1,s}^{h_q}\mathcal{L}_{2,s}\mathcal{L}_{1,s}^{h_q-1}\mathcal{L}_{2,s})^l & \mathit{for} \quad n = l\kappa_q. \end{cases}$$

Proof. Since the proof for odd q is completely analogous we restrict ourselves to the case $q = 2h_q + 2$. Induction on i shows that for $1 \le j \le h_q$ one has for $\overrightarrow{g} = (g)_{1 \le j \le h_q}$

$$((\mathcal{L}_s^{\mathcal{O}_+})^i \overrightarrow{g})_j = \mathcal{L}_{1,s}^i\, g_{i+j}, \quad 1 \le j \le h_q - i$$
$$((\mathcal{L}_s^{\mathcal{O}_+})^i \overrightarrow{g})_j = \mathcal{L}_{1,s}^{h_q-j}\, \mathcal{L}_{2,s}\, \mathcal{L}_{1,s}^{i+j-h_q-1}\, g_{i+j-h_q}, \quad h_q - i + 1 \le j \le h_q.$$

But this shows that

$$((\mathcal{L}_s^{\mathcal{O}_+})^{h_q} \overrightarrow{g})_1 = \mathcal{L}_{1,s}^{h_q-1}\mathcal{L}_{2,s}\, g_1, \quad \textit{respectively}$$
$$((\mathcal{L}_s^{\mathcal{O}_+})^{h_q} \overrightarrow{g})_j = \mathcal{L}_{1,s}^{h_q-j}\, \mathcal{L}_{2,s}\, \mathcal{L}_{1,s}^{j-1}\, g_j, \quad 2 \le j \le h_q.$$

and therefore

$$\mathrm{trace}\,(\mathcal{L}_s^{\mathcal{O}_+})^n = 0 \quad \text{if} \quad h_q \nmid n \quad \text{and}$$
$$\mathrm{trace}\,(\mathcal{L}_s^{\mathcal{O}_+})^n = h_q\, \mathrm{trace}\,(\mathcal{L}_{1,s}^{h_q-1}\, \mathcal{L}_{2,s})^l \quad \text{if} \quad n = l\, h_q = l\, \kappa_q.$$

This proves the Lemma. □

Now, since trace $(\mathcal{L}_{1,s}^{h_q-1}\,\mathcal{L}_{2,s})^l = \text{trace}\,(\mathcal{L}_{2,s}\,\mathcal{L}_{1,s}^{h_q-1})^l$ and

$$((\mathcal{L}_{2,s}\,\mathcal{L}_{1,s}^{h_q-1})^l\,g)(z) = \left(\frac{d}{dz}(\vartheta_{1^{h-1},2})^l\,(z)\right)^s g\left((\vartheta_{1^{h-1},2})^l(z)\right)$$

(recall that $\vartheta_{1^{h-1},2} = \vartheta_1 \circ \ldots \circ \vartheta_1 \circ \vartheta_2$) we find that

$$\text{trace}\,(\mathcal{L}_{2,s}\,\mathcal{L}_{1,s}^{h_q-1})^l = \left(\frac{d}{dz}(\vartheta_{1^{h-1},2})^l\,(z^\star)\right)^s \frac{1}{1-\frac{d}{dz}(\vartheta_{1^{h-1},2})^l(z^\star)},$$

where $z^\star$ is the attractive fixed point of $\vartheta_1 \circ \ldots \circ \vartheta_1 \circ \vartheta_2$, i.e. $z^\star = r_q$. Thus

$$\begin{aligned}\text{trace}\,(\mathcal{L}_{2,s}\,\mathcal{L}_{1,s}^{h_q-1})^l - \text{trace}\,(\mathcal{L}_{2,s+1}\,\mathcal{L}_{1,s+1}^{h_q-1})^l &= \left(\frac{d}{dz}(\vartheta_{1^{h-1},2})^l\,(z^*)\right)^s \\ &= \left(\frac{d}{dz}(\vartheta_{1^{h-1},2})\,(z^*)\right)^{l\,s}\end{aligned}$$

Lemma 6.2. *The partition function $Z_n^{\mathcal{O}_+}$ in (16) can be expressed in terms of the transfer operators $\mathcal{L}_s^{\mathcal{O}_+}$ in (17) and (18) as $Z_n^{\mathcal{O}_+}(s) = \text{trace}\,(\mathcal{L}_s^{\mathcal{O}_+})^n - \text{trace}\,(\mathcal{L}_{s+1}^{\mathcal{O}_+})^n$.*

Proof. We restrict ourselves again to the case $q = 2h_q+2$. The case of odd q is analogous. Since $(f_q^{\kappa_q})'(z^\star) = ((\vartheta_1 \circ \ldots \circ \vartheta_1 \circ \vartheta_2)'(z^\star))^{-1}$ and $Z_n^{\mathcal{O}_+} = \kappa_q \exp(-sl)\, r_{\mathcal{O}_+}$ with $r_{\mathcal{O}_+} = \sum_{k=0}^{\kappa_q-1} \ln f_q'(f_q^k z^\star) = \ln \prod_{k=0}^{\kappa_q-1} f_q'(f_q^k(z^\star)) = \ln(f_q^{\kappa_q})'(z^\star)$ we find that

$$\begin{aligned}Z_n^{\mathcal{O}_+} =& \kappa_q\left((f_q^{\kappa_q})'(z^*)\right)^{-l\,s} = \kappa_q((\vartheta_1 \circ \ldots \circ \vartheta_1 \circ \vartheta_2)'((z^\star)))^{l\,s} \\ =& \kappa_q\,(\text{trace}\,(\mathcal{L}_{2,s}\,\mathcal{L}_{1,s}^{h_q-1})^l - \text{trace}\,(\mathcal{L}_{2,s+1}\,\mathcal{L}_{1,s+1}^{h_q-1})^l). \\ =& \text{trace}\,(\mathcal{L}_s^{\mathcal{O}_+})^n - \text{trace}\,(\mathcal{L}_{s+1}^{\mathcal{O}_+})^n.\end{aligned}$$

□

Hence the Ruelle zeta function $\zeta_R^{\mathcal{O}_+}(s) = \exp(\sum_{n=1}^\infty \frac{1}{n} Z_n^{\mathcal{O}_+}(s))$ for the orbit $\mathcal{O}_+$ of the point r_q can be expressed as

$$\zeta_R^{\mathcal{O}_+}(s) = \frac{\det\,(1-\mathcal{L}_{s+1}^{\mathcal{O}_+})}{\det\,(1-\mathcal{L}_s^{\mathcal{O}_+})}.$$

We can furthermore show

Lemma 6.3. *The Fredholm determinant $\det(1-\mathcal{L}_s^{\mathcal{O}_+})$ coincides with the Fredholm determinant $\det(1-\mathcal{L}_{1,s}^{h_q-1}\,\mathcal{L}_{2,s})$ in the case $q = 2h_q+2$ and $r_q = [\![0; \overline{(1)^{h_q-1}, 2}]\!]$, respectively with $\det(1-\mathcal{L}_{1,s}^{h_q}\,\mathcal{L}_{2,s}\,\mathcal{L}_{1,s}^{h_q-1}\,\mathcal{L}_{2,s})$ in the case $q = 2h_q+3$ and $r_q = [\![0; \overline{(1)^{h_q}, 2, (1)^{h_q-1}, 2}]\!]$.*

Proof. This lemma follows immediately from Lemma 6.1 and the following formula which holds for any trace class operator L: $-\ln\det(1-L) = \sum_{n=1}^\infty \frac{1}{n}\text{trace}L^n$. □

Remark 2. The spectra of the two operators $\mathcal{L}_s^{\mathcal{O}_+}$ and $\mathcal{L}_{1,s}^{h_q-1}\,\mathcal{L}_{2,s}$ are related as follows: definition ((17)) implies that any eigenfunction $\overrightarrow{g} = (g_i)_{1\le i\le h_q}$ with eigenvalue ρ of $\mathcal{L}_s^{\mathcal{O}_+}$ fulfills the equation $\rho^{h_q-1} g_1 = \mathcal{L}_{1,s}^{h_q-1} g_{h_q}$ and hence also the equation $\rho^{h_q} g_1 = \mathcal{L}_{1,s}^{h_q-1}\,\mathcal{L}_{2,s} g_1$. Therefore every eigenvalue ρ of the operator $\mathcal{L}_s^{\mathcal{O}_+}$ determines an eigenvalue ρ^{h_q} of the operator $\mathcal{L}_{1,s}^{h_q-1}\,\mathcal{L}_{2,s}$. Given on the other hand an eigenfunction g of $\mathcal{L}_{1,s}^{h_q-1}\,\mathcal{L}_{2,s}$ with eigenvalue $\rho = \mid \rho \mid \exp(i\alpha)$, the function

$\overrightarrow{g}^{(j)} = (g)^{(j)}_{1\leq j\leq h_q}$ with $g^{(j)}_1 = g$ and $g^{(j)}_i = \rho_j^{-(h_q+1-i)} \mathcal{L}_{1,s}^{h_q-i}\,\mathcal{L}_{2,s}\,g$, $2 \leq i \leq h_q$ is an eigenfunction of the operator $\mathcal{L}_s^{\mathcal{O}_+}$ with eigenvalue ρ_j for $\rho_1,\ldots,\rho_{h_q}$ the h_q-the roots of ρ . This shows that the numbers $\rho_j = \sqrt[h_q]{|\,\rho\,|}\exp(i\frac{\alpha}{h_q})\ exp(2\pi i\frac{j}{h_q})$, $0 \leq j \leq h_q - 1$ are eigenvalues of this operator. This shows again that $\det(1 - \mathcal{L}_s^{\mathcal{O}_+}) = \det(1 - \mathcal{L}_{1,s}^{h_q-1}\,\mathcal{L}_{2,s})$.

The contribution of the periodic orbit of the geodesic flow corresponding to the periodic orbit $\mathcal{O}_+$ of the point r_q which appears twice in the Fredholm determinant of the transfer operator $\mathcal{L}_s$ for the map f_q is given by $\det(1 - \mathcal{L}_{1,s}^{h_q-1}\,\mathcal{L}_{2,s})$ for $q = 2h_q + 2$ respectively by $\det(1 - \mathcal{L}_{1,s}^{h_q}\,\mathcal{L}_{2,s}\,\mathcal{L}_{1,s}^{h_q-1}\,\mathcal{L}_{2,s})$ for $q = 2h_q + 3$. We then arrive at the following theorem.

Theorem 6.4. *The Selberg zeta function $Z_S(s)$ for the Hecke triangle group G_q can be written as*

$$Z_S(s) = \frac{det\,(1 - \mathcal{L}_s)}{det\,(1 - \mathcal{K}_s)} = \frac{det\,(1 - \mathcal{L}_{s,+})(1 - \mathcal{L}_{s,-})}{det\,(1 - \mathcal{K}_s)},$$

where $\mathcal{L}_s$ denotes the transfer operator of the Hurwitz-Nakada map $f_q : I_q \to I_q$ in Theorem 4.10, $\mathcal{L}_{s,\pm}$ denote the reduced transfer operators in (8, 5.1, 9) and $\mathcal{K}_s = \mathcal{L}_s^{\mathcal{O}_+}$ is the transfer operator in (17) for even q , respectively (18) for odd q . The spectrum $\sigma(\mathcal{K}_s)$ is given by

$$\sigma(\mathcal{K}_s) = \{\prod_{l=0}^{\kappa_q-1}(f_q^l(r_q))^{2s+2n},\ n = 0,1,2,\ldots\}$$

where κ_q denotes the period of the point r_q

Proof. The spectrum $\sigma(L)$ of a composition operator of the general form $Lf(z) = \varphi(z)f(\psi(z))$ on a Banach space $B(D)$ of holomorphic functions on a domain D with $\psi(\overline{D}) \subset D$ is given by [11] $\sigma(L) = \{\varphi(z^*)\psi'(z^*)^n,\ n = 0,1,\ldots\}$ where z^* is the unique fixed point of ψ in D. For $q = 2h_q + 2$ the operator $\mathcal{K}_s$ has this form with $\varphi(z) = ((\vartheta_2 \circ \vartheta_1^{h_q-1})')^s(z)$ and $\psi(z) = \vartheta_2 \circ \vartheta_1^{h_q-1}(z)$. Therefore $z^* = [\![0;\overline{2,1^{h_q-1}}]\!]$. But $(\vartheta_2\circ\vartheta_1^{h_q-1})'(z^*) = \vartheta_2'(\vartheta_1^{h_q-1}(z^*))\prod_{l=1}^{h_q-1}\vartheta_1'((\vartheta_1)^{h_q-1-l}(z^*))$. Since $\vartheta_m(\vartheta_m^{-1}(z)) = z^2$ for any $z \in \mathbb{C}$, $m \in \mathbb{N}$, $\vartheta_1^{h_q-1}(z^*) = r = \vartheta_2^{-1}(z^*)$ and $(\vartheta_1)^{h_q-1-l}(z^*) = (\vartheta_1)^{-1}\vartheta_1^{h_q-l}(z^*)$ it follows immediately that $(\vartheta_2 \circ \vartheta_1^{h_q-1})'(z^*) = (z^*)^2\prod_{l=1}^{h_q-1}(\vartheta_1^{h_q-l}(z^*))^2 = \prod_{l=0}^{h_q-1}(f_q^l(z^*))^2 = \prod_{l=0}^{h_q-1}(f_q^l(r_q))^2$. □

Remark 3. Using the explicit form of the maps which fix r_q, cf. e.g [13] Remark 27 (where the upper right entry of the matrix for even q should read $\lambda - \lambda^3$) one can prove that the spectrum of the operator $\mathcal{K}_s$ can also be written as $\{\mu_n = l^{2s+2n},\ n = 0,1,\ldots\}$ where

$$l = \frac{\sqrt{4-\lambda_q^2}}{R\lambda_q + 2} = \sqrt{\frac{2-\lambda_q}{2+\lambda_q}} \quad \text{for even } q \text{ and}$$

$$l = \frac{2-\lambda_q}{R\lambda_q + 2} = \frac{2-\lambda_q}{2+R\lambda_q} \quad \text{for odd } q.$$

The Selberg zeta function $Z_S(s)$ for Hecke triangle groups G_q and small q has been calculated numerically using the transfer operator $\mathcal{L}_s$ by one of us in [21].

Besides the case $q = 3$, that is the modular group $G_3 = SL(2, \mathbb{Z})$ [2], we do not yet know how the eigenfunctions of the transfer operator $\mathcal{L}_s$ with eigenvalue $\rho = 1$ are related to the automorphic functions for a general Hecke group G_q. The divisor of $Z_S(s)$ is closely related to the automorphic forms on G_q (see for instance [6],p. 498). One would therefore expect that there exist explicit relationships also for $q > 3$ similar to those obtained for modular groups between eigenfunctions of the transfer operator $\mathcal{L}_s$ with eigenvalue one and automorphic forms related to the divisors of Z_S at these s-values.

Another interesting problem would be to understand the behavior of our transfer operator $\mathcal{L}_s$ in the limit when q tends to ∞. In this limit the Hecke triangle group G_q tend to the theta group Γ_θ, generated by $Sz = \frac{-1}{z}$ and $Tz = z + 2$. This group is conjugate to the Hecke congruence subgroup $\Gamma_0(2)$, for which we have constructed a transfer operator in [7], [5]. One should understand how these two different transfer operators are related to each other. The limit $q \to \infty$ is quite singular, since the group Γ_θ has two cusps whereas all the Hecke triangle groups have only one cusp. Therefore one expects in this limit all the singular behavior Selberg predicted already in [20]. Understanding the limit $q \to \infty$ could also shed new light on the Phillips-Sarnak conjecture [15] on the existence of Maass wave forms for general non-arithmetic Fuchsian groups.

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