Nearest λ_q-multiple fractions

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A . We discuss the nearest λ_q–multiple continued fractions and their duals for $\lambda_q = 2\cos\left(\frac{\pi}{q}\right)$ which are closely related to the Hecke triangle groups G_q, $q = 3, 4, \ldots$. They have been introduced in the case $q = 3$ by Hurwitz and for even q by Nakada. These continued fractions are generated by interval maps f_q respectively $f_q^\star$ which are conjugate to subshifts over infinite alphabets. We generalize to arbitrary q a result of Hurwitz concerning the G_q- and f_q-equivalence of points on the real line. The natural extension of the maps f_q and $f_q^\star$ can be used as a Poincaré map for the geodesic flow on the Hecke surfaces $G_q\backslash\mathbb{H}$ and allows to construct the transfer operator for this flow.

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1. Introduction

In the transfer operator approach to Selberg's zeta-function for Fuchsian groups G [12], [13] this functions is expressed through the Fredholm-determinant of the generalized Perron-Frobenius operator $\mathcal{L}_\beta$ for the geodesic flow on the corresponding surface $G\backslash\mathbb{H}$ of constant negative curvature. This operator is constructed through an expanding interval map $f : I \to I$ closely related to a Poincaré map of the flow. In all the cases treated up to now this interval map generates some kind of continued fraction expansion like the Gauss expansion or its extensions such that the length spectrum of the flow can be completely characterized by the periodic orbits of f respectively the purely periodic continued fraction expansions. This program has been carried out in full detail for the modular surfaces $\Gamma \setminus \mathbb{H}$ defined by subgroups $\Gamma \subset \mathrm{PSL}(2,\mathbb{Z})$ of the full modular group.

For these groups the transfer operator has another rather important property: its eigenfunctions with eigenvalue 1 can be directly related to their automorphic forms, that is real analytic Eisenstein series and Maass wave forms respectively the holomorphic modular forms. This relation gave rise to the theory of periodic functions [10], [2] which generalize the Eichler-Manin-Shimura cohomology theory for holomorphic modular forms.

The physical interpretation of these relations between the transfer operator and the spectral properties of the Laplacian for these groups G is within the theory of quantum chaos [16], [19]: the transfer operator encodes the classical length spectrum of the geodesic flow and relates these data to the quantum data, namely eigenvalues and eigenfunctions respectively resonances of its quantized system. In this sense this transfer operator approach extends the more common approach to quantum chaos via the Selberg-Gutzwiller trace formula [7, Theorem 13.8, p. 209], [11].

Obviously it is necessary to work out the transfer operator for more general Fuchsian groups, especially non-arithmetic ones, for which the Hecke triangle groups G_q are good examples, since up to the cases $q = 3, 4, 6$ all of

them are indeed non-arithmetic. In [14] the authors constructed a symbolic dynamics for the geodesic flow on the Hecke surfaces $G_q\backslash\mathbb{H}$ for arbitrary q, the case $q = 3$ was treated earlier in [9]. In both cases the authors used the nearest λ_q-multiple continued fraction expansion, denoted for short by λ_q-CF, and their dual expansion. Another approach was discussed also in [20]. Some of the ergodic properties of these λ_q-CF's for q even have been worked out in [15] by H. Nakada. In the present paper we discuss the λ_q-CF's and their duals for arbitrary q via their generating interval maps f_q and $f_q^\star$, which allow us to derive also a transfer operator for the Hecke triangle groups G_q, whose Fredholm determinant is closely related to the Selberg function for the groups G_q as we will discuss in a forthcoming paper.

In [8] Hurwitz introduced nearest integer continued fraction expansions of the form

$$a_0 + \cfrac{-1}{a_1 + \cfrac{-1}{a_2 + \cfrac{-1}{a_3 + \cfrac{-1}{\cdots}}}} \tag{1.1}$$

where a_0 is an arbitrary integer and the a_i, $i \geq 1$, are integers satisfying $|a_i| \geq 2$ and $a_i\, a_{i+1} < 0$ if $|a_i| = 2$. They are generated by the interval map

$$f_3 \colon I_3 \to I_3; \quad x \mapsto \frac{-1}{x} - \left\langle \frac{-1}{x} \right\rangle,$$

where $I_3 = \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\langle x \rangle$ denotes the nearest integer to x, by the usual algorithm:

(0) $a_0 = \langle x \rangle$ and $x_1 := x - a_0$,
(1) $a_1 = \left\langle \frac{-1}{x_1} \right\rangle$ and $x_2 := \frac{-1}{x_1} - a_1 = f_3(x_1)$,
(i) $a_i = \left\langle \frac{-1}{x_i} \right\rangle$ and $x_{i+1} := \frac{-1}{x_i} - a_i = f_3(x_i)$.
$(\star)$ The algorithm terminates if $x_{i+1} = 0$.

Let $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z}) \bmod \{\pm\mathbf{1}\}$ denote the the full modular group. Elements of the group can be identified with 2×2-matrices with integer entries and determinant 1, up to a common sign. The group acts on the projective real line $\mathbb{R} \cup \{\infty\}$ by Möbius transformations $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$. The group $\mathrm{PSL}(2,\mathbb{Z})$ is generated by the elements S and T corresponding to the actions $z \mapsto \frac{-1}{z}$ and $z \mapsto z + 1$. The generators satisfy the relations $S^2 = (ST)^3 = \mathbf{1}$. In particular, the elements T^a and ST^a correspond to the actions $z \mapsto z + a$ and $z \mapsto \frac{-1}{a+z}$. Hence we can write the continued fraction expansion in (1.1) in terms of a (formal) Möbius transformation as $T^{a_0} S T^{a_1} S T^{a_2} S T^{a_3} \cdots 0$.

Hurwitz found in [8], that equivalence of two points $x, y \in \mathbb{R}$ under the generating map f_3 is not the same as equivalence under the group action of $\mathrm{PSL}(2,\mathbb{Z})$. This is obviously in contrast with the case of the Gauss map $f_{\mathrm{G}} \colon [0,1] \to [0,1]$ with $f_G(x) = \frac{1}{x} \mod 1$ and the modular group $\mathrm{PSL}(2,\mathbb{Z})$.

In [15] Nakada introduced for even integers $q \geq 4$ the nearest λ_q-multiple continued fractions with $\lambda_q = 2\cos\frac{\pi}{q}$, which we will denote by λ_q-CF's. They are similar to the Rosen contined fractions introduced in [21] and discussed in detail in [1]. The extension to the case $q \geq 3$ odd is straightforward, where $q = 3$ corresponds to the nearest integer continued fractions of Hurwitz. These λ_q-CF's and their dual expansions, introduced for $q = 3$ also by Hurwitz, can be generated by interval maps f_q and $f_q^\star$ closely related to the Hecke triangle groups G_q. Both maps are conjugate to subshifts over infinite alphabets, which when reduced to certain sofic systems, determine completely the properties of the corresponding λ_q-CF and its dual expansion. It turns out, that Hurwitz's result on equivalence of points on the real axis under f_3 and the group action of $\mathrm{PSL}(2,\mathbb{Z})$ is true for general $q \geq 3$: there exists for every $q \geq 3$ exactly one pair of points $(r_q, -r_q)$ which are equivalent under G_q but not under the map f_q. The natural extension F_q of the interval map $f_q\colon I_q \to I_q$ can be easily constructed from the symbolic dynamics of the maps f_q and $f_q^\star$ as sofic systems. It can be used to construct a Poincare section for the geodesic flow on the Hecke surface $G_q\backslash\mathbb{H}$, and hence also a transfer operator for the group G_q and its Selberg zeta function. The properties of this operator will be discussed elsewere.

The structure of this article is as follows: In Section 2 we introduce the Hecke triangle groups and the λ_q-CF's respectively the dual λ_q-CF's. In Section 3 we discuss the interval maps f_q and $f_q^\star$ generating the nearest λ-multiple continued fractions and construct Markov partitions for these maps. In Section 4 we show that the maps f_q and $f_q^\star$ are conjugate to subshifts over infinite alphabets and introduce sofic systems closely related to the λ_q-CF and its dual. This allows a simple construction of the natural extension F_q of the map f_q. In Section 5 we relate the natural extension F_q to the geodesic flow on the Hecke surfaces $G_q\backslash\mathbb{H}$ and derive the transfer operator for this flow. The final Section 6 contains a discussion of the convergence properties of the λ_q-CF's by relating them to reduced Rosen λ-fractions as discussed in [21].

2. Nearest λ_q-multiple continued fractions

2.1. Hecke triangle groups. Hecke triangle groups are Fuchsian groups of the first kind, all except three are non-arithmetic. The reader may wish to consult [6, pp. 591, 627] for a discussion of Hecke triangle groups and their relation to Dirichlet series.

Denote by $\mathrm{PSL}(2,\mathbb{R})$ the projective special linear group given by

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm\mathbf{1}\} \tag{2.1.1}$$

where $\pm\mathbf{1} = \begin{pmatrix}\pm1 & 0\\ 0 & \pm1\end{pmatrix}$. We denote by $\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}, \begin{pmatrix}-a & -b\\ -c & -d\end{pmatrix}\right\}$ the elements of $\mathrm{PSL}(2,\mathbb{R})$, but identify often elements of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$.

For a given integer $q \geq 3$ the q^{th} *Hecke triangle group* G_q is generated by

$$S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad T_q := \begin{bmatrix} 1 & \lambda_q \\ 0 & 1 \end{bmatrix} \tag{2.1.2}$$

with relations

$$S^2 = \left(S\,T_q\right)^q = \mathbf{1}, \tag{2.1.3}$$

where λ_q is given by

$$\lambda_q := 2\cos\left(\frac{\pi}{q}\right) \tag{2.1.4}$$

Lateron we also need the element

$$T_q' := \begin{bmatrix} 1 & 0 \\ \lambda_q & 1 \end{bmatrix} = S\,T_q^{-1}S \in G_q. \tag{2.1.5}$$

We may suppress the q-dependence in our notation when we work with a fixed value of q.

The Hecke triangle group G_q is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and its limit set is the projective line $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. It acts on the upper half-plane, the lower half-plane and on $\mathbb{P}^1_{\mathbb{R}}$ by *Möbius transformations*

$$G_q \times \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}}; \quad \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, x\right) \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} x := \begin{cases} \frac{ax+b}{cx+d} & \text{if } x \in \mathbb{R} \text{ and} \\ \frac{a}{c} & \text{if } x = \infty. \end{cases} \tag{2.1.6}$$

The points $x, y \in \mathbb{P}^1_{\mathbb{R}}$ are G_q-*equivalent* denoted by $x \sim_{G_q} y$, if there exists an element $g \in G_q$ such that $y = g\,x$. Obviously, this is an equivalence relation.

2.2. Nearest λ_q-multiple continued fractions and their duals. Consider finite or infinite sequences $(a_i)_i$. We denote periodic parts of the sequences by overlining the period part and finitely repeated patterns are denoted by a power where a 0^{th} power vanishes:

$$\begin{aligned} (a_1, \overline{a_2, a_3}) &= (a_1,\ a_2, a_3,\ a_2, a_3,\ a_2, a_3,\ \dots), \\ (a_1, (a_2, a_3)^i, a_4, \dots) &= (a_1, \underbrace{a_2, a_3, a_2, a_3, \dots, a_2, a_3}_{i \text{ times } a_2, a_3}, a_4, \dots) \quad \text{and} \\ (a_1, (a_2)^0, a_3, \dots) &= (a_1, a_3, \dots). \end{aligned}$$

We use also $-(a_1, \dots) = (-a_1, \dots)$.

Put

$$h_q := \begin{cases} \frac{q-2}{2} & \text{for } q \text{ even and} \\ \frac{q-3}{2} & \text{for } q \text{ odd.} \end{cases} \tag{2.2.1}$$

We define the set $\mathcal{B}_q$ of *forbidden blocks* as

$$\mathcal{B}_q := \begin{cases} \{(\pm 1)\} \cup \bigcup_{m=1}^{\infty}\{(\pm 2, \pm m)\} & \text{for } q = 3, \\ \{((\pm 1)^{h_q+1})\} \cup \bigcup_{m=1}^{\infty}\{((\pm 1)_q^h, \pm m)\} & \text{for } q \text{ even and} \\ \{((\pm 1)^{h_q+1})\} \cup & \\ \quad \cup \bigcup_{m=1}^{\infty}\{((\pm 1)^{h_q}, \pm 2, (\pm 1)^{h_q}, \pm m)\} & \text{for } q \text{ odd, } q \geq 5. \end{cases} \tag{2.2.2}$$

The choice of the sign is the same within each block. For example $(2, 3)$, $(-2, -3) \in \mathcal{B}$ and $(2, -3) \notin \mathcal{B}$ for $q = 3$.

We call a sequence $(a_1, a_2, a_3, \dots)$ *q-regular* if $(a_l, a_{l+1}, \dots, a_L) \notin \mathcal{B}_q$ for all $1 \leq l < L$ and *dual q-regular* if $(a_L, a_{L-1}, \dots, a_l) \notin \mathcal{B}_q$ for all $1 \leq l < L$. Denote by $\mathcal{A}_q^{\mathrm{reg}}$ respectively by $\mathcal{A}_q^{\mathrm{dreg}}$ the set of infinite q-regular respectively dual q-regular sequences $(a_i)_{i\in\mathbb{N}}$.

A *nearest λ_q-multiple continued fraction*, or λ_q-CF, is a formal expansion of the type

$$[a_0; a_1, a_2, a_3, \dots] := a_0\lambda_q + \cfrac{-1}{a_1\lambda_q + \cfrac{-1}{a_2\lambda_q + \cfrac{-1}{a_3\lambda_q + \dots}}} \tag{2.2.3}$$

with $a_i \in \mathbb{Z}_{\neq 0}$, $i \geq 1$ and $a_0 \in \mathbb{Z}$.

We say that $[a_0; a_1, a_2, a_3, \dots]$ *converges* if either $[a_0; a_1, a_2, a_3, \dots] = [a_0; a_1, a_2, a_3, \dots, a_L]$ has finite length or $\lim_{L\to\infty}[a_0; a_1, a_2, a_3, \dots, a_L]$ exists in $\mathbb{R}$.

We adopt the same notations as introduced for sequences earlier. For example we write $[a_0; a_1, \overline{a_2, a_3}]$ for a periodic tail of the expansion and $-[a_0; a_1, \dots]$ for $[-a_0; -a_1, \dots]$.

A λ_q-CF is *regular* respectively *dual regular* if the sequence $(a_1, a_2, a_3, \dots)$ is q-regular respectively dual q-regular. Regular and dual regular λ_q-CF's are denoted by $[\![a_0; a_1, \dots]\!]$ respectively $[\![a_0; a_1, \dots]\!]^\star$.

P 2.2.1. *Regular and dual regular λ_q-CF's converge.*

P . The proposition follows immediately from Lemmas 4 and 34 in [14]. □

An alternative proof of Proposition 2.2.1 for infinite regular and dual regular expansions with leading 0 follows also from the lemmas in the Sections 4.2 and 4.4.

Converging λ_q-CF's can be rewritten in terms of elements of the Hecke triangle group G_q: if the expansion (2.1.4) is finite it can be written as follows

$$\begin{aligned} [a_0; a_1, a_2, a_3, \dots, a_L] &= a_0\lambda_q + \cfrac{-1}{a_1\lambda_q + \cfrac{-1}{a_2\lambda_q + \cfrac{-1}{a_3\lambda_q + \dots \cfrac{-1}{a_L\lambda_q}}}} \\ &= T^{a_0} S T^{a_1} S T^{a_2} S T^{a_3} \cdots S T^{a_L}\, 0, \end{aligned} \tag{2.2.4}$$

since $\frac{-1}{a\lambda_q+x} = S\,T^a\,x$. For infinite converging λ_q-CF the expansion has to be interpreted as

$$\begin{aligned}[a_0; a_1, a_2, a_3, \ldots] &= \lim_{L\to\infty}[a_0; a_1, a_2, a_3, \ldots, a_L] \\ &= \lim_{L\to\infty} T^{a_0}\,S\,T^{a_1}\,S\,T^{a_2}\,S\,T^{a_3}\,\cdots\,S\,T^{a_L}\,0 \\ &= T^{a_0}\,S\,T^{a_1}\,S\,T^{a_2}\,S\,T^{a_3}\,\cdots\,0.\end{aligned}$$

An immediate consequence of this is

L 2.2.2. *For a finite regular λ_q-CF one finds for q even*

$$[\![a_0; a_1, \ldots, a_n, (1)^h]\!] = [\![a_0; a_1, \ldots, a_n - 1, (-1)^h]\!]$$

respectively for q odd

$$[\![a_0; \ldots, a_n, (1)^h, 2, (1)^h]\!] = [\![a_0; \ldots, a_n - 1, (-1)^h, -2, (-1)^h]\!].$$

P . Assume the left hand side to be regular. This implies $a_n \neq 1$ and hence the right hand side is regular, too. Conversely, assume the right hand side to be regular and hence $a_n - 1 \neq -1$. Therefore the expansions on the left hand side are regular.

The identity now follows by writing λ_q-CF's in terms of Möbius transformations and using the identity $(S\,T)^{h_q}\;0 = T^{-1}\,(S\,T^{-1})^{h_q}\;S\,T^{-1}S\;0 = T^{-1}\,(S\,T^{-1})^{h_q}\;0$ since 0 is a fixed point of $S\,T^{-1}S = T'$. □

R 2.2.3. For $q = 3$ the nearest λ_q-multiple continued fractions are in fact the well-known nearest integer fractions extensively discussed by Hurwitz in [8]. In particular, Theorem 2.1.2 for $q = 3$ was proved by him there. We include his results for the sake of completeness and to show how this special case $q = 3$ fits well into the discussion of the case of odd $q \geq 5$. See also Remark 2.3.2.

R 2.2.4. For $q \geq 4$ the regular λ_q-CF's correspond to Rosen's λ_q-fractions introduced in [21] and discussed in [1]. We will discuss this relation in more detail in §5.1.

2.3. Special values and their expansions. The following results are shown in [14]:

The point $x = \mp\frac{\lambda_q}{2}$ has the regular λ_q-CF

$$\mp\frac{\lambda_q}{2} = \begin{cases}[\![0; (\pm1)^{h_q}]\!] & \text{for even } q \text{ and} \\ [\![0; (\pm1)^{h_q}, \pm2, (\pm1)^{h_q}]\!] & \text{for odd } q.\end{cases} \tag{2.3.1}$$

Put

$$R_q := \lambda_q + r_q \quad \text{with}$$

$$r_q := \begin{cases}[\![0; \overline{3}]\!] & \text{for } q = 3, \\ [\![0; \overline{(1)^{h_q-1}, 2}]\!] & \text{for } q \text{ even and} \\ [\![0; \overline{(1)^{h_q}, 2, (1)^{h_q-1}, 2}]\!] & \text{for } q \text{ odd, } q \geq 5.\end{cases} \tag{2.3.2}$$

whose expansion is periodic of length κ_q with

$$\kappa_q := \begin{cases} h_q = \frac{q-2}{2} & \text{for even } q \text{ and} \\ 2h_q + 1 = q - 2 & \text{for odd } q, \end{cases} \tag{2.3.3}$$

The regular respectively dual regular λ_q-CF of the point $x = R_q$ has the form

(2.3.4)

$$R_q = \begin{cases} [\![1; \overline{(1)^{h_q-1}, 2}]\!] & \text{for even } q, \\ [\![1; \overline{(1)^{h_q}, 2, (1)^{h_q-1}, 2}]\!] & \text{for odd } q \geq 5 \text{ and} \\ [\![1; \overline{3}]\!] & \text{for } q = 3. \end{cases}$$

$$= \begin{cases} [\![0; (-1)^h, \overline{-2, (-1)^{h_q-1}}]\!]^\star & \text{for even } q, \\ [\![0; (-1)^h, \overline{-2, (-1)^{h_q}, -2, (-1)^{h_q-1}}]\!]^\star & \text{for odd } q \geq 5 \text{ and} \\ [\![0; -2, \overline{-3}]\!]^\star & \text{for } q = 3. \end{cases} \tag{2.3.5}$$

Moreover,

$$R_q = 1 \quad \text{and} \quad -R_q = S\,R_q \qquad \text{for even } q \text{ and} \tag{2.3.6}$$

$$R_q^2 + (2 - \lambda_q)R_q = 1 \quad \text{and} \quad -R_q = (TS)^{h_q+1}\,R_q \qquad \text{for odd } q \tag{2.3.7}$$

and R_q satisfies the inequality

$$\frac{\lambda_q}{2} < R_q \leq 1. \tag{2.3.8}$$

R 2.3.1. For R_3 one finds

$$1 + R_3 = \frac{1 + \sqrt{5}}{2}.$$

R 2.3.2. The form of the λ_q-CF of r_3 in (2.3.2) can be obtained from the expansions for q odd, $q \geq 5$ by interpreting it as a Möbius transformation with $(1)^{-1}$ as ST^{-1}:

$$\begin{aligned} r_3 &= [\![0; \overline{1^{h_3}, 2, (1)^{h_3-1}, 2}]\!] = [\![0; \overline{2, (1)^{-1}, 2}]\!] \\ &= ST^2ST^{-1}ST^2 \cdot ST^2ST^{-1}ST^2 \cdots 0 \\ &= ST^2TSTS\,ST^2 \cdot ST^2TSTS\,ST^2 \cdots 0 \\ &= ST^3ST^3 \cdot ST^3ST^3 \cdots 0 = [\![0; \overline{3}]\!]. \end{aligned}$$

2.4. A lexiographic order. Let $x, y \in Ir := \left[-R_q, R_q\right]$ have the regular λ_q-CF's $x = [\![a_0; a_1, \ldots]\!]$ and $y = [\![b_0; b_1, \ldots]\!]$. Denote by $l(x) \leq \infty$ respectively $l(y) \leq \infty$ the number of entries in the above λ_q-CF's. We introduce a *lexiographic order* "$\prec$" for λ_q-CF's in the following way: For $n \in \mathbb{Z}_{\geq 0}$ being the number of equal digits at the head of the λ_q-CF's, i.e., $a_i = b_i$ for all

$0 \le i \le n$ and $l(x), l(y) \ge n$, we define
(2.4.1)

$$x \prec y :\Longleftrightarrow \begin{cases} a_0 < b_0 & \text{if } n = 0, \\ a_n > 0 > b_n & \text{if } n > 0, \text{ both } l(x), l(y) \ge n+1 \text{ and } a_n b_n < 0, \\ a_n < b_n & \text{if } n > 0, \text{ both } l(x), l(y) \ge n+1 \text{ and } a_n b_n > 0, \\ b_n < 0 & \text{if } n > 0 \text{ and } l(x) = n \text{ or} \\ a_n > 0 & \text{if } n > 0 \text{ and } l(y) = n. \end{cases}$$

We also write $x \preceq y$ for $x \prec y$ or $x = y$.

This is indeed an order on regular λ_q-CF's, since Lemmas 22 and 23 in [14] imply:

L 2.4.1. *Let x and y have regular λ_q-CF's. Then $x \prec y$ if and only if $x < y$.*

The authors of [14] introduce a process called "rewriting" of λ_q-CF's where forbidden blocks in the λ_q-CF are replaced by allowed ones without changing its value. The rules for "rewriting" are based on the interpretation of a λ_q-CF in terms of Möbius transformations given by group elements of the Hecke group, see (2.2.4), and applying the group identities (2.1.3). We refer in particular to Lemma 11 and Lemma 13 in [14] for the details. A simple example for this rewriting is used in the proof of Lemma 2.2.2.

It follows from the proof of Lemma 34 in [14] that every dual regular λ_q-CF can be rewritten into a regular λ_q-CF.

L 2.4.2. *The lexiographic order $\prec$ in (2.4.1) can be extended to dual regular λ_q-CF's with leading digit 0. Rewriting two dual regular λ_q-CF's satisfying $[\![0; a_1, \ldots]\!]^\star \prec [\![0; b_1, \ldots]\!]^\star$ into regular λ_q-CF's does not change their order.*

R 2.4.3. The lexiographic order " $\prec$ " however cannot be defined for all dual regular λCF's with arbitrary leading coefficient as the following example shows: consider the dual regular λ_3-CF's of R_3 in (2.3.4) and (2.3.5). Obviously $R = [\![0; -2, \overline{-3}]\!]^\star = [\![1; \overline{3}]\!]^\star$. Extending naively "$\prec$" in (2.4.1) to this case would lead to $[\![0; -2, \overline{-3}]\!]^\star \prec [\![1; \overline{3}]\!]^\star$ and hence $[\![0; -2, \overline{-3}]\!]^\star \prec [\![0; -2, \overline{-3}]\!]^\star$.

P L 2.4.2. The λ_q-CF's $[\![0; a_1, \ldots]\!]^\star$ and $[\![0; b_1, \ldots]\!]^\star$ are dual regular. No rewriting is necessary if both are also regular.

Assume $[\![0; b_1, b_2, \ldots]\!]^\star$ starts with a forbidden block. If it is of the form

$$[\![0; b_1, b_2, \ldots]\!]^\star = \begin{cases} [\![0; (1)^{h_q}, m]\!]^\star & \text{for even } q, \\ [\![0; (1)^{h_q}, 2, (1)^{h_q}, m]\!]^\star & \text{for odd } q \ge 5 \text{ and} \\ [\![0; 2, m]\!]^\star & \text{for } q = 3 \end{cases}$$

with $m \geq 2$ for $q \geq 4$ respectively $m \geq 3$ for $q = 3$, then $[\![0; a_1, a_2, \ldots]\!]^\star$ must be of the form

$$[\![0; a_1, a_2, \ldots]\!]^\star = \begin{cases} [\![0; (1)^{h_q}, n]\!]^\star & \text{for even } q, \\ [\![0; (1)^{h_q}, 2, (1)^{h_q}, n]\!]^\star & \text{for odd } q \geq 5 \text{ and} \\ [\![0; 2, n]\!]^\star & \text{for } q = 3 \end{cases}$$

with $n < m$ and $n \geq 2$ for $q \geq 4$ respectively $n \geq 3$ for $q = 3$. Using the rewriting rules in Lemmas 11 and 13 in [14] we find

$$\begin{aligned} [\![0; a_1, a_2, \ldots]\!]^\star &\to [\![\tilde{a}_0; \tilde{a}_1, \tilde{a}_2, \ldots]\!] \\ &:= \begin{cases} [\![-1; (-1)^{h_q}, n-1, \ldots]\!] & \text{for even } q, \\ [\![-1; (-1)^{h_q}, -2, (-1)^{h_q}, n-1, \ldots]\!] & \text{for odd } q \geq 5, \\ [\![-1; -2, n-1, \ldots]\!] & \text{for } q = 3 \text{ and} \end{cases} \end{aligned}$$

respectively

$$\begin{aligned} [\![0; b_1, b_2, \ldots]\!]^\star &\to [\![\tilde{b}_0; \tilde{b}_1, \tilde{b}_2, \ldots]\!] \\ &:= \begin{cases} [\![-1; (-1)^{h_q}, m-1, \ldots]\!] & \text{for even } q, \\ [\![-1; (-1)^{h_q}, -2, (-1)^{h_q}, m-1, \ldots]\!] & \text{for odd } q \geq 5, \\ [\![-1; -2, m-1, \ldots]\!] & \text{for } q = 3 \end{cases} \end{aligned}$$

hence by (2.4.1) $[\![-1; \tilde{a}_1, \tilde{a}_2, \ldots]\!] \prec [\![-1; \tilde{b}_1, \tilde{b}_2, \ldots]\!]$.

If $[\![0; b_1, b_2, b_3, \ldots]\!]^\star$ is of the form

(2.4.2)

$$[\![0; b_1, b_2, \ldots]\!]^\star = \begin{cases} [\![0; (-1)^{h_q}, -m, \ldots]\!]^\star & \text{for even } q, \\ [\![0; (-1)^{h_q}, -2, (-1)^{h_q}, -m, \ldots]\!]^\star & \text{for odd } q \geq 5, \\ [\![0; (-2, -m), \ldots]\!]^\star & \text{for } q = 3 \end{cases}$$

with $m \geq 2$ for $q \geq 4$ respectively $m \geq 3$ for $q = 3$, and $[\![0; a_1, a_2, \ldots]\!]^\star$ does not contain a forbidden block starting with a_1, the rewriting rules in [14] give

$$\begin{aligned} [\![0; b_1, b_2, \ldots]\!]^\star &\to [\![\tilde{b}_0; \tilde{b}_1, \tilde{b}_2, \ldots]\!] \\ &= \begin{cases} [\![1; (1)^{h_q}, 1-m, \ldots]\!] & \text{for even } q, \\ [\![1; (1)^{h_q}, 2, (1)^{h_q}, 1-m, \ldots]\!] & \text{for odd } q \geq 5, \\ [\![1; 2, 1-m, \ldots]\!] & \text{for } q = 3. \end{cases} \end{aligned}$$

Therefore (2.4.1) implies $[\![0; a_1, a_2, \ldots]\!]^\star \prec [\![1; \tilde{b}_1, \tilde{b}_2, \ldots]\!]$.

If $[\![0; b_1, b_2, b_3, \ldots]\!]^\star$ is of the form (2.4.2) and $[\![0; a_1, a_2, \ldots]\!]^\star$ is of the form

$$[\![0; a_1, a_2, \ldots]\!]^\star = \begin{cases} [\![0; (-1)^{h_q}, -n]\!]^\star & \text{for even } q, \\ [\![0; (-1)^{h_q}, 2, (1)^{h_q}, -n]\!]^\star & \text{for odd } q \geq 5 \text{ and} \\ [\![0; 2, -n]\!]^\star & \text{for } q = 3 \end{cases}$$

with $n > m$ then the rewriting rules in [14] lead to

$$[\![1;\tilde{a}_1,\tilde{a}_2,\ldots]\!] = \begin{cases} [\![1;(1)^{h_q},1-n,\ldots]\!] & \text{for even } q,\\ [\![1;(1)^{h_q},2,(1)^{h_q},1-n,\ldots]\!] & \text{for odd } q\geq 5,\\ [\![1;2,1-n,\ldots]\!] & \text{for } q=3,\end{cases}$$

$$[\![1;\tilde{b}_1,\tilde{b}_2,\ldots]\!] = \begin{cases} [\![1;(1)^{h_q},1-m,\ldots]\!] & \text{for even } q,\\ [\![1;(1)^{h_q},2,(1)^{h_q},1-m,\ldots]\!] & \text{for odd } q\geq 5,\\ [\![1;2,1-m,\ldots]\!] & \text{for } q=3,\end{cases}$$

and hence $[\![1;\tilde{a}_1,\tilde{a}_2,\ldots]\!] \prec [\![1;\tilde{b}_1,\tilde{b}_2,\ldots]\!]$.

Completely analogous are the cases when $[\![0;a_1,\ldots]\!]^\star$ with a forbidden block or the first forbidden block starts at a_n, $n>1$ and $[\![0;b_1,b_2,\ldots]\!]^\star$ is a regular λ_q-CF. If both $[\![0;a_1,\ldots]\!]^\star$ and $[\![0;b_1,\ldots]\!]^\star$ have the same forbidden block starting at a_1 respectively b_1 then both dual regular λ_q-CF's are rewritten in the same way and the forbidden block does not influence the order "$\prec$". □

2.5. Equivalence relations and continued fractions. Let $x,y\in\mathbb{R}$ have infinite regular λ_q-CF's $x=[\![a_0;a_1,\ldots]\!]$ and $y=[\![b_0;b_1,\ldots]\!]$. We say that x and y are *regular λ_q-CF-equivalent*, denoted by $x\sim_{\text{reg}} y$, if the regular λ_q-CF's of x and y have the same tail, i.e., there exists $m,n\in\mathbb{N}$ such that the sequences $(a_m,a_{m+1},\ldots)$ and $(b_n,b_{n+1},\ldots)$ coincide. Obviously, this is an equivalence relation. We can extend this equivalence relation to all regular λ_q-CF's by declaring all finite regular λ_q-CF's to be regular λ_q-CF-equivalent.

T 2.5.1 (Equivalence relations). *For $x,y\in\mathbb{R}$ the following properties are equivalent:*

(1) $x\sim_{G_q} y$.
(2) *x and y satisfy:*
   - $x\sim_{reg} y$ *or*
   - $x\sim_{reg}\pm r$ *and* $y\sim_{reg}\mp r$.

To prove the proposition, we need the following lemmas:

L 2.5.2. *If x has an infinite regular λ_q-CF and $g\in G_q$ satisfies $gx\in\mathbb{R}$, then gx has an infinite λ_q-CF with at most h_q consecutive digits ±1. Its tail coincides with the tail of the regular λ_q-CF of x.*

P . [-1] Let x have the regular λ_q-CF $x=[\![a_0;a_1,\ldots]\!]$. We can write g as a word in the generators S and T_q: $g=T_q^{b_0}ST_q^{b_1}ST_q^{b_2}\cdots ST_q^{b_m}S^\delta$ with $b_0\in\mathbb{Z}$, $b_i\in\mathbb{Z}_{\neq0}$, $i=1,\ldots,m$, and $\delta\in\{0,1\}$. Then gx can formally be written as $gx=T^{b_0}ST^{b_1}ST^{b_2}\cdots ST^{b_m}S^\delta T^{a_0}ST^{a_1}ST^{a_2}\cdots 0$. Change -1

Consider for $n>m$ sufficiently large the element $g_n\in G_q$ given by

$$g_n := T_q^{b_0}ST_q^{b_1}ST_q^{b_2}\cdots ST_q^{b_m}S^\delta T_q^{a_0}ST_q^{a_1}ST_q^{a_2}\cdots ST_q^{a_n}\in G_q. \tag{2.5.1}$$

[-1] Beweis leicht umformuliert.

even q: $(m \geq 1)$		
$\pm[a_{i-1}, \underline{1^{h_q+1}}, 2, 1^{h_q-1}, m]$	$\to$	$\pm[a_{i-1}-1, (-1)^{h_q-1}, \underline{(1)^{h_q}, m}]$
$\pm[a_{i-1}, \underline{1^{h_q}}, 2, 1^{h_q-1}, m]$	$\to$	$\pm[a_{i-1}-1, (-1)^{h_q}, \underline{1^{h_q}, m}]$
odd $q \geq 5$: $(m \geq 2)$		
$\pm[a_{i-1}, \underline{1^{h_q+1}}, 2, 1^{h_q}]$	$\to$	$\pm[a_{i-1}-1, (-1)^{h_q}, \underline{1^{h_q+1}}]$
$\pm[a_{i-1}, \underline{1^{h_q+1}}, 2, 1^{h_q-1}, 2, 1^{h_q}, m]$	$\to$	$\pm[a_{i-1}-1, (-1)^{h_q}, \underline{1^{h_q}, 2, 1^{h_q}, m}]$
$\pm[a_{i-1}, \underline{1^{h_q}, 2, 1^{h_q}, 2}, 1^{h_q}]$	$\to$	$\pm[a_{i-1}-1, (-1)^{h_q}, -2, (-1)^{h_q}, \underline{1^{h_q+1}}]$
$\pm[a_{i-1}, \underline{1^{h_q}, 2, 1^{h_q}, 2}, 1^{h_q-1}, 2, 1^{h_q}, m]$	$\to$	$\pm[a_{i-1}-1, (-1)^{h_q}, -2, (-1)^{h_q}, \underline{1^{h_q}, 2, 1^{h_q}, m}]$
$q = 3$: ($m \geq 3$ and $n \geq 0$)		
$\pm[a_{i-1}, \underline{2, 3}, m]$	$\to$	$\pm[a_{i-1}-1, -2, \underline{2, m}]$
$\pm[a_{i-1}, \underline{2, 2}, 2^n, 3, m]$	$\to$	$\pm[a_{i-1}-1, -(3+n), \underline{2, m}]$

T 1. Under assumptions of Lemma 2.5.3 we list all possibilities where the rewriting of a forbidden block generates a new forbidden block. The forbidden blocks are underlined.

The identities $S^2 = \mathbf{1}$, $\left(S\,T_q^{\pm 1}\right)^q = \mathbf{1}$, $T_q^a \left(S\,T_q^{\pm 1}\right)^{q-1} S\,T_q^b = T_q^{a+b\mp 1}$ and $T_q^a \left(S\,T_q^{\pm 1} S\,T_q^b\right)^l = T_q^{a\mp 1} \left(S\,T_q^{\mp 1}\right)^{q-l-2} S\,T_q^{b\mp 1}$ for $h_q + 1 \leq l \leq q-2$ and a, b arbitrary follow from (2.1.3). But $q - l - 2 \leq q - (h_q + 1) - 2 = h_q - 1$ for q even and $q - l - 2 \leq h_q$ for q odd. We apply these identities recursively on g_n in (2.5.1). After a finite number of steps one arrives at a word representing g_n which contains blocks of at most h_q consecutive digits ± 1. Indeed, since each application of one of these identities reduces the length of the word, the process of applying the identities has to stop after a finite number of steps. And, since the λ_q-CF of x is reduced, there are no blocks of more than h_q consecutive ± 1 to the right of the right of part "$S\,T^{b_m}\,S^{\delta}\,T^{a_0}\,S\,T^{a_1}$" of the word g_n in (2.5.1).

Hence $g\,x$ can be written as a λ_q-CF of the form (2.2.4) without blocks of more than h_q consecutive digits ± 1 and with a tail identical to the regular tail in the λ_q-CF of x. □

Change -2

L 2.5.3. [-2] *Let $[a_0; a_1, a_2, \dots]$ be an infinite λ_q-CF containing no blocks of more than $h_q + 1$ consecutive digits ± 1 and at most one block of $h_q + 1$ consecutive digits ± 1 for $q \geq 4$ respectively no forbidden digits ± 1 for $q = 3$. If the block $[(\pm 1)^{h_q+1}]$ exists the block has to be the first forbidden block of the λ_q-CF and has to be preceded by a digit of alternate sign.*

If the first forbidden block starts at a_i, $i \geq 1$ and its rewriting leads to a new forbidden block then this forbidden block and its rewritten version must have the form given in Table 1. The new forbidden block will appear to the right of a_i. If the new forbidden block is of the form $[(\pm 1)^{h_q+1}]$ then its preceding digit is negative.

P . W.l.o.g. assume, the forbidden block starting at a_i has positive digits and hence $a_{i-1} \neq 1$. For q even the forbidden block must have the

[-2] Lemma leicht umformuliert

form $[1^{h_q+1}]$ or $[1^{h_q}, m]$ with $m \geq 2$. The rewriting rules in Lemma 11 of [14] lead to

$$[\ldots, a_{i-1}, 1^{h_q+1}, a_{i+h_q+1}, \ldots] \to [\ldots, a_{i-1}-1, (-1)^{h_q-1}, a_{i+h_q+1}-1, \ldots] \text{ and}$$
$$[\ldots, a_{i-1}, 1^{h_q}, m, a_{i+h_q+1}, \ldots] \to [\ldots, a_{i-1}-1, (-1)^{h_q}, m-1, a_{i+h_q+1}, \ldots]$$

for $m \geq 2$. Changing a_{i-1} to $a_{i-1}-1$ cannot introduce a forbidden block since $a_{i-1} \neq 1$ and a digit $a_{i-1} \geq 2$ cannot follow a block of the from $[1^{h_q}]$. Hence any new forbidden block has to start with the digit $a_{i+h_q+1}-1$ respectively $m-1$. Two cases are possible: $[a_{i+h_q+1}, \ldots] = [2, 1^{h_q-1}, l, \ldots]$ respectively $[m, \ldots] = [2, 1^{h_q-1}, l, \ldots]$ with $l \geq 1$. This shows that the block $[a_i, \ldots]$ must have the form $[1^{h_q+1}, 2, 1^{h_q-1}, l]$ respectively $[1^{h_q}, 2, 1^{h_q-1}, l]$ with its rewriting leading to the form as stated in the lemma.

For q odd, $q \geq 5$, the forbidden block $[a_i, \ldots]$ has either the form $[1^{h_q}, 2, 1^{h_q}, m]$ with $m \geq 2$ or the form $[1^{h_q+1}]$. Rewriting rules in Lemma 13 of [14] then give

$$[\ldots, a_{i-1}, 1^{h_q+1}, a_{i+h_q+1}, \ldots] \to [\ldots, a_{i-1}-1, (-1)^{h_q}, a_{i+h_q+1}-1, \ldots] \text{ and}$$
$$[\ldots, a_{i-1}, 1^{h_q}, 2, 1^{h_q}, m, \ldots] \to [\ldots, a_{i-1}-1, (-1)^{h_q}, -2, (-1)^{h_q}, m-1, \ldots]$$

for $m \geq 2$, and similar arguments as for q even show that the forbidden block and the digits following it are either of the form $[1^{h_q+1}]$ followed by $[2, 1^{h_q}]$ or $[2, 1^{h_q-1}, 2, 1^{h_q}, l]$, $l \geq 2$, or $[1^{h_q}, 2, 1^{h_2}, 2]$ followed by $[1^{h_q}]$ respectively $[1^{h_q-1}, 2, 1^{h_q}, l]$, $l \geq 2$. The rewritten form is the as given in the lemma. For $a_{i-1} = 2$ rewriting cannot lead to a new forbidden block to the left of a_i contradicting otherwise the first forbidden block to start with a_i.

The case $q = 3$ with forbidden block $[2, 2^n, m]$, $m \geq 2$ and $n \in \mathbb{Z}_{\geq 0}$ can be handled in complete analogy by using the rewriting rule $[a, 2, 2^n, b] \to [a-1, -2-n, b-1]$ with $a, b \neq 2$. □

P P 2.5.1. We show first the implication (2) ⇒ (1). If $x \sim_{\text{reg}} y$ then x and y have regular λ_q-CF's with the same tail: [-3] Change -3

$$x = [\![a_0; a_1, \ldots, a_m, a_{m+1}, \ldots]\!] \text{ and } y = [\![b_0; b_1, \ldots, b_n, a_{m+1}, \ldots]\!].$$

Put $g := T^{a_0} S T^{a_1} \cdots S T^{a_m} \left(T^{b_0} S T^{b_1} \cdots S T^{b_n}\right)^{-1} \in G_q$. Writing x and y in terms of Möbius transformations as explained in (2.2.4) we find,

$$\begin{aligned} g\,y &= g\,T^{b_0} S T^{b_1} \cdots S T^{b_n} S T^{a_{m+1}} \cdots 0 \\ &= T^{a_0} S T^{a_1} \cdots S T^{a_m} S T^{a_{m+1}} \cdots 0 = x \end{aligned}$$

and hence x and y are G_q-equivalent [-4]. Change -4

Assume next $x \sim_{\text{reg}} r$ and $y \sim_{\text{reg}} -r$ and hence $x \sim_{G_q} r$ and $y \sim_{G_q} -r$. Since $R_q = T_q\, r_q$ according to (2.3.2) and $-R_q = S R_q$ according to (2.3.6) for even q respectively $-R_q = (T_q)^{h_q+1} R_q$ according to (2.3.7) for odd q obviously $r \sim_{G_q} -r$ and hence $x \sim_{G_q} y$.

[-3] Formelnummer der Formelzeile entfernt.

[-4] Formelzeile weggelassen, da sie unsinnig ist.

Change -5
Remark -6

To show implication (1) $\Rightarrow$ (2), assume there exists $g \in G_q$ with $g\,x = y$ with x and y having infinite [-5] [-6] regular λ_q-CF's. Lemma 2.5.2 shows that $g\,x$ can be written as an infinite λ_q-CF with regular tail satisfying the assumptions of Lemma 2.5.3. Using the rewriting rules in Lemmas 11 and 13 of [14] we can recursively rewrite the λ_q-CF of $g\,x$ into a regular λ_q-CF from the left to the right. We procede to the next forbidden block if rewriting does not lead to a new forbidden block. Lemma 2.5.3 implies that a new forbidden block can only appear to the right of the original one which we process next. If this rewriting process stops after finitely many steps then $y = g\,x$ and x have the same tail in their λ_q-CF's and $x \sim_{\text{reg}} y$.

Hence assume, the rewriting process has to be repeated again and again. Then after a sufficiently large but finite number of rewriting steps one arrives at the situation where the λ_q-CF of $g\,x$ is regular up to one forbidden block. Denote this λ_q-CF by $[a_0; a_1, \ldots]$ with the remaining forbidden block starting at digit a_i, $i \geq 1$ and assume w.l.o.g. the forbidden block has positive digits.

Consider first the case q even: By Lemma 2.5.3 the forbidden block and the following digits have the form $[a_i, \ldots] = [B_0, B_1, B_2, \ldots]$ with the block $B_0 \in \left\{[1^{h_q+1}, 2], [1^{h_q}, 2]\right\}$ and the blocks $B_j \in \left\{[1^{h_q}, 2], [1^{h_q-1}, 2]\right\}$, for all $j \geq 1$. Since by assumption B_0 was the last forbidden block in the λ_q-CF, necessarily $B_j \neq [1^{h_q}, 2]$. Hence the λ_q-CF of $g\,x$ has the form

$$g\,x = [a_0; a_1, \ldots, a_{i-1}, \underline{1^l, 2, \overline{1^{h_q-1}, 2}}] \quad \text{with } l = h_q, h_q + 1$$

where the forbidden block at digit a_i is underlined, and whose tail, determining also the tail of x, is regular λ_q-CF-equivalent to r_q. After infinitely many further rewritings one arrives at the regular λ_q-CF of y whose tail is regular λ_q-CF-equivalent to $-r_q$.

Consider next the case $q \geq 5$ odd: Lemma 2.5.3 again determines the form of the forbidden block and the following digits as

$$[a_i, \ldots] = [B_0, B_1, B_2, \ldots]$$

with the block $B_0 \in \left\{[1^{h_q+1}, 2], [1^{h_q}, 2, 1^{h_q}, 2]\right\}$ and the blocks

$$B_j \in \left\{A_1 := [1^{h_q}, 2], A_2 := [1^{h_q-1}, 2, 1^{h_q}, 2]\right\}, \qquad j \geq 1.$$

Since the blocks $[A_1, A_1]$ and $[A_2, A_1]$ are forbidden blocks, necessarily $B_j = A_2$ for all $j \geq 2$, since otherwise B_0 would not be the last forbidden block in the λ_q-CF of $g\,x$. Hence the λ_q-CF of $g\,x$ has the form

$$g\,x = [a_0; a_1, \ldots, a_{i-1}, \underline{B_0}, B_1, \overline{1^{h_q-1}, 2, 1^{h_q}, 2}]$$

where the forbidden block at digit a_i is again underlined. As in the previous case, we find x is regular λ_q-CF-equivalent to r_q and y is regular λ_q-CF-equivalent to $-r_q$.

[-5] "Infinite" hinzugefgt.

[-6] Wie beweise ich den Fall der endlichen Λ-CF's?

Consider finally the case $q = 3$: Lemma 2.5.3 gives again the form of the forbidden block and the following digits as $[a_i, \ldots] = [B_0, B_1, \ldots]$ with $B_0 = [2, 2^n, 3]$, $n \geq 0$, and the blocks $B_j \in \{[2], [3]\}$, $j \geq 1$. Since the blocks $[2, 2]$ and $[2, 3]$ are forbidden, necessarily $B_j = [3]$ for all $j \geq 1$. The λ_q-CF of $g\,x$ hence has the form

$$g\,x = [a_0; a_1, \ldots, a_{i-1}, \underline{2, 2^n, 3}, \overline{3}]$$

where the forbidden block at digit a_i is underlined. Again x is regular λ_q-CF-equivalent to r_q and y is regular λ_q-CF-equivalent to $-r_q$.

□

3. Generating maps for the λ_q-continued fractions and their duals

Similar to the Gauss continued fractions also the λ_q-continued fractions and their duals, which for $q = 3$ have been introduced by Hurwitz in [8], can be generated by interval maps with strong ergodic properties like in the case of the Gauss maps.

3.1. The interval maps f_q and $f_q^\star$. Denote by I_q respectively I_{R_q} the intervals

$$I_q = \left[-\frac{\lambda_q}{2}, \frac{\lambda_q}{2}\right] \quad \text{respectively} \quad I_{R_q} = \left[-R_q, R_q\right] \tag{3.1.1}$$

with λ_q as in (2.1.4) and $R_q = \lambda_q + r_q$ as in (2.3.2). The *nearest λ_q-multiple map* $\langle\cdot\rangle_q$ is given by

$$\langle\cdot\rangle_q : \mathbb{R} \to \mathbb{Z}; \quad x \mapsto \langle x\rangle_q := \left\lfloor \frac{x}{\lambda_q} + \frac{1}{2} \right\rfloor \tag{3.1.2}$$

where $\lfloor\cdot\rfloor$ is the (modified) floor function

$$\lfloor x \rfloor = n \iff \begin{cases} n \leq x < n+1 & \text{if } x > 0 \text{ and} \\ n < x \leq n+1 & \text{if } x \leq 0. \end{cases} \tag{3.1.3}$$

We also need the map $\langle\cdot\rangle_q^\star$ given by

$$\langle\cdot\rangle_q^\star : \mathbb{R} \to \mathbb{Z}; \quad x \mapsto \langle x\rangle_q^\star := \begin{cases} \left\lfloor \frac{x}{\lambda_q} + 1 - \frac{R_q}{\lambda_q} \right\rfloor & \text{if } x \geq 0 \text{ and} \\ \left\lfloor \frac{x}{\lambda_q} + \frac{R_q}{\lambda_q} \right\rfloor & \text{if } x < 0. \end{cases} \tag{3.1.4}$$

The interval maps $f_q \colon I_q \to I_q$ and $f_q^\star \colon I_{R_q} \to I_{R_q}$ are defined as follows:

$$f_q(x) = \begin{cases} -\frac{1}{x} - \left\langle \frac{-1}{x} \right\rangle_q \lambda_q & \text{if } x \in I_q \backslash \{0\}, \\ 0 & \text{if } x = 0 \text{and} \end{cases} \tag{3.1.5}$$

$$f_q^\star(y) = \begin{cases} -\frac{1}{y} - \left\langle \frac{-1}{y} \right\rangle_q^\star \lambda_q & \text{if } y \in I_{R_q} \backslash \{0\}, \\ 0 & \text{if } y = 0. \end{cases} \tag{3.1.6}$$

3.2. λ_q-CF's and their generating interval maps. The maps f_q and $f_q^\star$ generate the regular respectively dual regular λ_q-CF's in the following sense:

For given $x, y \in \mathbb{R}$ the coefficients a_i and b_1, $i \in \mathbb{Z}_{\geq 0}$ are determined by the following algorithms:

(0) $a_0 = \langle x \rangle_q$ and $x_1 := x - a_0\lambda_q \in I_q$,
(1) $a_1 = \left\langle \frac{-1}{x_1} \right\rangle_q$ and $x_2 := \frac{-1}{x_1} - a_1\lambda_q = f_q(x_1) \in I_q$,
(i) $a_i = \left\langle \frac{-1}{x_i} \right\rangle_q$ and $x_{i+1} := \frac{-1}{x_i} - a_i\lambda_q = f_q(x_i) \in I_q$,
$(\star)$ The algorithm terminates if $x_{i+1} = 0$

and

(0) $b_0 = \langle x \rangle_q^\star$ and $y_1 := y - b_0\lambda_q \in I_{R_q}$,
(1) $b_1 = \left\langle \frac{-1}{y_1} \right\rangle_q^\star$ and $y_2 := \frac{-1}{y_1} - b_1\lambda_q = f_q^\star(y_1) \in I_{R_q}$,
(i) $b_i = \left\langle \frac{-1}{y_i} \right\rangle_q^\star$ and $y_{i+1} := \frac{-1}{y_i} - b_i\lambda_q = f_q^\star(y_i) \in I_{R_q}$,
$(\star)$ The algorithm terminates if $y_{i+1} = 0$.

By construction the coefficients form λ_q-CF's in the sense of (2.2.3):

$$x = [a_0; a_1, a_2, \ldots] \quad \text{and} \quad y = [b_0; b_1, b_2, \ldots]. \tag{3.2.1}$$

P 3.2.1. *The λ_q-CF of x in (3.2.1) is unique for all x not in $\bigcup_{n=0}^\infty f_q^{-n}(\pm\frac{\lambda_q}{2})$ and regular whereas the one of y is unique for all $y \notin \bigcup_{n=0}^\infty \left(f_q^\star\right)^{-n}(\pm r_q)$ and dual regular.*

P . A simple calculation shows that the regular λ_q-CF of all points $x = \pm\frac{2}{2m-1)\lambda_q}$, $m = 2, 3, \ldots$ and their preimages is not unique. But these points belong to the preimages of the points $\pm\frac{\lambda_q}{2}$ On the other hand the dual λ_q-CF of the points $y = \pm\frac{1}{r_q + m\lambda_q}$, $m = 1, 2, \ldots$ and their preimages is not unique. But these points are all the preimages of the points $\pm r_q$. □

R 3.2.2. The non-uniqueness of certain finite regular λ_q-CF's in Lemma 2.2.2 can also be derived from Proposition 3.2.1.

3.3. Markov partitions for f_q and $f_q^\star$. Obviously f_q is locally expanding, that means $\left|f_q'(x)\right| > 1$ for all $x \in I_q$, if one takes the one-sided derivatives at the points of discontinuity. The same holds true for the map $f_q^\star$ for q odd. For q even $f_q^{\star\prime}(\pm R_q) = 1$ but $\left|\left(f_q^{\star 2}\right)'(y)\right| > 1$ for all $y \in I_{R_q}$, and hence both maps f_q and $f_q^\star$ are locally smooth, expanding maps. Indeed both maps have the Markov property, that means that they allow for Markov partitions. To construct these partitions we use the orbits of the boundary points of the two intervals I_q and I_{R_q} respectively the monotonicity intervals of the maps f_q and $f_q^\star$.

Define the orbit of x under f_q respectively $f_q^\star$ as

$$\text{orbit}\,(x) = \left\{x,\, f_q(x),\, f_q^2(x) := f_q(f_q(x)),\, f_q^3(x), \ldots\right\} \tag{3.3.1}$$
$$= \left\{f_q^n(x);\ n = 0, 1, 2, \ldots\right\} \quad \text{respectively}$$
$$\text{orbit}^\star\,(x) = \left\{\left(f_q^\star\right)^n(x);\ n = 0, 1, 2, \ldots\right\}. \tag{3.3.2}$$

The orbits $\text{orbit}\left(-\frac{\lambda_q}{2}\right)$ and $\text{orbit}^\star\left(-R_q\right)$ are both finite. If $\sharp\{S\}$ denotes the cardinality of the set S, we have

$$\sharp\text{orbit}\left(-\frac{\lambda_q}{2}\right) = \sharp\text{orbit}^\star\left(-R_q\right) = \kappa_q + 1,$$

as can be seen from the regular λ_q-CF of $-\frac{\lambda_q}{2}$ in (2.3.1) and the dual regular λ_q-CF of $-R_q$ in (2.3.5). We denote the elements of $\text{orbit}\left(-\frac{\lambda_q}{2}\right)$ by ϕ_i respectively of $\text{orbit}^\star\left(-R_q\right)$ by ψ_i, $i = 0, \ldots, \kappa_q$, such that

(3.3.3)

$$-R_q = -\psi_0 < -\frac{\lambda_q}{2} = \phi_0 < \psi_1 < \phi_1 < \psi_2 < \phi_2 < \ldots$$
$$\ldots < \psi_{\kappa_q-2} < \phi_{\kappa_q-2} < \psi_{\kappa_q-1} < \phi_{\kappa_q-1} = -\frac{1}{\lambda_q} < \psi_{\kappa_q} < \phi_{\kappa_q} = 0$$

holds. By using the regular λ_q-CF of $-\frac{\lambda_q}{2}$ and the dual regular $-\lambda_q$-CF of R_q respectively the order "$\prec$" in §2.4 one easily verifies

L 3.3.1. *The order in* (3.3.3) *is achieved for q even by defining*

$$\phi_i = f_q^i\left(-\frac{\lambda_q}{2}\right) \quad \textit{and} \quad \psi_i = \left(f_q^\star\right)^i(-R_q), \quad 0 \le i \le h = \kappa_q, \tag{3.3.4}$$

respectively for q odd by defining

(3.3.5)

$$\phi_{2i} = f_q^i\left(-\frac{\lambda_q}{2}\right), \qquad \phi_{2i+1} = f_q^{h+i+1}\left(-\frac{\lambda_q}{2}\right) \quad \textit{and}$$
$$\psi_{2i} = \left(f_q^\star\right)^i(-R_q), \qquad \psi_{2i+1} = \left(f_q^\star\right)^{h+i+1}(-R_q), \quad 0 \le i \le h_q = \frac{\kappa_q - 1}{2}.$$

In the case $q = 3$ one has $\kappa_3 = 1$ and $h_3 = 0$. Therefore

$$\phi_0 = -\frac{1}{2}, \quad \phi_1 = 0, \quad \psi_0 = -R_3 = \frac{1 - \sqrt{5}}{2} \text{ and } \psi_1 = R_3 - 1 = \frac{\sqrt{5} - 3}{2}.$$

Define next $\phi_{-i} = -\phi_i$, $0 \le i \le \kappa_q$, respectively $\psi_{-i} = -\psi_i$ for $0 \le i \le \kappa_q + 1$ with $\psi_{\kappa_q+1} = 0$.

Obviously the intervals

$$\Phi_i := [\phi_{i-1}, \phi_i] \quad \text{and} \quad \Phi_{-i} := [\phi_{-i}, \phi_{-(i-1)}] \qquad 1 \le i \le \kappa_q \tag{3.3.6}$$

respectively

$$\Psi_i := [\psi_{i-1}, \psi_i] \quad \text{and} \quad \Psi_{-i} := [\psi_{-i}, \psi_{-(i-1)}], \tag{3.3.7}$$

$1 \le i \le \kappa_q + 1$, define Markov partitions of the intervals I_q and I_{R_q}: this means that

$$\bigcup_{\varepsilon=+,-} \bigcup_{i=1}^{\kappa_q} \Phi_{\varepsilon i} = I_q, \quad \Phi_{\varepsilon i}^\circ \cap \Phi_{\delta j}^\circ = \emptyset \quad \text{for} \quad \varepsilon i \neq \delta j$$

$$\bigcup_{\varepsilon=+,-} \bigcup_{i=1}^{\kappa_q+1} \Psi_{\varepsilon i} = I_{R_q}, \quad \Psi_{\varepsilon i}^\circ \cap \Psi_{\delta j}^\circ = \emptyset \quad \text{for} \quad \varepsilon i \neq \delta j.$$

where S° denotes the interior of the set S. To get a reasonable symbolic dynamics for the two maps f_q and $f_q^\star$ we have to construct finer partitions using the monotonicity intervals of the two maps. Consider first the case $q = 3$ such that $\lambda_3 = 1$. Define for $m = 2, 3, 4, \ldots$ the intervals J_m as

$$J_2 = \left[-\frac{1}{2}, -\frac{2}{5}\right] \quad \text{and} \quad J_m = \left[-\frac{2}{2m-1}, -\frac{2}{2m+1}\right], \quad m = 3, 4, \ldots, \tag{3.3.8}$$

and set $J_{-m} := -J_m$ for $m = 2, 3, 4, \ldots$. Since $f_3(J_{\pm 2}) = \mp\left[0, \frac{1}{2}\right]$ and $f_3(J_{\pm m}) = I_3$ for $m = 3, 4, \ldots$ the partition satisfies

$$\bigcup_{\varepsilon=+,-} \bigcup_{m=2}^{\infty} J_{\varepsilon m} = I_3 \quad \text{and} \quad J_{\varepsilon m}^\circ \cap J_{\delta k}^\circ = \emptyset \quad \text{for} \quad \varepsilon m \neq \delta k.$$

Hence this partition, which we denote by $\mathcal{M}(f_3)$, is Markovian. The maps $f_3\big|_{J_m}$ are monotone with $f_3\big|_{J_m}(x) = -\frac{1}{x} - m$ and locally invertible with $\left(f_3\big|_{J_m}\right)^{-1}(y) = -\frac{1}{y+m}$ for $y \in f_3(J_m)$.

For $q \ge 4$ define intervals J_m, $m = 1, 2, \ldots$, as

$$\begin{aligned} J_1 &= \left[-\frac{\lambda_q}{2}, -\frac{2}{3\lambda_q}\right] \quad \text{and} \\ J_m &= \left[-\frac{2}{(2m-1)\lambda_q}, -\frac{2}{(2m+1)\lambda_q}\right], \quad m = 2, 3, \ldots, \end{aligned} \tag{3.3.9}$$

and set $J_{-m} := -J_m$ for $m \in \mathbb{N}$. For even q, $q \ge 4$, the points in orbit $\left(-\frac{\lambda_q}{2}\right)$ do not fall onto a boundary point of any of the intervals J_m, $m \in \mathbb{N}$. Indeed from the regular λ_q-CF of $-\frac{\lambda_q}{2}$ in (2.3.1) and the order "$\prec$" in §2.4 one sees easily that

$$-\frac{\lambda_q}{2} = \phi_0 < \phi_1 < \ldots < \phi_{\kappa_q - 1} < -\frac{2}{3\lambda_q} < \phi_{\kappa_q} = 0$$

with $\phi_i = f_q^i\left(-\frac{\lambda_q}{2}\right)$. If we hence define the intervals $J_{\pm 1_i}$ as

$$J_{\varepsilon 1_i} := J_{\varepsilon 1} \cap \Phi_{\varepsilon i} \quad \text{for } \varepsilon = +, -, \quad 1 \le i \le \kappa_q \tag{3.3.10}$$

and therefore $J_{\varepsilon 1_i} = \Phi_{\varepsilon i}$ for $1 \le i \le \kappa_q - 1$ we get the partition $\mathcal{M}(f_q)$, defined as

$$I_q = \bigcup_{\varepsilon=+,-} \left(\bigcup_{i=1}^{\kappa_q} J_{\varepsilon 1_i} \cup \bigcup_{m=2}^{\infty} J_{\varepsilon m} \right), \tag{3.3.11}$$

which is obviously again Markovian, since

$$\begin{aligned} f_q(J_{\varepsilon 1_i}) &= J_{\varepsilon 1_{i+1}}, \quad \varepsilon = +,-,\ i = 1, \dots, \kappa_q - 2, \\ f_q(J_{\varepsilon 1_{\kappa_q - 1}}) &= J_{\varepsilon 1_{\kappa_q}} \cup \bigcup_{m=2}^{\infty} J_{\varepsilon m}, \quad \varepsilon = +,- \quad \text{and} \\ f_q(J_{\varepsilon \kappa_q}) &= \varepsilon \left[0, \frac{\lambda_q}{2}\right], \quad \varepsilon = +,- \quad \text{respectively} \\ f_q(J_{\varepsilon m}) &= I_q, \quad \varepsilon = +,-,\ m = 2, 3, \dots . \end{aligned}$$

The maps $f_q\big|_{J_m}$ are monotone increasing with $f_q\big|_{J_m}(x) = -\frac{1}{x} - m\lambda_q$ and $\left(f_q\big|_{J_m}\right)^{-1}(y) = -\frac{1}{y+m\lambda_q}$ for $m = \pm 1, \pm 2, \pm 3, \dots$.

Consider next the case q odd, $q \ge 5$. In this case one has, using again the regular λ_q-CF of $-\frac{\lambda_q}{2}$ in (2.3.1) and the order "$\prec$" in §2.4,

$$-\frac{\lambda_q}{2} = \phi_0 < \phi_1 < \ldots < \phi_{\kappa_q - 2} < -\frac{2}{3\lambda_q} < \phi_{\kappa_q - 1} < -\frac{2}{5\lambda_q} < \phi_{\kappa_q} = 0,$$

with $\kappa_q = 2h_q + 1$ and the ϕ_i's given in (3.3.5), (3.3.6). Hence for $\varepsilon = +,-$ one finds $\phi_{\varepsilon i} \in J_{\varepsilon 1}$ for $1 \le i \le \kappa_q - 2$ and $\phi_{\varepsilon(\kappa_q - 1)} \in J_{\varepsilon 2}$. If we then define for $\varepsilon = +,-$ the intervals

(3.3.12)

$$\begin{aligned} &J_{\varepsilon 1_i} := J_{\varepsilon 1} \cap \Phi_{\varepsilon i} \quad 1 \le i \le \kappa_q - 1 \text{ and hence } J_{\varepsilon 1_i} = \Phi_{\varepsilon i},\ 1 \le i \le \kappa_q - 2 \\ &J_{\varepsilon 2,i} := J_{\varepsilon 2} \cap \Phi_{\varepsilon i},\ i = \kappa_q - 1, \kappa_q, \end{aligned}$$

we find that the partition $\mathcal{M}(f_q)$ defined by

$$I_q = \bigcup_{\varepsilon=+,-} \left(\bigcup_{i=1}^{\kappa_q - 1} J_{\varepsilon 1_i} \cup \bigcup_{i=\kappa_q - 1}^{\kappa_q} J_{\varepsilon 2_i} \cup \bigcup_{m=3}^{\infty} J_{\varepsilon m} \right) \tag{3.3.13}$$

is Markovian. Indeed for $\varepsilon = +,-$ one finds $f_q(\Phi_{\varepsilon 2i}) = \Phi_{\varepsilon(2i+2)}$, $1 \le i \le h_q - 2$, $f_q\left(J_{\varepsilon 1_{\kappa_q - 1}}\right) = \varepsilon\left[0, \frac{\lambda_q}{2}\right]$, $f_q\left(J_{\varepsilon 2_{\kappa_q - 1}}\right) = \varepsilon\left[-\frac{\lambda_q}{2}, \phi_1\right]$, $f_q\left(\Phi_{\varepsilon(2i-1)}\right) = \Phi_{\varepsilon(2i+1)}$ for $1 \le i \le h$, $f_q\left(J_{\varepsilon 2_{\kappa_q}}\right) = \varepsilon\left[\phi_1, \frac{\lambda_q}{2}\right]$ and $f_q\left(J_{\varepsilon m}\right) = I_q$ for $m = 3, 4, \dots$. The maps $f_q\big|_{J_m}$ are monotone with $f_q\big|_{J_m}(x) = -\frac{1}{x} - m\lambda_q$ and $\left(f_q\big|_{J_m}\right)^{-1}(y) = -\frac{1}{y - m\lambda_q}$ for $m = \pm 1, \pm 2, \pm 3, \dots$.

Consider next the map $f_q^\star$ in (3.1.6). In the case $q = 3$ and $\lambda_q = 1$ define the intervals $J_{\pm m}^\star$, $m = 2, 3, \ldots$ as

$$\begin{aligned} J_m^\star &:= \left[\frac{-1}{r_q+m}, \frac{-1}{r_q+m+1}\right] \quad \text{respectively} \\ J_{-m}^\star &:= -J_m^\star = \left[\frac{1}{r_q+m+1}, \frac{1}{r_q+m}\right]. \end{aligned} \tag{3.3.14}$$

Since $-R_q = -\frac{1}{2+r_q} = -1 + r_q$ and $r_q = -\frac{1}{3+r_q}$, see §2.3, we find

$$I_{R_q} = [-R_q, R_q] = \bigcup_{\varepsilon=+,-} \bigcup_{m=2}^{\infty} J_{\varepsilon m}^\star \quad \text{with} \quad J_m^{\star\circ} \cap J_n^{\star\circ} = \emptyset \text{ for all } m \neq n. \tag{3.3.15}$$

An easy calculation shows that

$$f_3^\star\left(J_{\varepsilon m}^\star\right) = \varepsilon[r_q, R_q] \quad \text{for all } m \geq 2,$$

where one uses $f_3^\star(-R_q) = r_q$, $\lim_{\varepsilon\searrow 0} f_3^\star(r_q+\varepsilon) = r_q$ and $\lim_{\varepsilon\searrow 0} f_3^\star(r_q-\varepsilon) = R_q$. Hence the intervals $\{J_m^\star\}$ define a Markov partition $\mathcal{M}(f_q^\star)$ and $f_3^\star$ is a locally expanding, smooth Markov map.

For even q define the intervals $J_{\pm m}^\star$ for $m \in \mathbb{N}$ as

$$\begin{aligned} J_m^\star &:= \left[\frac{-1}{r_q+m\lambda_q}, \frac{-1}{r_q+(m+1)\lambda_q}\right] \quad \text{respectively} \\ J_{-m}^\star &:= -J_m^\star = \left[\frac{1}{r_q+(m+1)\lambda_q}, \frac{1}{r_q+m\lambda_q}\right]. \end{aligned} \tag{3.3.16}$$

Since according to (2.3.6) $R_q = 1$ for q even, a simple calculation shows that $f_q^\star(-R_q) = r_q = 1 - \lambda_q$. But according to (2.3.2) $r_q = [\![0; \overline{(1)^{h-1}, 2}]\!]$ and hence $\left(f_q^\star\right)^{h_q-1}(r_q) = [\![0; \overline{2, (1)^{h_q-1}}]\!]$. This with (2.3.5) shows that

$$\left(f_q^\star\right)^{h_q}(R_q) = \left(f_q^\star\right)^{h_q-1}(r_q) = -\frac{1}{2\lambda_q + r_q}.$$

The order of the points $\psi_i \in \text{orbit}^\star\left(-R_q\right)$ in (3.3.3) is given by

$$-R_q = \psi_0 < \psi_1 < \ldots < \psi_{\kappa_q} = \frac{-1}{2\lambda_q + r_q}$$

where $\kappa_q = h_q$ by (2.3.3). Hence $\psi_i \in J_1^\star$ for $0 \leq i \leq \kappa_q - 1$ whereas ψ_{κ_q} is just the common boundary of $J_1^\star$ and $J_2^\star$. Define therefore the intervals $J_{\varepsilon 1_i}^\star$ as

$$J_{\varepsilon 1_i}^\star := J_{\varepsilon 1}^\star \cap \Psi_{\varepsilon i} \quad \text{for all } 1 \leq i \leq \kappa_q \text{ and } \varepsilon = +, -, \tag{3.3.17}$$

such that $J_{\varepsilon 1}^\star = \bigcup_{i=1}^{\kappa_q} J_{\varepsilon 1_i}^\star$.

Then the partition $\mathcal{M}(f_q^\star)$, defined by

$$I_{R_q} = \bigcup_{\varepsilon=+,-} \left(\bigcup_{i=1}^{\kappa_q} J^\star_{\varepsilon 1_i} \cup \bigcup_{m=2}^{\infty} J^\star_{\varepsilon m} \right), \tag{3.3.18}$$

is a Markov partition, since

$$\begin{aligned} f_q^\star \left(J^\star_{\varepsilon 1_i} \right) &= J^\star_{\varepsilon 1_{i+1}} \quad \text{for } 1 \le i \le \kappa_q - 1, \\ f_q^\star \left(J^\star_{\varepsilon 1_{\kappa_q}} \right) &= \varepsilon \left[\frac{-1}{2\lambda_q + r_q}, R_q \right] \quad \text{and} \\ f_q^\star \left(J^\star_{\varepsilon m} \right) &= \varepsilon \left[r_q, R_q \right] \quad \text{for } m \ge 2. \end{aligned}$$

The restriction $f_q^\star\big|_{J_m^\star}$ of $f_q^\star$ to the interval $J_m^\star$ is given by

$$f_q^\star\big|_{J_m}(x) = \frac{-1}{x} - m\lambda_q \qquad \text{for } m \in \mathbb{Z}_{\neq 0}$$

and its inverse by

$$\left(f_q^\star\big|_{J_m} \right)^{-1}(y) = \frac{-1}{y + m\lambda_q} \qquad \text{for } y \in f_q^\star(J_m).$$

Remains the case q odd, $q \ge 5$. The intervals $J^\star_{\varepsilon m}$ for $\varepsilon = +, -$, $m \ge 2$ are defined as for q even in (3.3.16):

$$J_m^\star := \left[\frac{-1}{r_q + m\lambda_q}, \frac{-1}{r_q + (m+1)\lambda_q} \right] \quad \text{and} \quad J^\star_{-m} := -J_m^\star. \tag{3.3.19}$$

The intervals $J^\star_{\varepsilon 1}$ are defined as

$$\begin{aligned} J_1^\star &:= \left[-R_q, \frac{-1}{r_q + 2\lambda_q} \right] \quad \text{respectively} \\ J^\star_{-1} &:= -J_1^\star = \left[\frac{1}{r_q + 2\lambda_q}, R_q \right]. \end{aligned} \tag{3.3.20}$$

According to (3.3.5)

$$\psi_{2h_q} = \left(f_q^\star \right)^{h_q}(-R_q) = \frac{-1}{r_q + 2\lambda_q} = [\![0; \overline{2, 1^{h_q}, 2, 1^{h_q - 1}}]\!]^\star$$

and

$$\begin{aligned} \psi_{2h_q+1} &= f_q^\star(-R_q) = [\![0; 2, \overline{1^{h_q-1}, 2, 1^{h_q}, 2}]\!]^\star \\ &\le [\![0; 3, \overline{1^{h_q}, 2, 1^{h_q-1}, 2}]\!]^\star = \frac{-1}{3\lambda_q + r_q}. \end{aligned}$$

Hence $\psi_i \in J_1^\star = \left[-R_q, \frac{-1}{r_q + 2\lambda_q} \right]$ for $1 \le i \le 2h_q = \kappa_q - 1$ whereas $\psi_{2h_q+1} = \psi_{\kappa_q} \in J_2^\star$. Define for $\varepsilon = +, -$ the intervals $J^\star_{\varepsilon 1_i}$ as

$$J^\star_{\varepsilon 1_i} := J^\star_{\varepsilon 1} \cap \Psi_{\varepsilon i} = \Psi_{\varepsilon i} \quad \text{for } 1 \le i \le \kappa_q - 1 \tag{3.3.21}$$

and the intervals $J^\star_{\varepsilon 2_i}$ for $i = \kappa_q, \kappa_q + 1$ as

$$J^\star_{\varepsilon 2_{\kappa_q}} := J^\star_{\varepsilon 2} \cap \Psi_{\varepsilon\kappa_q} = \Psi_{\varepsilon\kappa_q} \quad \text{respectively}$$
$$J^\star_{\varepsilon 2_{\kappa_q+1}} := J^\star_{\varepsilon 2} \setminus \Psi_{\varepsilon\kappa_q} = \varepsilon\left[\psi_{\kappa_q}, \frac{-1}{3\lambda_q + r_q}\right].$$

Then the partition

$$I_{R_q} = \bigcup_{\varepsilon=+,-} \left(\bigcup_{i=1}^{\kappa_q-1} J^\star_{\varepsilon 1_i} \cup \bigcup_{i=\kappa_q}^{\kappa_q+1} J^\star_{\varepsilon 2_i} \cup \bigcup_{m=3}^{\infty} J^\star_{\varepsilon m} \right) \tag{3.3.22}$$

is a Markov partition. This follows from the following identities, which can be easily verified:

$$f_q^\star\left(J^\star_{\varepsilon m}\right) = \varepsilon\left[r_q, R_q\right] \quad \text{for all } m = 3, 4, \ldots,$$
$$f_q^\star\left(J^\star_{\varepsilon 1_{2i}}\right) = J^\star_{\varepsilon 1_{2i+2}} \quad \text{for all} 1 \le i \le h_q - 1,$$
$$f_q^\star\left(J^\star_{\varepsilon 1_{2i-1}}\right) = J^\star_{\varepsilon 1_{2i+1}} \quad \text{for all } 1 \le i \le h_q,$$
$$f_q^\star\left(J^\star_{\varepsilon 1_{2h}}\right) = \varepsilon\left[\psi_{\kappa_q}, R_q\right] = J^\star_{\varepsilon 2_{\kappa_q+1}} \cup \bigcup_{\delta=+,-} \bigcup_{m=3}^{\infty} J^\star_{\delta m} \cup J^\star_{-\varepsilon 2 m},$$
$$f_q^\star\left(J^\star_{\varepsilon 2_{\kappa_q}}\right) = J_{\varepsilon 1_2} \quad \text{and}$$
$$f_q^\star\left(J^\star_{\varepsilon 2_{\kappa_q+1}}\right) = \varepsilon\left[\psi_2, R_q\right] = \bigcup_{i=3}^{\kappa_q-1} J_{\varepsilon 1_i} \cup \bigcup_{i=\kappa_q}^{\kappa_q+1} J_{\varepsilon 2_i} \cup \bigcup_{\delta=+,-} \bigcup_{m=3}^{\infty} J^\star_{\delta m} \cup J_{-\varepsilon 2} \cup J_{-\varepsilon 1}.$$

4. The maps f_q and $f_q^\star$ and regular respectively dual regular λ_q-CF's

We are going to use the Markov partitions $\mathcal{M}(f_q)$ respectively $\mathcal{M}(f_q^\star)$ constructed in the forgoing section for the maps $f_q\colon I_q \to I_q$ and $f_q^\star : I_{R_q} \to I_{R_q}$ to show that these maps can be conjugated to subshifts over infinite alphabets. By introducing sofic systems closely related to these subshifts the symbolic dynamics of the above two maps are directly related to the regular respectively dual regular λ_q-CF's.

4.1. Symbolic dynamics for f_q and a subshift of infinite type. For $q = 3$ and $f_3\colon I_3 \to I_3$ let F be the alphabet $F = \mathbb{Z} \setminus \{0, \pm 1\}$. Define the transition matrix $\mathbb{A} = \left(\mathbb{A}_{i,j}\right)_{i,j\in F}$ with $\mathbb{A}_{i,j} \in \{0, 1\}$ for $\varepsilon = +, -$ as follows:

$$\begin{aligned} \mathbb{A}_{\varepsilon 2,\varepsilon m} &= 0, \quad m \ge 2, \\ \mathbb{A}_{\varepsilon 2,-\varepsilon m} &= 1, \quad m \ge 2, \\ \mathbb{A}_{\varepsilon k,m} &= 1, \quad k \ge 3 \quad \text{and all } m \in F. \end{aligned} \tag{4.1.1}$$

Denote by $(F_{\mathbb{A}}^{\mathbb{N}}, \tau)$ the subshift over the alphabet F with

$$F_{\mathbb{A}}^{\mathbb{N}} = \left\{\underline{\xi} = (\xi_i)_{i\in\mathbb{N}},\ \xi_i \in F, \mathbb{A}_{\xi_i,\xi_{i+1}} = 1,\ i \in \mathbb{N}\right\}$$

and $\left(\tau(\underline{\xi})\right)_i = \xi_{i+1}$ the shift map.

Let $\partial\mathcal{M}(f_3) := \left\{x \in I_q : \exists\, n \in \mathbb{Z}_{\geq 0} : f_q^n(x) = 0\right\}$. The projection map $\pi\colon I_3 \setminus \partial\mathcal{M}(f_3) \to F_{\mathbb{A}}^{\mathbb{N}}$ defined by

$$\pi(x) = \underline{\xi} = (\xi_i)_{i\in\mathbb{N}} \quad \text{if } f_3^{i-1}(x) \in J_{\xi_i} \quad \text{for } i \in \mathbb{N}$$

is bijective with inverse $\pi^{-1}(\underline{\xi}) = x$, where x is the unique point with $x \in J_{\xi_1} \cap \bigcap_{l=1}^{\infty}(f_{\xi_l} \circ \ldots \circ f_{\xi_1})^{-1} J_{\xi_{l+1}}$. That the point x is uniquely defined follows from the expanding property of the local branches $f_{J_m} = f_3\big|_{J_m}$ of the map f_3, given on the interval J_m by $f_{J_m} = -\frac{1}{x} - m$, $m \in \mathbb{Z} \setminus \{0, \pm 1\}$. Obviously one has $\pi \circ f_3 = \tau \circ \pi$ on $I_3 \setminus \partial\mathcal{M}(f_3)$.

For q even with $q = 2h_q + 2$ define the alphabet F as $F = \{\varepsilon 1_i,\ \varepsilon = +,-,\ 1 \leq i \leq \kappa_q\} \cup \mathbb{Z} \setminus \{0, \pm 1\}$. The transition matrix $\mathbb{A} = \left(\mathbb{A}_{i,j}\right)_{i,j\in F}$ is defined in this case as follows:

$$\begin{aligned}
\mathbb{A}_{\varepsilon 1_l, \varepsilon 1_{l+1}} &= 1, \quad 1 \leq l \leq \kappa_q - 1,\\
\mathbb{A}_{\varepsilon 1_{\kappa_q - 1}, \varepsilon m} &= 1, \quad m = 2, 3, \ldots,\\
\mathbb{A}_{\varepsilon 1_{\kappa_q}, -\varepsilon 1_l} &= 1, \quad 1 \leq l \leq \kappa_q,\\
\mathbb{A}_{\varepsilon 1_{\kappa_q}, -m} &= 1, \quad m = 2, 3, \ldots,\\
\mathbb{A}_{m,n} &= 1, \quad m \in \mathbb{Z} \setminus \{0, \pm 1\}, \quad n \in F,
\end{aligned} \tag{4.1.2}$$

and all the other matrix elements vanishing.

Define the set $\partial\mathcal{M}(f_q)$ and the map $\pi\colon I_q \setminus \partial\mathcal{M}(f_q) \to F_{\mathbb{A}}^{\mathbb{N}}$ in analogy to the case $q = 3$. The same arguments as there show that this map is bijective and conjugates f_q to the shift map τ with $\pi \circ f_3 = \tau \circ \pi$ on $I_q \setminus \partial\mathcal{M}(f_q)$.

For $q = 2h_q + 3 = \kappa_q + 1$ finally define the alphabet F as $F = \{\varepsilon 1_i,\ \varepsilon = +,-,\ 1 \leq i \leq \kappa_q - 1\} \cup \{\varepsilon 2_i,\ \varepsilon = +,-,\ \kappa_q - 1 \leq i \leq \kappa_q\} \cup \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}$. The transition matrix $\mathbb{A} = \left(\mathbb{A}_{i,j}\right)_{i,j\in F}$ is given in Table 2.

The set $\partial\mathcal{M}(f_q)$ and the map $\pi\colon I_q \setminus \partial\mathcal{M}(f_q) \to F_{\mathbb{A}}^{\mathbb{N}}$ are defined similarly as in the foregoing cases $q = 3$ and q even and have the same properties. The inverse $(\pi)^{-1}\colon F_{\mathbb{A}}^{\mathbb{N}} \to I_q \setminus \partial\mathcal{M}(f_q)$ is given by $\pi^{-1}(\underline{\xi}) = x$ with $x \in J_{\xi_1} \cap \bigcap_{l=1}^{\infty}(f_{\xi_l} \circ \ldots \circ f_{\xi_1})^{-1} J_{\xi_{l+1}}$ where $f_{\xi_i} = f\big|_{J_{\varepsilon 1}}$ for $\xi_i = \varepsilon 1_l$, $1 \leq l \leq \kappa_q - 1$ respectively $f_{\xi_i} = f\big|_{J_{\varepsilon 2}}$ for $\xi_i = \varepsilon 2_l$, $\kappa_q \leq l \leq \kappa_q + 1$. Hence also in this case the map f_q gets conjugated by π to the shift map τ on the space $F_{\mathbb{A}}^{\mathbb{N}}$ of symbol sequences and therefore is itself a subshift of infinite type.

4.2. A sofic system related to the map f_q and the regular λ_q-CF. The transition matrix $\mathbb{A}$ in (4.1.1) for the subshift $f_3 : I_3 \to I_3$ shows that a symbol sequence $\underline{a} = (a_i)_{i\in\mathbb{N}} \in F_{\mathbb{A}}^{\mathbb{N}} = \pi(I_3 \setminus \partial\mathcal{M}(f_3))$ if and only if $(a_i, a_{i+1}) \neq (\varepsilon 2, \varepsilon m)$, $m \geq 2$ for all $i \in \mathbb{N}$. Hence this sequence is q-regular for $q = 3$ and $F_{\mathbb{A}}^{\mathbb{N}} = \mathcal{A}_3^{reg}$. The inverse map $\pi^{-1}\colon F_{\mathbb{A}}^{\mathbb{N}} \to I_q \setminus \partial\mathcal{M}(f_q)$ therefore has the form $\pi^{-1}(\underline{a}) = [\![0; a_1, a_2, \ldots]\!]$. This follows from

L 4.2.1. *For $\underline{a} = (a_i)_{i\in\mathbb{N}} \in \mathcal{A}_3^{reg}$ a 3-regular sequence the limit $\lim_{n\to\infty}[\![0; a_1, a_2, \ldots, a_n]\!]$ exists and defines a point $x \in \mathbb{R}$.*

$$
\begin{aligned}
\mathbb{A}_{\varepsilon 1_{2i},\varepsilon 1_{2i+1}} &= 1 \quad 1 \le i \le h_q - 2,\\
\mathbb{A}_{\varepsilon 1_{2h_q-2},\varepsilon 1_{2h_q}} &= 1,\\
\mathbb{A}_{\varepsilon 1_{2h_q-2},\varepsilon 2_{\kappa_q}} &= 1,\\
\mathbb{A}_{\varepsilon 1_{2h_q},-\varepsilon 1_i} &= 1 \quad 1 \le i \le \kappa_q - 1,\\
\mathbb{A}_{\varepsilon 1_{2h_q},-\varepsilon 2_i} &= 1 \quad \kappa_q \le i \le \kappa_q + 1,\\
\mathbb{A}_{\varepsilon 1_{2h_q},-\varepsilon m} &= 1 \quad m \ge 3,\\
\mathbb{A}_{\varepsilon 1_{2i-1},\varepsilon 1_{2i+1}} &= 1, \quad 1 \le i \le h_q - 1,\\
\mathbb{A}_{\varepsilon 1_{2h_q-1},\varepsilon 2_{\kappa_q+1}} &= 1,\\
\mathbb{A}_{\varepsilon 1_{2h_q-1},-\varepsilon m} &= 1, \quad m \ge 3,\\
\mathbb{A}_{\varepsilon 2_{\kappa_q},-\varepsilon 1_1} &= 1,\\
\mathbb{A}_{\varepsilon 2_{\kappa_q+1},\delta 1_i} &= 1, \quad \delta = +,-, \quad 2 \le i \le \kappa_q - 1,\\
\mathbb{A}_{\varepsilon 2_{\kappa_q+1},\delta 2_i} &= 1, \quad \delta = +,-, \quad 2 \le i \le \kappa_q - 1,\\
\mathbb{A}_{\varepsilon 2_{\kappa_q+1},\delta n} &= 1, \quad \delta = +,-, \quad n \ge 3,\\
\mathbb{A}_{\varepsilon 2_{\kappa_q+1},-\varepsilon 1_1} &= 1,\\
\mathbb{A}_{m,n} &= 1, \quad m \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}, \quad n \in F,
\end{aligned}
$$

T 2. The transition matrix $\mathbb{A} = (\mathbb{A}_{i,j})_{i,j\in F}$ with $\varepsilon = +,-$ for q-regular sequences and q odd, $q \ge 5$. All other matrix elements vanish.

P . Set $x_n := [\![0; a_1, a_2, \ldots, a_n]\!]$ and denote by $J_{(a_1,\ldots,a_n)}$ the closed interval $J_{(a_1,\ldots,a_n)} := J_{a_1} \cap \bigcap_{l=1}^{n-1} (f_{a_l} \circ \ldots f_{a_1})^{-1} J_{a_{l+1}}$ with $f_{a_i} := f_q\big|_{J_{a_i}}$. Obviously $x_n \in J_{(a_1,\ldots,a_n)}$. All these intervals are nonempty and $J_{(a_1,\ldots,a_{n+1})} \subset J_{(a_1,\ldots,a_n)}$ for all n. Hence $\bigcap_{n=1}^{\infty} J_{(a_1,\ldots,a_n)}$ is not empty. Because the map f_3 is strictly expanding this set contains exactly one point x. But this shows that $\lim_{n\to\infty} [\![0; a_1, a_2, \ldots, a_n]\!] = x$. □

In the case $q \neq 3$ the relation between the symbolic dynamics with respect to the Markov partitions $\mathcal{M}(f_q)$ and the λ_q-CF is more complicated. Indeed one has to introduce a corresponding sofic system, namely in the alphabet F the letters $\varepsilon 1_i$ respectively $\varepsilon 2_i$ have to be replaced by the letters $\varepsilon 1$ respectively $\varepsilon 2$ for all i. This corresponds to replacing the Markov partition $\mathcal{M}(f_q)$ defined in Section 3.3 by the partition $\mathcal{J}(f_q)$ defined as $I_q = \bigcup_{\varepsilon=+,-} \bigcup_{m=1}^{\infty} J_{\varepsilon m}$ with J_m given in (3.3.9). It is not difficult to see that this partition is generating that means $\bigcap_{i=1}^{\infty} f_q^{-(i-1)} J_{m_i}$ is either empty or consists of exactly one point. This follows again from the fact that all branches of f_q are expanding. Denote by $\partial\mathcal{J}(f_q)$ the boundary points of the intervals J_m including the point $x = 0$ together with all their preimages under the map f_q. Since $x = 0$ belongs to the orbit of $-\frac{\lambda_q}{2}$ the boundaries $\partial\mathcal{M}(f_q)$ and $\partial\mathcal{J}(f_q)$ coincide. Denote by $\hat{\pi}\colon I_q \setminus \partial\mathcal{J}(f_q) \to \hat{F}^{\mathbb{N}}$ with $\hat{F} = \mathbb{Z} \setminus \{0\}$ the map

$\hat\pi(x) = \underline{a} = (a_i)_{i\in\mathbb{N}}$ when $f_q^{i-1}(x) \in J_{a_i}$. If $\pi(x) = \underline{\xi}$, then obviously $a_i = m$ if $\xi_i = m \in \mathbb{Z}$ and $a_i = \varepsilon 1$ if $\xi_i = \varepsilon 1_l$ for some l respectively $a_i = \varepsilon 2$ if $\xi_i = \varepsilon 2_l$ for some l. The following Lemma then holds

L 4.2.2. *The map $\hat\pi\colon I_q \setminus \partial\mathcal{J}(f_q) \to \mathcal{A}_q^{reg} \subset \hat F^{\mathbb{N}}$ is bijective. The inverse map $\hat\pi^{-1}\colon \mathcal{A}_q^{reg} \to I_q \setminus \partial\mathcal{J}(f_q)$ is given by $\hat\pi^{-1}(\underline{a}) = [\![0; a_1, a_2, \ldots]\!]$.*

P . Consider first q even: From the definition of the transition matrix $\mathbb{A}$ in (4.1.2) for the case $q = 2h_q+2$ it follows that there cannot be more than h_q consecutive symbols $\varepsilon 1$ in $\underline{a} = (a_i)_{i\in\mathbb{N}} = \hat\pi(x)$ since $(\mathbb{A})_{\varepsilon 1_{\kappa_q}, \varepsilon m} = 0$ for all $m = 2, 3, \ldots$: indeed h_q consecutive symbols $\varepsilon 1$ are only possible for points x with $f_q^{i-1}(x) \in J_{\varepsilon 1_i}$ for $i = 1, \ldots, \kappa_q$ and $f_q^{\kappa_q} \in J_{-\varepsilon 1_l}$ for some $1 \le l \le \kappa_q$ or $f_q^{\kappa_q} \in J_{-\varepsilon m}$ for some $m \ge 2$. This shows that $\underline{a} = (a_i)_{i\in\mathbb{N}} = \hat\pi(x)$ defines a q-regular sequence in $\mathcal{A}_q^{\mathrm{reg}}$.

Given on the other hand such a q-regular sequence $\underline{a} = (a_i)_{i\in\mathbb{N}}$ there exists a unique point $x \in I_q \setminus \partial\mathcal{M}(f_q)$ with $\hat\pi(x) = \underline{a}$: indeed if for some $l \ge 1$ and $k \ge 0$ one has $a_l = a_{l+1} = \ldots = a_{l+k} = \varepsilon 1$ and $a_{l+k+1} = m \ne \varepsilon 1$ consider the sequence $\underline{\xi} \in F_{\mathbb{A}}^{\mathbb{N}}$ with $\xi_{l+k} = \varepsilon 1_{\kappa_q}$, $\xi_{l+k-1} = \varepsilon 1_{\kappa_q - 1}, \ldots, \xi_l = \varepsilon 1_{\kappa_q - l}$ if $\operatorname{sign}(m) \ne \varepsilon$ respectively $\xi_{l+k} = \varepsilon 1_{\kappa_q - 1}$, $\xi_{l+k-1} = \varepsilon 1_{\kappa_q - 2}$, …, $\xi_l = \varepsilon 1_{\kappa_q - l - 1}$ if $\operatorname{sign}(m) = \varepsilon$, whereas $\xi_i = a_i$ for all $a_i \ne \varepsilon 1$. Since $k \le \kappa_q - 1$ respectively $k \le \kappa_q - 2$ in the second case, the sequence $\underline{\xi}$ belongs indeed to $F_{\mathbb{A}}^{\mathbb{N}}$ and hence there exists a point $x \in I_q \setminus \partial\mathcal{M}(f_q)$ with $\pi(x) = \underline{\xi}$ and hence also $\hat\pi(x) = \underline{a}$. The inverse map $\hat\pi^{-1}$ is again given by $\hat\pi^{-1}(\underline{a}) = [\![0; a_1, a_2, \ldots]\!]$. Since for any $\underline{a} \in \mathcal{A}_q^{\mathrm{reg}}$ there exists an unique $\underline{\xi} \in F_{\mathbb{A}}^{\mathbb{N}}$ which is related to $\underline{a}$ when replacing the symbols $\pm 1_i$ by the symbol ± 1, there exists therefore $x \in I_q \setminus \partial\mathcal{J}(f_q)$ with $\pi(x) = \underline{\xi}$. But $x_n := [\![0; a_1, a_2, \ldots a_n]\!] \in J_{(\xi_1, \ldots, \xi_n)}$ and hence $\lim_{n\to\infty} x_n = x \in \bigcap_{n=1}^{\infty} J_{(\xi_1, \ldots, \xi_n)}$ and hence $\hat\pi^{-1}(\underline{a}) = [\![0; a_1, a_2, \ldots]\!]$.

The same reasoning can be applied in the case $q = 2h_q + 3$ odd to show that the map $\hat\pi\colon I_q \setminus \partial\mathcal{J}(f_q) \to \mathcal{A}_q^{\mathrm{reg}} \in \hat F^{\mathbb{N}}$ is bijective with inverse $\hat\pi^{-1}(\underline{a}) = [\![0; a_1, a_2, \ldots]\!]$. □

4.3. Symbolic dynamics for $f_q^\star$ and a subshift of infinite type. Let us start again with the case $q = 3$ and recall the Markov partition $\mathcal{M}(f_q^\star)$ defined in (3.3.15) by $I_{R_q} = \bigcup_{\varepsilon=+,-} \bigcup_{m=2}^{\infty} J_{\varepsilon m}^\star$ with $J_{\varepsilon m}^\star = \varepsilon\left[-\frac{1}{r_3+m}, -\frac{1}{r_3+m+1}\right]$. Denote by F the alphabet $F = \mathbb{Z}\setminus\{0, \pm 1\}$ and by $\mathbb{A} = (\mathbb{A}_{i,j})_{i,j\in F}$ the transition matrix with

$$(\mathbb{A})_{m,n} = 1 \quad \text{for all } m, n \in F \text{ with } n \ne 2\operatorname{sign}(m), \tag{4.3.1}$$

and all the other matrix elements vanishing. Denote by $\partial\mathcal{M}(f_3^\star)$ the set

$$\partial\mathcal{M}(f_3^\star) = \left\{ y \in I_{R_q} : \exists\, n \in \mathbb{Z}_{\ge 0} \text{ with } (f_3^\star)^n(y) = \pm r_3 \text{ or } \left(f_q^\star\right)^n(y) = 0 \right\}$$

and by $\pm\underline{r}_3$ the sequence $\pm\underline{r}_3 = (\pm\overline{3})$. Then one has for $f_{\xi_i}^\star := f_3^\star\big|_{J_{\xi_i}^\star}$

L 4.3.1. *The map*

$$\pi\colon I_{R_3} \setminus \partial\mathcal{M}(f_3^\star) \to F_{\mathbb{A}}^{\mathbb{N}} \setminus \left\{\underline{\xi} \in F_{\mathbb{A}}^{\mathbb{N}} : \exists n \in \mathbb{Z}_{\geq 0} : \tau^n(\underline{\xi}) = \pm\underline{r}_3\right\},$$

given by $\pi(x) = \underline{\xi} = (\xi_i)_{i\in\mathbb{N}}$ *if* $(f_3^\star)^{i-1} \in J_{\xi_i}^\star$ *for* $i \in \mathbb{N}$*, is bijective, and* $\pi \circ f_3^\star = \tau \circ \pi$*. Its inverse, the map* $\pi^{-1} : F_{\mathbb{A}}^{\mathbb{N}} \to I_{R_3}$ *can be defined on the entire set* $F_{\mathbb{A}}^{\mathbb{N}}$ *and is given by* $\pi^{-1}(\underline{\xi}) = x$ *with* x *the unique point in* I_{R_3} *with* $x \in J_{\xi_1}^\star \bigcap_{l=1}^\infty (f_{\xi_l}^\star \circ \ldots \circ f_{\xi_1}^\star)^{-1} J_{\xi_{l+1}}^\star$.

P . Obviously all the preimages of the point $x = 0$ have a finite symbol sequence $(\xi_1, \ldots, \xi_N)$, whereas the points $\pm r_q \in J_{\pm 2}^\star \cup J_{\pm 3}^\star$ have the two different symbol sequences $\pi(\pm r_q) = \underline{\xi} = (\pm\overline{3})$ respectively $\pi(\pm r_q) = (\pm 2, \mp\overline{3})$. The same holds then true for all the preimages of these points. The point $x \in I_{R_q}$ is again uniquely determined because of the expansive nature of the local branches of the map $f_3^\star$. □

The map $f_3^\star\colon I_{R_q} \to I_{R_q}$ is hence a subshift of infinite type.

Consider next the case q even with $q = 2h_q + 2$ and $\kappa_q = h_q$. Recall the Markov partition $\mathcal{M}(f_q^\star)$ in (3.3.18). We define the alphabet F as

$$F = \left\{\varepsilon 1_i,\ \varepsilon = +,-,\ 1 \leq i \leq \kappa_q\right\} \cup \mathbb{Z} \setminus \{0, \pm 1\}$$

and by $\mathbb{A} = (\mathbb{A}_{i,j})_{i,j\in F}$ the transition matrix with matrix elements

$$\begin{aligned}
(\mathbb{A})_{\varepsilon 1_i, \varepsilon 1_{i+1}} &= 1 \quad \text{for} \quad \varepsilon = +,- ; \quad 1 \leq i \leq \kappa_q - 1, \\
(\mathbb{A})_{\varepsilon 1_{\kappa_q}, m} &= 1 \quad \text{for} \quad m \in F,\ m \neq \varepsilon 1_i \quad 1 \leq i \leq \kappa_q, \quad \text{and} \\
(\mathbb{A})_{m,n} &= 1 \quad \text{for} \quad |m| \geq 2 \quad \text{and all} \quad n \neq \operatorname{sign}(m) 1_1,
\end{aligned} \tag{4.3.2}$$

whereas all other matrix elements vanish. If we define again $\partial\mathcal{M}(f_q^\star)$ by

$$\partial\mathcal{M}(f_q^\star) = \left\{y \in I_{R_q} : \exists\, n \in \mathbb{Z}_{\geq 0} \text{ with } (f_q^\star)^n(y) = \pm r_q \text{ or } (f_q^\star)^n(y) = 0\right\}$$

one shows in complete analogy with Lemma 4.3.1.

L 4.3.2. *The map*

$$\pi\colon I_{R_q} \setminus \mathcal{M}(f_q^\star) \to F_{\mathbb{A}}^{\mathbb{N}} \setminus \left\{\underline{\xi} \in F_{\mathbb{A}}^{\mathbb{N}} : \exists\, n \in \mathbb{Z}_{\geq 0} : \tau^n(\underline{\xi}) = \pm\underline{r}_q\right\},$$

given by $\pi(x) = \underline{\xi} = (\xi_i)_{i\in\mathbb{N}}$ *if* $(f_q^\star)^{i-1} \in J_{\xi_i}^\star$ *for* $i \in \mathbb{N}$*, is bijective, and* $\pi \circ f_q^\star = \tau \circ \pi$*. Its inverse, the map* $\pi^{-1} : F_{\mathbb{A}}^{\mathbb{N}} \to I_{R_q}$ *can be defined on the entire set* $F_{\mathbb{A}}^{\mathbb{N}}$ *and is given by* $\pi^{-1}(\underline{\xi}) = x$ *with* x *the unique point in* I_{R_q} *with* $x \in J_{\xi_1}^\star \bigcap_{l=1}^\infty (f_{\xi_l}^\star \circ \ldots \circ f_{\xi_1}^\star)^{-1} J_{\xi_{l+1}}^\star$.

This shows that the map $f_q^\star\colon I_{R_q} \to I_{R_q}$ is a subshift of infinite type also for even q.

$$
\begin{aligned}
(\mathbb{A})_{\varepsilon 1_{2i-1},\varepsilon 1_{2i+1}} &= 1 \quad \text{for} \quad 1 \le i \le h_q - 1,\\
(\mathbb{A})_{\varepsilon 1_{2h_q-1},\varepsilon 2_{\kappa_q}} &= 1\\
(\mathbb{A})_{\varepsilon 1_{2i},\varepsilon 1_{2i+2}} &= 1 \quad \text{for} \quad 1 \le i \le h_q - 1,\\
(\mathbb{A})_{\varepsilon 1_{2h_q},\varepsilon 2_{\kappa_q+1}} &= 1\\
(\mathbb{A})_{\varepsilon 1_{2h_q},m} &= 1 \quad m \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\},\\
(\mathbb{A})_{\varepsilon 1_{2h_q},-2} &= 1,\\
(\mathbb{A})_{\varepsilon 2_{\kappa_q},\varepsilon 1_2} &= 1,\\
(\mathbb{A})_{\varepsilon 2_{\kappa_q+1},\varepsilon 1_i} &= 1 \quad \text{for} \quad 2 \le i \le \kappa_q - 1,\\
(\mathbb{A})_{\varepsilon 2_{\kappa_q+1},\varepsilon 2_i} &= 1 \quad \text{for} \quad i = \kappa_q,\ \kappa_q + 1,\\
(\mathbb{A})_{\varepsilon 2_{\kappa_q+1},m} &= 1 \quad \text{for} \quad m \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\},\\
(\mathbb{A})_{\varepsilon 2_{\kappa_q+1},-\varepsilon 1_i} &= 1 \quad \text{for} \quad 1 \le i \le \kappa_q - 1,\\
(\mathbb{A})_{\varepsilon 2_{\kappa_q+1},-\varepsilon 2_i} &= 1 \quad \text{for} \quad i = \kappa_q,\ \kappa_q + 1,\\
(\mathbb{A})_{\varepsilon m,\varepsilon 1_i} &= 1 \quad \text{for} \quad 2 \le i \le \kappa_q - 1, \quad \varepsilon = +.-, \quad m \in \mathbb{N} \setminus \{0, 1, 2\},\\
(\mathbb{A})_{\varepsilon m,-\varepsilon 1_1} &= 1, \quad \varepsilon = +, -,\\
(\mathbb{A})_{\varepsilon m,-\varepsilon 2_i} &= 1, \quad \varepsilon = +, -, \quad I = \kappa_q,\ \kappa_q + 1.
\end{aligned}
$$

T 3. The transition matrix $\mathbb{A} = (\mathbb{A}_{i,j})_{i,j\in F}$ with $\varepsilon = +, -$ for q-dual regular sequences and q odd, $q \ge 5$. All other matrix elements vanish.

Consider finally the case $q = 2h_q + 3$ and $\kappa_q = 2h_q + 1$. The Markov partition $\mathcal{M}(f_q^\star)$ was given in this case in (3.3.22). Define the alphabet F as

$$
\begin{aligned}
F = &\left\{\varepsilon 1_i,\ \varepsilon = +, -,\ 1 \le i \le \kappa_q - 1\right\} \cup\\
&\cup \left\{\varepsilon 2_i,\ \varepsilon = +, -,\ \kappa_q \le i \le \kappa_q + 1\right\} \cup \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}.
\end{aligned}
$$

The transition matrix $\mathbb{A} = (\mathbb{A}_{i,j})_{i,j\in F}$ has now the form given in Table 3.

If $\partial\mathcal{M}(f_q^\star)$ denotes again the set of preimages of the points $x = \pm r_q$ and the point $x = 0$ one shows as in the former cases that the map

$$
\pi\colon I_{R_q} \setminus \partial\mathcal{M}(f_q^\star) \to F_{\mathbb{A}}^{\mathbb{N}} \setminus \left\{\underline{\xi} \in F_{\mathbb{A}}^{\mathbb{N}} :\ \exists n \in \mathbb{Z}_{\ge 0} :\ \tau^n(\underline{\xi}) = \pm \underline{r}_q\right\}
$$

is bijective and the map $f_q^\star$ is conjugated therefore on $I_{R_q} \setminus \partial\mathcal{M}(f_q^\star)$ to the shift τ on $F_{\mathbb{A}}^{\mathbb{N}} \setminus \left\{\underline{\xi} \in F_{\mathbb{A}}^{\mathbb{N}} :\ \exists n \in \mathbb{Z}_{\ge 0} :\ \tau^n(\underline{\xi}) = \pm \underline{r}_q\right\}$.

Hence also in the case q is odd the map $f_q^\star$ is conjugate to a subshift of infinite type.

4.4. A sofic system related to $f_q^\star$ and the dual regular λ_q-CF. In the case of the map $f_3^\star$ the subshift $\tau : F_{\mathbb{A}}^{\mathbb{N}} \to F_{\mathbb{A}}^{\mathbb{N}}$ can be easily related to the dual λ_3-CF: from the form of the transition matrix $(\mathbb{A})$ in (4.3.1) it follows that the sequence $\underline{b} \in F_{\mathbb{A}}^{\mathbb{N}}$ with $\underline{b} = \pi(x)$ can be characterized by the fact that

$(b_i, b_{i+1}) \neq (m, 2\,\mathrm{sign}\,(m))$ for all $i \in \mathbb{N}$ and hence $\underline{b} \in \mathcal{A}_q^{\mathrm{dreg}}$. On the other hand any such sequence $\underline{b} \in \mathcal{A}_3^{\mathrm{dreg}}$ belongs to $F_{\mathbb{A}}^{\mathbb{N}}$ and defines a unique point $x \in I_{R_q}$ through $J_{b_1}^\star \bigcap_{l=1}^\infty (f_{b_l}^\star \circ \ldots \circ f_{b_1}^\star)^{-1} J_{b_{l+1}}^\star$. Since $x_n := [\![0; b_1, \ldots, b_n]\!]^\star \in J_{b_1}^\star \bigcap_{l=1}^n (f_{b_l}^\star \circ \ldots \circ f_{b_1}^\star)^{-1} J_{b_{l+1}}^\star$ for all n we find $\lim_{n\to\infty} x_n = x$ and hence $\pi^{-1}(\underline{b}) = [\![0; b_1, b_2, \ldots,]\!]^\star$.

To connect the subshift for the map $f_q^\star$ in the case $q = 2h_q + 2$ with the dual λ_q-CF one has to introduce the sofic systems by replacing in $\underline{\xi}$ all the symbols $\pm 1_i$, $1 \leq i \leq \kappa_q$ by the symbol ± 1. This corresponds to replacing the Markov partition $\mathcal{M}(f_q^\star)$ by the generating partition $\mathcal{J}(f_q^\star)$ determined by $I_{R_q} = \bigcup_{\varepsilon=+,-} \bigcup_{m=1}^\infty J_{\varepsilon m}^\star$ with $J_{\varepsilon m}^\star$ defined in (3.3.16). Denote by $\partial\mathcal{J}(f_q^\star)$ the set

$$\partial\mathcal{J}(f_q^\star) = \left\{ y \in I_{R_q}, : \exists\, n \in \mathbb{Z}_{\geq 0} \text{ with } (f_q^\star)^n(y) = \pm r_q \text{ or } (f_q^\star)^n(y) = 0 \right\}$$

which obviously coincides with the set $\partial\mathcal{M}(f_q^\star)$. Then for the alphabet $\hat{F} = \mathbb{Z} \setminus \{0\}$ one shows again

L 4.4.1. *The map*

$$\hat{\pi} : I_{R_q} \setminus \partial\mathcal{J}(f_q^\star) \to \mathcal{A}_q^{dreg} \setminus \left\{ \underline{b} \in \mathcal{A}_q^{dreg} : \exists\, n \in \mathbb{Z}_{\geq 0} : \tau^n(\underline{b}) = \pm \underline{r}_q \right\} \subset \hat{F}^{\mathbb{N}}$$

defined by $\hat{\pi}(x) = \underline{b} = (b_i)_{i\in\mathbb{N}}$ *if* $(f_q^\star)^{i-1} \in J_{b_i}^\star$ *for* $i \in \mathbb{N}$, *is bijective, and* $\hat{\pi} \circ f_q^\star = \tau \circ \hat{\pi}$. *Its inverse, the map* $\hat{\pi}^{-1} : \mathcal{A}_q^{dreg} \to I_{R_q}$ *can be defined on the entire set* $\mathcal{A}_q^{dreg}$ *and is given by* $\hat{\pi}^{-1} : (\underline{b}) = [\![0; b_1, b_2, \ldots,]\!]^\star$.

P . Since $\pm\psi_{\kappa_q} = (f_q^\star)^{h_q-1}(\pm r_q) \in J_{\pm 1}^\star \cap J_{\pm 2}^\star$ this point has two different dual regular sequences $\underline{b} = \pm(1, (-1)^{h_q}, -2, (-1)^{h_q} - 1)$ respectively $\underline{b} = \pm(\overline{1^{h_q-1}, 2})$. Hence also all preimages of this point have two different dual regular sequences, but these points all have the same tail as the point $\pm r_q$. If $\underline{b} = (b_i)_{i\in\mathbb{N}} = \hat{\pi}(x)$ assume that $\underline{b}$ contains for some $k \geq 0$ and some $l \geq 0$ a subsequence $b_{k+1} = \cdots = b_{k+l} = \varepsilon 1$ with either $k = 0$ or $b_k \neq \varepsilon 1$ and $b_{k+l+1} \neq \varepsilon 1$. Then the sequence $\underline{\xi} = (\xi_i)_{i\in\mathbb{N}}$ related to $\underline{b}$ by replacing the symbols $\pm 1_i$ by the symbol ± 1 must be of the form $\xi_{k+l} = \varepsilon 1_{\kappa_q}$ and hence $\xi_{k+i} = \varepsilon 1_{\kappa_q-(l-i)}$ for $1 \leq i \leq l$. This shows that $l \leq \kappa_q$. The case $l = \kappa_q$ however is only possible if either $k = 0$ or $b_k = -\varepsilon m$, $m \geq 1$. This shows that $\underline{b} = \hat{\pi}(x) \in \mathcal{A}_q^{\mathrm{dreg}}$.

Given on the other hand $\underline{b} \in \mathcal{A}_q^{\mathrm{dreg}}$ with a subsequence $b_{k+1} = \ldots = b_{k+l} = \varepsilon 1$ and $b_{k+l+1} \neq \varepsilon 1$ for some k and some l then define the sequence $\underline{\xi}$ such that $\xi_{k+i} = \varepsilon 1_{\kappa_q-(l-i)}$, $1 \leq i \leq l$. Since $l \leq \kappa_q$ respectively $l \leq \kappa_q - 1$ the sequence $\underline{\xi}$ belongs to $F_{\mathbb{A}}^{\mathbb{N}}$ and hence there exists a point $x \in I_{R_q}$ with $\pi(x) = \underline{\xi}$ and therefore by construction also $\hat{\pi}(x) = \underline{b}$. The inverse map $\hat{\pi}^{-1}$ is obviously defined for all $\underline{b} \in \mathcal{A}_q^{\mathrm{dreg}}$. An argument completely analogous to the one in the case $q = 3$ then shows that $\hat{\pi}^{-1}(\underline{b}) = [\![0; b_1, b_2, \ldots]\!]^\star$. □

Introduce finally in the case $q = 2h_q + 3$ for the map $f_q^\star$ the sofic system defined by replacing in $\underline{\xi} \in F_{\mathbb{A}}^{\mathbb{N}}$ all the symbols $\pm 1_i$ by the symbol ± 1 and the symbols $\pm 2_i$ by the symbol ± 2. Denote by $\mathcal{J}(f_q^\star)$ the corresponding generating partition $I_{R_q} = \bigcup_{\varepsilon=+,-} \bigcup_{m=1}^{\infty} J_{\varepsilon m}^\star$ and by $\partial\mathcal{J}(f_q^\star)$ the set of preimages of the points $x = \pm r_q$ and $x = 0$ which obviously coincides with the set $\partial\mathcal{M}(f_q^\star)$. As in the previous cases one shows also for $q = 2h_q + 3$

L 4.4.2. *The map*

$$\hat{\pi} : I_{R_q} \setminus \partial\mathcal{J}(f_q^\star) \to \mathcal{A}_q^{dreg} \setminus \left\{\underline{b} \in \mathcal{A}_q^{dreg} : \underline{b} \text{ has the tail } \pm\underline{r}_q\right\} \subset \hat{F}^{\mathbb{N}}$$

defined by $\hat{\pi}(x) = \underline{b} = (b_i)_{i\in\mathbb{N}}$ *if* $(f_q^\star)^{i-1} \in J_{b_i}^\star$ *for* $i \in \mathbb{N}$, *is bijective and* $\hat{\pi} \circ f_q^\star = \tau \circ \hat{\pi}$. *Its inverse, the map* $\hat{\pi}^{-1} : \mathcal{A}_q^{dreg} \to I_{R_q}$ *can be defined on the entire set* $\mathcal{A}_q^{dreg}$ *and is given by* $\hat{\pi}^{-1} : (\underline{b}) = [\![0; b_1, b_2, \ldots]\!]^\star$.

P . From the form of the transition matrix $\mathbb{A}$ in Table 3 it is clear that there are only restrictions on the symbol sequence $\underline{b}$ for $\underline{b} = (b_i)_{i\in\mathbb{N}} = \hat{\pi}(x)$ if it contains subsequences of consecutive symbols ± 1 since $(\mathbb{A})_{i,j} = 1$ for all $j \in F$ if $|i| \geq 3$. Assume $b_k = m \neq 1$ and $b_{k+1} = \ldots = b_{k+l} = \pm 1$, $b_{k+l+1} \neq \pm 1$ for some $k \geq 0$ and some $l \geq 1$, where $k = 0$ means that $b_1 = \pm 1$. Then either $(f_q^\star)^{k+l-1}(x) \in J_{\pm 1_{2h_q-1}}^\star$ or $(f_q^\star)^{k+l-1}(x) \in J_{\pm 1_{2h_q}}^\star$ and hence $\xi_{k+l} = \pm 1_{2h_q-1}$ or $\xi_{k+l} = \pm 1_{2h_q}$. In the first case $\xi_{k+i} = \pm 1_{2h_q-2(l-i)-1}$, $1 \leq i \leq l$ and hence $\xi_{k+1} = \pm 1_{2h_q-(2l-1)}$. If $m = \pm n$ for some $n \geq 3$ then necessarily $l \leq h_q - 1$ since $(\mathbb{A})_{m,\pm 1_1} = 0$ for all $m \in F$. If on the other hand $m = \mp n$ for some $n \geq 3$ or $k = 0$ then $l \leq h_q$ with $l = h_q$ iff $\xi_{k+1} = \pm 1_1$. In the case $(f_q^\star)(x) \in J_{\pm 1_{2h_q}}^\star$ we find $\xi_{k+i} = \pm 1_{2h_q-2(l-i)}$, $1 \leq i \leq l$ and hence $\xi_{k+1} = \pm 1_{2+2(h_q-l)}$. This shows that also in this case $l \leq h_q$. In the symbol sequence $\underline{b}$ there can appear therefore no subsequence of more than h_q consecutive symbols ± 1.

Assume next that there exists in $\underline{b}$ a subsequence of h_q consecutive symbols ± 1 such that $b_{k+1} = \ldots = b_{k+h_q} = \pm 1$ and $b_{k+h_q+1} \neq= \pm 1$. Then either $k = 0$, that means $b_{k+1} = b_1$, or $b_k = \mp n$ for some $n \neq 1$. Then $(f_q^\star)^{k+h_q-1}(x) \in J_{\pm 1_{2h_q-1}}^\star$ or $(f_q^\star)^{k+h_q-1}(x) \in J_{\pm 1_{2h_q}}^\star$ and hence $\xi_{k+h_q} = \pm 1_{2h_q-1}$ respectively $\xi_{k+h_q} = \pm 1_{2h_q}$. The transition matrix $\mathbb{A}$ in Table 3 then shows that in the first case $\xi_{k+h_q+1} = \pm 2_{\kappa_q}$ and in the second case

$$\xi_{k+h_q+1} \in \left\{n \in F,\ n \neq \pm 1_i,\ 1 \leq i \leq \kappa_q - 1,\ n \neq \pm 2_{\kappa_q}\right\}.$$

If $\xi_{k+h_q+1} = \pm 2_{\kappa_q}$ then $\xi_{k+h_q+1+i} = \pm 1_{2i}$ for $1 \leq i \leq h_q$ and hence $\xi_{k+2h_q+1} = \pm 1_{2h_q}$.

If in the second case $\xi_{k+h_q+1} = \pm 2_{\kappa_q+1}$ then the maximal number of consecutive symbols ± 1 in $\underline{b}$ is $h_q - 1$ since in this case $\xi_{k+h_q+2} = \pm 1_i$ for some $i \geq 3$ and only for $i = 3$ one has $\xi_{k+2h_q} = \pm 1_{2h_q-1}$. In all other cases when $b_{k+h_q+1} \neq \pm 2$ the number of consecutive symbols ± 1 is certainly bounded by h_q. This shows that in the sequence $\underline{b} = \hat{\pi}(x)$ the subsequence $(\pm m, (\pm 1)_q^h, \pm 2, (\pm 1)_q^h, \pm 2)$ and the subsequence $(\pm 1)^{h_q+1}$ cannot appear. Hence $\underline{b} = \hat{\pi}(x) \in \mathcal{A}_q^{\mathrm{dreg}}$. Since also in this case to every $\underline{b} \in \mathcal{A}_q^{\mathrm{dreg}}$

there exist a unique $\underline{\xi} \in F_{\mathbb{A}}^{\mathbb{N}}$ which is related to $\underline{b}$ by replacing the symbols $\pm 1_i$ respectively the symbols $\pm 2_i$ by the symbols ± 1 respectively ± 2, the same arguments as in the previous cases apply to show, that the inverse map $\hat{\pi}^{-1}$ is given by $\hat{\pi}^{-1}(\underline{b}) = [\![0; b_1, b_2, \ldots]\!]^\star$. □

4.5. Symbolic dynamics and the natural extension F_q of the map f_q. Consider the maps f_q and $f_q^\star$. Since

$$f_q([\![0; a_1, a_2, a_3, a_4, \ldots]\!]) = [\![0; a_2, a_3, a_4, \ldots]\!] \quad \text{and}$$
$$f_q^\star([\![0; a_0, a_{-1}, a_{-2}, a_{-3}, \ldots]\!]^\star) = [\![0; a_{-1}, a_{-2}, a_{-3}, \ldots]\!]^\star,$$

f_q and $f_q^\star$ are equivalent to the shift map τ on one sided infinite sequences $\underline{a}_> := (a_i)_{i\in\mathbb{N}} \in \mathcal{A}_q^{\text{reg}}$ respectively $\underline{a}_< := (a_i)_{i\in\mathbb{Z}_{\le 0}} \in \mathcal{A}_q^{\text{dreg}}$. Denote by $\mathcal{A}_q$ the set of two-sided infinite sequences

$$\mathcal{A}_q = \left\{\underline{a} = (a_i)_{i\in\mathbb{Z}} : \forall l \in \mathbb{Z}, \forall k > 0 : (a_l, a_{l+1}, \ldots, a_{l+k}) \notin \mathcal{B}_q\right\},$$

where $\mathcal{B}_q$ was defined in (2.2.2). The natural extension of the one-sided shift map τ is the two-sided shift $\tau : \mathcal{A}_q \to \mathcal{A}_q$ with

$$(\tau(\underline{a}))_i = a_{i+1},\ i \in \mathbb{Z} \quad \text{respectively its inverse}$$
$$(\tau^{-1}(\underline{a}))_i = a_{i-1},\ i \in \mathbb{Z} \quad \text{if } \underline{a} = (a_i)_{i\in\mathbb{Z}}. \tag{4.5.1}$$

The natural extension F_q of the map f_q respectively its inverse F_q^{-1} can then be identified simply with the corresponding induced maps on pairs of points (x, y) with regular respectively dual regular λ_q-CF

$$x = [\![0; a_1, a_2, a_3, a_4, \ldots]\!] \text{ and } y = [\![0; a_0, a_{-1}, a_{-2}, a_{-3}, a_{-4}, \ldots]\!]^\star,$$

as long as the two-sided sequence $\underline{a} = (a_i)_{i\in\mathbb{Z}}$ belongs to $\mathcal{A}_q$. Then, F_q and F_q^{-1} satisfy

(4.5.2)
$$F_q([\![0; a_1, a_2, \ldots]\!], [\![0; a_0, \ldots]\!]^\star) = ([\![0; a_2, \ldots]\!], [\![0; a_1, a_0, \ldots]\!]^\star) \text{ and}$$
$$F_q^\star([\![0; a_1, \ldots]\!], [\![0; a_0, a_{-1}, \ldots]\!]^\star) = ([\![0; a_0, a_1, \ldots]\!], [\![0; a_{-1}, \ldots]\!]^\star).$$

To characterize the set Ω_q of pairs (x, y) with the above property, define in a first step $I_q^\star := I_q \setminus \left\{x \text{ has a finite regular} \lambda_q - CF\right\}$. Obviously $I_q^\star$ has full measure. Denote next by $\Pi_1 : \mathcal{A}_q \to I_q^\star$ the map

$$\Pi_1(\ldots, a_{-1}, a_0; a_1, a_2, a_3 \ldots) = [\![0; a_1, a_2, a_3, \ldots]\!]. \tag{4.5.3}$$

By construction the following lemma holds.

L 4.5.1. *The map Π_1 is surjective and satisfies $\Pi_1 \circ f_q = \tau \circ \Pi_1$.*

Next, define $I_{R_q}^\star := I_{R_q} \setminus \left\{y \text{ has a finite dual regular } \lambda_q\text{-CF}\right\}$ which has full measure. Similar to (4.5.3) the map $\Pi_2 : \mathcal{A}_q \to I_{R_q}^\star$ given by

$$\Pi_2(\ldots, a_{-2}, a_{-1}, a_0; a_1, \ldots) = [\![0; a_0, a_{-1}, a_{-2}, \ldots]\!]^\star. \tag{4.5.4}$$

is well defined, surjective and satisfies $\Pi_2 \circ f_q^\star = \tau^{-1} \circ \Pi_2$. The following Lemma characterizes the domain of definition Ω_q of the natural extension F_q:

L 4.5.2. *For $\underline{a} \in \mathcal{A}_q$ we have*

$$\Pi_2(\underline{a}) \in \pm\left[\psi_{\kappa_q-i+1}, R_q\right] \quad \text{if } \Pi_1(\underline{a}) \in \pm\Phi_i = \pm\left[\phi_{i-1}, \phi_i\right],\ i \in \{1, \dots, \kappa_q\}.$$

The set $\Omega_q \subset I_q \times I_{R_q}$ hence is given by

$$\Omega_q = \bigcup_{i=1}^{\kappa_q} \Big(\left[\phi_{i-1}, \phi_i\right] \times \left[\psi_{\kappa_q-i+1}, R_q\right]\Big) \cup \Big(\left[-\phi_i, -\phi_{i-1}\right] \times \left[-R_q, -\psi_{\kappa_q-i+1}\right]\Big).$$

P . W.l.o.g. assume $\Pi_1(\underline{a}) \in \Phi_i \subset I_q$ where Φ_i is defined in (3.3.6). For q even Lemma 3.3.1 and the λ_q-CF of $-\frac{\lambda}{2}$ in (2.3.1) show

$$[\![0; (1)^{h_q-i+1}]\!] \le \Pi_1(\underline{a}) \le [\![0; (1)^{h_q-i}]\!].$$

Hence $\Pi_1(\underline{a})$ has a λ_q-CF of the form $\Pi_1(\underline{a}) = [\![0; (1)^{h_q-i}, m, \dots]\!]$ for some $m \ge 2$. Since $\underline{a}$ is q-regular we have

$$\underline{a} = (\dots, a_{-i-1}, a_{-i}, a_{-i+1}, \dots, a_0; (1)^{h_q-i}, m, a_{h_q-i+2}, \dots)$$

with at most $i-1$ consecutive 1's in the sequence $(a_{-i+2}, \dots, a_0)$. The point $\Pi_2(\underline{a})$ hence is bounded by the largest and smallest number whose dual regular λ_q-CF starts with at most $i-1$ consecutive 1's and hence

$$[\![0; (1)^{i-1}, 2, \overline{(1)^{h_q-i}, 2}]\!]^\star \le \Pi_2(\underline{a}) \le [\![0; (-1)^{h_q}, -2, \overline{(-1)^{h_q-1}, -2}]\!]^\star.$$

But (3.3.4) and (2.3.5) show that these bounds are just ψ_{κ_q-i+1} and R_q.

The case q odd, $q \ge 5$, is slightly more complicated. First, assume i to be even and put $j = \frac{i}{2}$. Then by Lemma 3.3.1 and the λ_q-CF of $-\frac{\lambda}{2}$ in (2.3.1)

$$\Pi_1(\underline{a}) \in \Phi_{2j} = \Big[[\![0; (1)^{h_q-j+1}]\!], [\![0; (1)^{h_q-j}, 2, (1)^{h_q}]\!]\Big].$$

Hence $\Pi_1(\underline{a})$ has a λ_q-CF of the form $\Pi_1(\underline{a}) = [\![0; (1)^{h_q-j}, 2, (1)^{h_q}, m, \dots]\!]$ for some $m \ge 2$. Since $\underline{a} \in \mathcal{A}_q$ the sequence $(a_{-j+1}, \dots, a_0)$ in

$$\underline{a} = (\dots, a_{-j}, a_{-j+1}, \dots, a_0; (1)^{h_q-j}, 2, (1)^{h_q}, m, a_{2h_q-j+2}, \dots)$$

cannot contain more than $j-1$ consecutive digits 1. Hence $\Pi_2(\underline{a})$ is bounded by the points

$$[\![0; (1)^{j-1}, 2, \overline{(1)^h, 2, (1)^{h-1}, 2}]\!]^\star \le \Pi_2(A) \le [\![0; (-1)^h, -2, \overline{(-1)^h, -2, (-1)^{h-1}, -2}]\!]^\star.$$

which by Lemma 3.3.1 and (2.3.5) are just ψ_{κ_q-2j+1} and R_q.

Next, consider the case i odd and put $j = \frac{i-1}{2}$ for $1 \le i \le h_q$. Again, by Lemma 3.3.1 and (2.3.5)

$$\Pi_1(\underline{a}) \in \Phi_{2j+1} = \Big[[\![0; (1)^{h_q-j}, 2, (1)^{h_q}]\!], [\![0; (1)^{h_q-j}]\!]\Big],$$

and therefore $\Pi_1(\underline{a})$ has a λ_q-CF of the form $\Pi_1(\underline{a}) = [\![0; (1)^{h_q-j}, m, \ldots]\!]$ for some $m \geq 2$. Hence there is again a restriction on the sequence $\underline{a}$:

$$\underline{a} = \Big(\ldots, a_{-h-j-1}, \underbrace{a_{-h-j}, \ldots, a_0; (1)^{h-j}, m}_{\text{not a forbidden block}}, a_{h-j+1}, \ldots\Big),$$

and therefore $\Pi_2(\underline{a})$ is bounded by

$$[\![0; (1)^j, 2, \overline{(1)^{h_q-1}, 2, (1)^{h_q}, 2}]\!]^\star \leq \Pi_2(A) \leq [\![0; (-1)^{h_q}, -2, [0; (-1)^{h_q}, -2, \overline{(-1)^{h_q}, -2, (-1)^{h_q-1}, -2}]\!]^\star,$$

which by Lemma 3.3.1 and (2.3.1) respectively (2.3.5) are just ψ_{κ_q-2j} and R_q .

Finally for $q = 3$ Lemma 3.3.1 and (2.3.5) show $[\![0; 2]\!] \leq \Pi_1(\underline{a}) \prec [\![0;]\!]$, and therefore $\Pi_1(\underline{a})$ has a λ_q-CF of the form $\Pi_1(\underline{a}) = [\![0; m, \ldots]\!]$ for some $m \geq 2$. Hence $\Pi_2(\underline{a})$ must not have a leading digit 2. This implies the bounds $r_3 = [0; \overline{3}] \leq \Pi_2(\underline{a}) \leq [0; -2, \overline{-3}] = R_3$.

The case $\Pi_1(\underline{a}) \in \Phi_{-i}$ for some $1 \leq i \leq \kappa_q$ follows from $\Pi_1(-\underline{a}) = -\Pi_1(\underline{a})$. □

Recall the definition of the domain Ω_q in Lemma 4.5.2 and define the set $\Omega_q^\star = \Omega_q \cap (I_q^\star \times I_{R_q}^\star)$, which obviously is dense in Ω_q. Then one has

L 4.5.3. *The map* $\Pi\colon \mathcal{A}_q \to \Omega_q^\star$ *with* $\Pi(\underline{a}) = \big(\Pi_1(\underline{a}), \Pi_2(\underline{a})\big)$ *is a bijection.*

For $F_q : \Omega_q^\star \to \Omega_q^\star$ *and* $F_q^{-1} : \Omega_q^\star \to \Omega_q^\star$ *given by* (4.5.2) *the diagrams*

$$\begin{array}{ccc} \mathcal{A}_q & \xrightarrow{\tau} & \mathcal{A}_q \\ {\scriptstyle\Pi}\downarrow & & \downarrow{\scriptstyle\Pi} \\ \Omega_q^\star & \xrightarrow{F_q} & \Omega_q^\star \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathcal{A}_q & \xrightarrow{\tau^{-1}} & \mathcal{A}_q \\ {\scriptstyle\Pi}\downarrow & & \downarrow{\scriptstyle\Pi} \\ \Omega_q^\star & \xrightarrow{F_q^{-1}} & \Omega_q^\star \end{array} \quad \textit{commute.}$$

P . Obviously, the map Π is well defined. Commutativity of the diagrams follows from combining Lemma 4.5.1 and

$$\Pi_2(\tau(\underline{a})) = \Pi_2(\tau(\ldots, a_{-1}, a_0; a_1, a_2, \ldots)) = \Pi_2(\ldots, a_{-1}, a_0, a_1; a_2, \ldots) = [\![0; a_1, a_0, a_{-1}, \ldots]\!]^\star = \frac{-1}{[\![a_0, a_{-1}, \ldots]\!]^\star + a_1\lambda_q} = \frac{-1}{\Pi_2(\underline{a}) + a_1\lambda_q}$$

respectively

$$\Pi_1(\tau^{-1}(\underline{a})) = \Pi_1(\tau^{-1}(\ldots, a_{-1}, a_0; a_1, \ldots)) = \Pi_1(\ldots, a_{-1}; a_0, a_1, \ldots) = [\![0; a_0, a_1, a_2, \ldots]\!] = \frac{-1}{[\![a_1, a_2, \ldots]\!] + a_0\lambda_q} = \frac{-1}{\Pi_1(\underline{a}) + a_0\lambda_q}.$$

Since the map $\Pi\colon \mathcal{A}_q \to \Omega_q^\star$ is obviously injective we only need to show $\Pi(\mathcal{A}) = \Omega_q^\star$. For this take $(x, y) \in \Omega_q^\star$ with

$$x = [\![0; a_1, a_2, a_3, \ldots]\!] \quad \text{and} \quad y = [\![0; a_0, a_{-1}, a_{-2}, \ldots]\!]^\star.$$

If $(x,y) \in \left([\phi_{i-1},\phi_i] \cap I_q^\star\right) \times \left([\psi_{\kappa_q-i+1},\psi_0] \cap I_{R_q}^\star\right)$ for some $i \in \{1,\dots,\kappa_q\}$ the definitions of ϕ_i and ψ_i as elements of orbit$\left(-\frac{\lambda_q}{2}\right)$ and orbit$\left(-R_q\right)$ respectively imply that the bi-infinite sequence

$$\underline{a} := (\dots, a_{-2}, a_{-1}, a_0; a_1, a_2, a_3, \dots)$$

does not contain forbidden blocks, and hence $\underline{a} \in \mathcal{A}_q$. □

It is well known that the map $\Pi\colon \mathcal{A}_q \to \mathbb{R}^2$ is continuous (see for instance [**18**]) when $\mathcal{A}_q$ is equipped with the usual metric of the shift space.

5. Some applications

5.1. Reduced geodesics on $G_q\backslash\mathbb{H}$ and the natural extension of f_q. The Poincaré upper half-plane, equipped with the hyperbolic metric $\mathrm{d}s^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$, is denoted by $\mathbb{H} = \{z \in \mathbb{C};\ \mathrm{Im}\,(z) > 0\}$. The group of isometries of this space is given by $\mathrm{PSL}(2,\mathbb{R})$. The boundary of $\mathbb{H}$ is the projective line $\mathbb{P}^1_{\mathbb{R}}$.

We consider oriented geodesics on $\mathbb{H}$. Geodesic lines on $\mathbb{H}$ are half-circles perpendicular to $\mathbb{R}$ or straight lines parallel to the imaginary axis $\mathrm{Re}\,(z) = 0$. An oriented geodesic ω on $\mathbb{H}$ will be represented by the two base points $\omega_-, \omega_+ \in \mathbb{R} \cup \{i\infty\}$ with its orientation from ω_- towards ω_+. We denote such a geodesic by $\omega = (\omega_-, \omega_+)$.

We call two oriented geodesics ω and υ G_q*-equivalent* if there exists an element $g \in G_q$ with $g\,\omega_- = \upsilon_-$ and $g\,\omega_+ = \upsilon_+$.

Then one can show

T 5.1.1. *Let $\omega = (\omega_-, \omega_+)$ be a geodesic with ω_- having an infinite regular respectively ω_+ having an infinite dual regular λ_q-CF. Then there exist a geodesic $\omega' = (\omega'_-, \omega'_+)$ such that*

- *ω and ω' are G_q-equivalent and*
- *$(S\,\omega'_+, -\omega'_-) \in \Omega_q$.*

P . We prove the Theorem first for $q \geq 4$. Using translations by powers of T_q we may assume that for $\omega = (\omega_-, \omega_+)$ either

- $\omega_+ > 0$ and $\omega_- \in \left[-R_q, -r_q\right] \subset I_{R_q}$ or
- $\omega_+ < 0$ and $\omega_- \in \left[r_q, R_q\right] \subset I_{R_q}$.

Assume $\omega_+ > 0$ and $\omega_- \in \left[-R_q, -r_q\right]$ with infinite λ_q-CF's

$$\omega_+ = [\![a_0; a_1, a_2, \dots]\!] \quad \text{and} \quad \omega_- = [\![0; b_1, b_2, \dots]\!]^\star. \tag{5.1.1}$$

For $x = S\,\omega_+$ and $y = -\omega_-$ we have

$$x = \begin{cases} [0; a_0, a_1, a_2, \dots] & \text{if } a_0 \neq 0, \\ [\![a_1; a_2 \dots]\!] & \text{if } a_0 = 0 \text{ and} \end{cases} \tag{5.1.2}$$
$$y = [\![0; -b_1, -b_2, \dots]\!]^\star.$$

The following three cases have to be discussed: "$a_0 \geq 2$", "$a_0 = 1$" and "$a_0 = 0$".

If $a_0 \geq 2$ then $\omega_+ \geq 2\lambda - \frac{\lambda}{2}$ and the λ_q-CF of x in (5.1.2) is regular. If the two-sided sequence

$$\underline{a} := (\ldots, -b_2, -b_1; a_0, a_1, \ldots) \tag{5.1.3}$$

belongs to $\mathcal{A}_q$ then by Lemma 4.5.3 $(S\ \omega_+, -\omega_-) = (x, y) = \Pi(\underline{a}) \in \Omega_q^\star \subset \Omega_q$.

Assume therefore $\underline{a} \notin \mathcal{A}_q$. The forbidden block must appear around ";" in (5.1.3). This can only happen for q even if $(-b_{h_q}, \ldots - b_1) = (1^{h_q})$ respectively for q odd if $(-b_{2h_q+1}, \ldots - b_1) = (1^{h_q}, 2, 1^{h_q})$. Using the lexicographic order "$\prec$" in Section 2.4 we find

$$\begin{aligned} -\omega_- = [\![0; -b_1, -b_2, \ldots]\!]^\star &= \begin{cases} [\![0; 1^{h_q}, \ldots]\!]^\star & \text{for } q \text{ even and} \\ [\![0; 1^{h_q}, 2, 1^{h_q}, \ldots]\!]^\star & \text{for } q \text{ odd} \end{cases} \\ &\prec \begin{cases} [\![0; \overline{(1)^{h_q-1}, 2}]\!] & \text{for } q \text{ even and} \\ [\![0; \overline{(1)^{h_q}, 2, (1)^{h_q-1}, 2}]\!] & \text{for } q \text{ odd, } q \geq 5. \end{cases} = r_q. \end{aligned}$$

Then Lemma 2.4.2 implies $[\![0; -b_1, -b_2, \ldots]\!]^\star < r_q$.

On the other hand, $-\omega_- \in \left[r_q, R_q\right]$ implies $-\omega_- \geq r_q$. This leads to a contradiction.

If $\mathbf{a_0} = \mathbf{0}$ then $\omega_+ \in \left(0, \frac{\lambda_q}{2}\right] \subset \left(0, \frac{2}{\lambda_q}\right)$. Hence $a_1 < 0$ in the λ_q-CF (5.1.1). For $m \in \mathbb{Z}_{\geq 2}$ the sequence $(-m, a_1, a_2, \ldots)$ is q-regular and also $(-m, b_1, b_2, \ldots)$ is dual q-regular, since $r_q \leq -\omega_-$ implies

$$\begin{aligned} r_q &= \begin{cases} [\![0; \overline{(1)^{h_q-1}, 2}]\!]^\star & \text{if } q \text{ is even and} \\ [\![0; \overline{(1)^{h_q}, 2, (1)^{h_q-1}, 2}]\!]^\star & \text{if } q \text{ is odd.} \end{cases} \\ &\leq [\![0; -b_1, -b_2, \ldots]\!]^\star = -\omega_-. \end{aligned}$$

For $g := S\,T_q^{-m}$ define $(\omega'_-, \omega'_+) = \omega' := g\ \omega$ with

$$\begin{aligned} \omega'_+ &= [\![0; -m, a_1, a_2, \ldots]\!] > 0 \quad \text{and} \\ \omega'_- &= [\![0; -m, b_1, b_2, \ldots]\!]^\star \in (0, -r_q] \subset [r_q, R_q]. \end{aligned}$$

The corresponding bi-infinite sequence $\underline{a}' := (\ldots, -b_1, m; -m, a_1, \ldots)$ is then q-regular and Lemma 4.5.3 hence implies $(S\ \omega'_+, -\omega'_-) = \Pi(\underline{a}') \in \Omega_q^\star \subset \Omega_q$.

For $\mathbf{a_0} = \mathbf{1}$

$$\omega_+ \geq \lambda_q - \frac{\lambda_q}{2} = \begin{cases} [\![1; (1)^{h_q}]\!] & \text{for } q \text{ even,} \\ [\![1; (1)^{h_q}, 2, (1)^{h_q}]\!] & \text{for } q \text{ odd and} \end{cases}$$

$$-R_q \leq \omega_- \leq -r_q = \begin{cases} [\![0; \overline{(-1)^{h_q-1}, -2}]\!]^\star & \text{for } q \text{ even,} \\ [\![0; \overline{(-1)^{h_q}, -2, (-1)^{h_q-1}, -2}]\!]^\star & \text{for } q \text{ odd.} \end{cases} \tag{5.1.4}$$

Then the dual regular λ_q-CF of y cannot start for q even with a block of the form (1^{h_q}) respectively for q odd $(1^{h_q}, 2, 1^{h_q})$.

Assume $\underline{a} \notin \mathcal{A}_q$, with $\underline{a}$ defined as in (5.1.3). Then a block of the form $(1^l; 1, 1^t)$ with $0 \le l \le h$ and $0 \le t \le h$ must exist around the ";" in $\underline{a}$ such that $a_{t+1} \ne 1$, $-b_{l+1} \ne 1$. As in the case $a_0 = 0$, choose an $m \in \mathbb{Z} \setminus \{-2, -1, 0, 1\}$ with $(m+1, a_1, a_2, \ldots)$ being q-regular and $(m, b_1, b_2, \ldots)$ being dual q-regular: indeed any m with $\operatorname{sign}(m) = -\operatorname{sign}(b_1)$ can be used. Define $g := S T_q^m$ and $(\omega'_-, \omega'_+) = \omega' := g\,\omega$ with

$$\begin{aligned} \omega'_+ &= [\![0; m+1, a_1, a_2, \ldots]\!] \\ \omega'_- &= [\![0; m, b_1, b_2, \ldots]\!]^\star \in \left[r_q, -r_q\right]. \end{aligned} \tag{5.1.5}$$

Then the bi-infinite sequence $\underline{a}' := (\ldots, -b_2, -b_1, -m; m+1, a_1, a_2, \ldots)$ is q-regular and by Lemma 4.5.3 $(S\,\omega'_+, -\omega'_-) = \Pi(\underline{a}') \in \Omega_q$.

The case $\omega_+ < 0$ and $\omega_- \in \left[r_q, R_q\right] \subset I_{R_q}$ can be treated in the same way.

The proof for $q = 3$ is similar to the case $q \ge 4$, however there are the four cases $a_0 \ge 3$, $a_0 = 2$, $a_0 = 1$ and $a_0 = 0$ to be considered.

If $a_0 \ge 3$ then we can argue as in the case $a_0 \ge 2$ before. Since $\omega_- < -r_3 = [\![0; \overline{-3}]\!]^\star$, the bi-infinite sequence $\underline{a}$ in (5.1.3) is q-regular and $(S\,\omega_+, -\omega_-) = (x, y) = \Pi(\underline{a}) \in \Omega_q$.

The cases $a_0 = 2$ and $a_0 = 1$ are similar to the case $a_0 = 1$ for $q \ge 4$: we just take the integer $|m| \ge 5$ with $m\,b_1 < 0$. Then ω' is defined as $(\omega'_-, \omega'_+) = \omega' := g\,\omega$ with $g := S\,T^m$ and hence

$$\begin{aligned} \omega'_+ &= \begin{cases} [\![0; m+2, a_1, a_2, \ldots]\!] & \text{if } a_0 = 2 \text{ and} \\ [\![0; m+1, a_1, a_2, \ldots]\!] & \text{if } a_0 = 1, \end{cases} \\ \omega'_- &= [\![0; m, b_1, b_2, \ldots]\!]^\star \in [r_3, -r_3]\,. \end{aligned}$$

The case $a_0 = 0$ is similar to the case $a_0 = 0$ for $q \ge 4$, if we choose there the integer $m \ge 3$ and recall $r_3 = [\![0; \overline{3}]\!]$ in (2.3.2). □

5.2. The transfer operator for G_q. The authors of [14] have constructed a Poincaré section Σ for the geodesic flow $\Phi_t \colon S_1 G_q \backslash \mathbb{H} \to S_1 G_q \backslash \mathbb{H}$ on the Hecke surfaces $G_q \backslash \mathbb{H}$ for which the Poincaré map $P \colon \Sigma \to \Sigma$ is basically given by the natural extension F_q of the map $f_q \colon I_q \to I_q$. The periodic orbits of this geodesic flow can therefore be characterized by the periodic orbits of F_q and therefore also by the periodic orbits of the map f_q respectively its periodic points which determine the ones of F_q uniquely. Indeed, Theorem 2.5.1 implies an almost one-to-one correspondence between the periodic orbits of the geodesic flow on the Hecke surfaces $G_q \backslash \mathbb{H}$ and the periodic orbits of the map f_q, only the periodic orbits of the points r_q and $-r_q$ which are not equivalent under the map f_q lead to the same periodic orbit of the geodesic flow since these points are G_q-equivalent. This shows already

that the Selberg zeta function Z_{G_q} for the Hecke triangle groups defined as

$$Z_{G_q}(\beta) = \prod_{k=0}^{\infty} \prod_{\gamma \text{ prime}} \left(1 - e^{-(\beta+k)l(\gamma)}\right),$$

where the product is over the prime periodic orbits γ of the geodesic flow and $l(\gamma)$ denotes its period (and hence the length of the corresponding closed geodesic), cannot be expressed in terms of the transfer operator for the map f_q alone. Indeed, to relate the above Selberg zeta function to the Poincaré map P one uses the following Lemma by Ruelle [17]:

L 5.2.1. $Z_{G_q}(\beta) = \prod_{k=0}^{\infty} e^{-\sum_{n=1}^{\infty} \frac{1}{n} Z_n(\beta+k)}$ *for* $\operatorname{Re}(\beta) > 1$ *where* $Z_n(\beta)$ *is the so called partition function* $Z_n(\beta) = \sum_{x \in \operatorname{Fix} P^n} e^{-\beta \sum_{l=0}^{\infty} r\left(P^l(x)\right)}$ *and* $r\colon \Sigma \to \mathbb{R}^+$ *denotes the recurrence time of the geodesic flow with respect to the Poincaré section* Σ.

In the transfer operator approach to the dynamical zeta functions the partition functions $Z_n(\beta)$ get expressed in terms of the traces of an operator constructed from the Poincaré map $P\colon \Sigma \to \Sigma$ respectively its restriction to the unstable directions. In our case the unstable direction is one-dimensional and the restriction of P to it is basically just the map $f_q : I_q \to I_q$. On the other hand one knows that the recurrence time $r\colon \Sigma \to \mathbb{R}^+$ in our case is given by $r(x) = \log \left| f_q'(x) \right|$. The Ruelle transfer operator $\mathcal{L}_\beta$ then has the following form

$$\mathcal{L}_\beta g(x) = \sum_{y \in f_q^{-1}(x)} e^{-\beta r(y)} g(y) \tag{5.2.1}$$

where $g\colon I_q \to \mathbb{C}$ is some complex valued function and $\operatorname{Re}(\beta) > 1$ to ensure convergence of the series. To get an explicit form for the operator $\mathcal{L}_\beta$ one has to determine the preimages y of any point $x \in I_q$. For this recall the Markov partition $I_q = \bigcup_{i \in A_{\kappa_q}} \Phi_i$ with $A_{\kappa_q} = \{\pm 1, \dots, \pm\kappa_q\}$ in (3.3.6), determined by the intervals Φ_i, and the local inverses $\vartheta_{\pm m}(x) := \left(f_q\big|_{J_{\pm m}}\right)^{-1}(x) = \frac{-1}{x \pm m\lambda_q}$ on the intervals $J_{\pm m}$, $1 \le m \le \infty$, respectively $2 \le m \le \infty$ for $q = 3$, defined in §3.3. For $1 \le i \le \kappa_q$ denote by $\mathcal{N}_i$ the set $\mathcal{N}_i := \left\{n \in \mathbb{Z} \setminus \{0\} \text{ such that there exists } j \in A_{\kappa_q} \text{ with } \vartheta_n(\Phi_i) \subset \Phi_j\right\}$. But then $\mathcal{N}_i = \bigcup_{j \in A_{\kappa_q}} \mathcal{N}_{i,j}$ with $\mathcal{N}_{i,j} := \left\{n \in \mathbb{Z} \setminus \{0\} \text{ such that } \vartheta_n(\Phi_i) \subset \Phi_j\right\}$. Using these sets we can rewrite the transfer operator $\mathcal{L}_\beta$ in (5.2.1) as

$$\mathcal{L}_\beta g(x) = \sum_{i \in A_{\kappa_q}} \chi_{\Phi_i}(x) \sum_{n \in \mathcal{N}_i} \left(\vartheta_n'(x)\right)^\beta g(\vartheta_n(x)), \tag{5.2.2}$$

with χ_{Φ_i} the characteristic function of the set Φ_i. With $g_i := g\big|_{\Phi_i}$ this can be written also as follows

$$(\mathcal{L}_\beta g)_i(x) = \sum_{j\in A_{\kappa_q}} \sum_{n\in\mathcal{N}_{i,j}} (\vartheta_n'(x))^\beta \; g_j(\vartheta_n(x)), \quad x \in I_i. \tag{5.2.3}$$

Thereby we used the Markov property of the partition $I_q = \bigcup_{i\in A_{\kappa_q}} \Phi_i$. If g_i is continuous on Φ_i for all $i \in A_{\kappa_q}$ then also $(\mathcal{L}_\beta g)_i$ is continuous on Φ_i, that means $\mathcal{L}_\beta$ maps piecewise continuous functions to piecewise continuous functions. Unfortunately on the Banach space $B = \oplus_{i\in A_{\kappa_q}} C(\Phi_i)$ of piecewise continuous functions the operator $\mathcal{L}_\beta$ is not trace class, it is even not compact. Much better spectral properties however can be achieved by defining $\mathcal{L}_\beta$ on a space of piecewise holomorphic functions. This is possible since all the maps $\vartheta_{\pm m}$, $m \geq 1$ have holomorphic extensions to a complex neighbourhood of I_q. Indeed one shows

L 5.2.2. *There exist open discs $D_i \subset \mathbb{C}$, $i \in A_{\kappa_q}$ with $\Phi_i \subset D_i$ such that for all $n \in \mathcal{N}_{i,j}$ we have $\vartheta_n(\overline{D_i}) \subset D_j$* [-7] . Change -7

Consider therefore the Banach space $B = \oplus_{i\in A_{\kappa_q}} B(D_i)$ with $B(D_i)$ the Banach space of holomorphic functions on the disc D_i with the sup norm. On this space the transfer operator $\mathcal{L}_\beta$ has the form

$$(\mathcal{L}_\beta g)_i(z) = \sum_{j\in A_{\kappa_q}} \sum_{n\in\mathcal{N}_{i,j}} \left(\frac{1}{z+n\lambda_q}\right)^{2\beta} g_j\left(\frac{-1}{z+n\lambda_q}\right), \qquad z \in D_i \tag{5.2.4}$$

which is well defined for $\mathrm{Re}\,(\beta) > \frac{1}{2}$. In a forthcoming paper we will discuss the spectral properties of this operator and its relation to the Selberg zeta function for the Hecke triangle groups G_q. Here we give the explicit form of this operator for the case $q = 3$ and $q = 4$.

For $q = 3$ one has $\kappa_3 = 1$ and therefore $A_{\kappa_3} = \{\pm 1\}$. The index sets $\mathcal{N}_{i,j}$, $i, j \in A_{\kappa_3}$ are given by $\mathcal{N}_{1,1} = \mathbb{Z}_{\geq 3}$, $\mathcal{N}_{1,-1} = \mathbb{Z}_{\leq -2}$, $\mathcal{N}_{-1,1} = \mathbb{Z}_{\geq 2}$ and $\mathcal{N}_{-1,-1} = \mathbb{Z}_{\leq -3}$.

For $q = 4$ one has also $\kappa_4 = 1$ and hence $A_{\kappa_4} = \{\pm 1\}$. The index sets $\mathcal{N}_{i,j}$, $i, j \in A_{\kappa_4}$ are given by $\mathcal{N}_{1,1} = \mathbb{Z}_{\geq 2}$, $\mathcal{N}_{1,-1} = \mathbb{Z}_{\leq -1}$, $\mathcal{N}_{-1,1} = \mathbb{Z}_{\geq 1}$ and $\mathcal{N}_{-1,-1} = \mathbb{Z}_{\leq -2}$. This leads in these two cases to the following transfer

[-7] Changed $\vartheta_n(D_i) \subset \overline{D_j}$ to $\vartheta_n(\overline{D_i}) \subset D_j$

operators

$$
\begin{aligned}
(\mathcal{L}_\beta g)_1(z) &= \sum_{n=3\,(2)}^{\infty} \left(\frac{1}{z+n\lambda_q}\right)^{2\beta} g_1\left(\frac{-1}{z+n\lambda_q}\right) \\
&\quad + \sum_{n=2\,(1)}^{\infty} \left(\frac{1}{z-n\lambda_q}\right)^{2\beta} g_{-1}\left(\frac{-1}{z-n\lambda_q}\right), \quad z \in D_1, \\
(\mathcal{L}_\beta g)_{-1}(z) &= \sum_{n=2\,(1)}^{\infty} \left(\frac{1}{z+n\lambda_q}\right)^{2\beta} g_1\left(\frac{-1}{z+n\lambda_q}\right) \\
&\quad + \sum_{n=3\,(2)}^{\infty} \left(\frac{1}{z-n\lambda_q}\right)^{2\beta} g_{-1}\left(\frac{-1}{z-n\lambda_q}\right), \quad z \in D_2
\end{aligned}
\tag{5.2.5}
$$

where $\lambda_3 = 1$ and $\lambda_4 = \sqrt{2}$ and the summation index in brackets belongs to the case $q = 4$. For $q = 3, 4$ the discs D_i, $i = \pm 1$ can be taken as $D_{\pm 1} = \pm\left\{z \in \mathbb{C};\ \left|z - \left(\frac{\lambda_q - 2}{4}\right)\right| < \frac{\lambda_q + 2}{4}\right\}$.

For $q = 3$ this operator and its eigenfunctions with eigenvalue $\rho = 1$ have been discussed in [3] where it was shown that these eigenfunctions are directly related to the eigenfunctions with eigenvalues $\rho = \pm 1$ of the transfer operator for the modular group G_3 derived from a symbolic dynamics for the geodesic flow using the Gauss continued fractions in [12].

6. λ_q-CF's and Rosen λ-fractions

6.1. Regular λ_q-CF's and reduced Rosen λ-fractions ($q \geq 4$). In [21] Rosen discussed continued fractions of the form

$$
[r_0; (\varepsilon_1 : r_1), (\varepsilon_2 : r_2), (\varepsilon_3 : r_3), \ldots] = r_0\lambda_q + \cfrac{\varepsilon_1}{r_1\lambda_q + \cfrac{\varepsilon_2}{r_2\lambda_q + \cfrac{\varepsilon_3}{r_3\lambda_q + \ldots}}}
\tag{6.1.1}
$$

with $r_0 \in \mathbb{Z}$ and $\varepsilon_i = \pm 1$, $r_i \geq 1$ for $i \in \mathbb{N}$. We call such expansions *Rosen λ-fraction*.

Rosen λ-fractions and λ_q-CF's can easily be transformed into each other using the relations

$$
\begin{aligned}
&[r_0; (\varepsilon_1 : r_1), (\varepsilon_2 : r_2), (\varepsilon_3 : r_3), \ldots] \\
&\quad = r_0\lambda_q + \cfrac{\varepsilon_1}{r_1\lambda_q + \cfrac{\varepsilon_2}{r_2\lambda_q + \cfrac{\varepsilon_3}{r_3\lambda_q + \ldots}}} = r_0\lambda_q + \cfrac{-1}{-\varepsilon_1 r_1\lambda_q + \cfrac{-1}{\varepsilon_1\varepsilon_2 r_2\lambda_q + \cfrac{-1}{-\varepsilon_1\varepsilon_2\varepsilon_3 r_3\lambda_q + \ldots}}} \\
&\quad = [r_0; -\varepsilon_1 r_1, \varepsilon_1\varepsilon_2 r_2, -\varepsilon_1\varepsilon_2\varepsilon_3 r_3, \ldots, (-1)^i \varepsilon_1 \cdots \varepsilon_i r_i, \ldots]
\end{aligned}
\tag{6.1.2}
$$

and

$$[a_0; a_1, a_2, a_3, \ldots] = a_0\lambda_q + \cfrac{-1}{a_1\lambda_q + \cfrac{-1}{a_2\lambda_q + \frac{-1}{a_3\lambda_q+\ldots}}} \tag{6.1.3}$$

$$= a_0\lambda_q + \cfrac{-\operatorname{sign}(a_1)}{|a_1|\lambda_q + \cfrac{-\operatorname{sign}(a_1)\operatorname{sign}(a_2)}{a_2\lambda_q + \frac{-\operatorname{sign}(a_2)\operatorname{sign}(a_3)}{a_3\lambda_q+\ldots}}}$$

$$= [a_0; (-\operatorname{sign}(a_1) : |a_1|), (-\operatorname{sign}(a_1)\operatorname{sign}(a_2) : |a_2|), (-\operatorname{sign}(a_2)\operatorname{sign}(a_3) : |a_3|), \ldots].$$

As claimed in [14, Remark 15] these relations imply directly

L 6.1.1. *Given a Rosen λ-fraction $[r_0; \ldots, (\varepsilon_i : r_i), (\varepsilon_{i+1} : r_{i+1}), \ldots]$ and its corresponding λ_q-CF $[r_0; \ldots, a_i, a_{i+1}, \ldots]$ in (6.1.3) we have*

$$\varepsilon_1 = -\operatorname{sign}(a_1) \quad \textit{and} \quad \varepsilon_{i+1} = -\operatorname{sign}(a_i\, a_{i+1}), \quad (i \in \mathbb{N}).$$

P . Equation 6.1.2 shows that the sign of the i^{th} digit $(-1)^i\, \varepsilon_1 \cdots \varepsilon_i\, r_i$ in the formal λ_q-CF is determined by $(-1)^i\, \varepsilon_1 \cdots \varepsilon_i$. Hence, the ratio

$$\frac{(-1)^{i+1}\, \varepsilon_1 \cdots \varepsilon_i \varepsilon_{i+1}}{(-1)^i\, \varepsilon_1 \cdots \varepsilon_i} = -\varepsilon_{i+1}$$

determines whether a_i and a_{i+1} have the same or opposite signs. □

Equations (6.1.2) and (6.1.3) indeed relate regular λ_q-CF's and reduced Rosen λ-fractions as we show next. Set

$$h_{\mathrm{R}} = h_{\text{Rosen}} := \left\lfloor \frac{q-3}{2} \right\rfloor = \begin{cases} h_q - 1 & \text{if } q \text{ is even and} \\ h_q & \text{if } q \text{ is odd.} \end{cases} \tag{6.1.4}$$

The above h_{R} coincides with "h" used in [21, above Definition 1]. Recall from [21, Page 555]

D 6.1.2 (Reduced Rosen λ-fractions). The Rosen λ-fraction in (6.1.1) is called *reduced* if it satisfies the following conditions:

(1) Blocks of the form

$$(* : 1), \underbrace{(-1 : 1), \ldots, (-1, 1)}_{h_{\mathrm{R}} \text{ times}}, (-1 : *)$$

do not appear.

(2) For q odd, blocks of the form

$$(* : 1), \underbrace{(-1 : 1), \ldots, (-1 : 1)}_{h_{\mathrm{R}} \text{ times}}$$

do not appear.

(3) For q odd, blocks of the form

$$(* : 1), \underbrace{(-1 : 1), \ldots, (-1 : 1)}_{h_{\mathbb{R}}-1 \text{ times}}, (-1 : 2), \underbrace{(-1 : 1), \ldots, (-1 : 1)}_{h_{\mathrm{R}} \text{ times}}, (-1, *)$$

do not appear.

(4) For q odd, a finite Rosen λ-fraction expansion does not terminate in a block of the form

$$(* : 1), \underbrace{(-1 : 1), \ldots, (-1 : 1)}_{h_R \text{ times}}.$$

(5) The value $\pm\frac{\lambda_q}{2}$ of the tail $[(* : r_i), (\varepsilon_{i+1} : r_{i+1}), \ldots, (\varepsilon_{i+k} : r_{i+k})]$ of a finite Rosen λ-fractions leads because of $r_{i-1}\lambda_q \pm \frac{\lambda_q}{2} = (r_{i-1} \pm 1)\lambda_q \mp \frac{\lambda_q}{2}$ to non-uniqueness of the expansion. We always choose the first possibility.

Then one shows

L 6.1.3. *The following three statements hold:*

- *The λ_q-CF associated to a reduced Rosen λ-fraction expansion in* (6.1.2) *is regular.*
- *The Rosen λ-fraction expansion corresponding to a regular λ_q-CF in* (6.1.3) *satisfies Properties (1)–(4) of Definition 6.1.2.*
- *The two expansions of the finite Rosen λ-fractions in (5) of Definition 6.1.2 correspond to the identities of the finite regular λ_q-CF's in Lemma 2.2.2.*

P . Let $x \in \mathbb{R}$ have the regular Rosen λ-fraction expansion (6.1.1). We have to show that the corresponding λ_q-CF in (6.1.2) does not contain any forbidden block from $\mathcal{B}_q$. We consider the cases q even and q odd separately.

Let q be even. Using Lemma 6.1.1 and the identity $h_R = h_q - 1$ in (6.1.4) we see that Property (1) of Definition 6.1.2 corresponds to the absence of blocks of the form $[(\pm 1)^{h_q}, \pm m]$ for any $m \in \mathbb{Z}_{\geq 1}$.

Consider next q odd. Using again Lemma 6.1.1 and the identity $h_R = h_q$ we see that Property (2) of Definition 6.1.2 corresponds to the absence of blocks of the form $[(\pm 1)^{h_q+1}]$. Similarly, Property (3) corresponds to the absence of blocks of the form $[(\pm 1)^{h_q}, \pm 2, (\pm 1)^{h_q}, \pm m]$ for any $m \in \mathbb{Z}_{\geq 1}$.

This shows that no forbidden block from $\mathcal{B}_q$ appears in the λ_q-CF in (6.1.2).

Next, let $x \in \mathbb{R}$ have the regular λ_q-CF $x = [\![a_0; a_1, a_2, a_3, \ldots]\!]$. We have to show that the corresponding formal Rosen λ-fraction in (6.1.3) satisfies properties (1)–(4) of Definition 6.1.2. Again, we discuss the cases q even and q odd separately.

Consider first q even. Using Lemma 6.1.1 and the identity $h_R = h_q - 1$ in (6.1.4) we find that forbidden blocks of the form $[(\pm 1)^{h_q}, \pm m]$ for any $m \in \mathbb{Z}_{\geq 1}$ imply Property (1). Property 5 corresponds just to the ambiguity of finite λ_q-CF's given in Lemma 2.2.2 since the tails $[(\pm 1)^{h_q}]$ correspond to $\mp\frac{\lambda_q}{2}$.

Consider next q odd. Using Lemma 6.1.1 and recalling the identity $h_R = h_q$ in (6.1.4) we see that forbidden blocks $[(\pm 1)^{h_q+1}]$ imply Property (2)

and Property (4) of Definition 6.1.2. Property (2) also implies Property (1). Property (3) follows from the forbidden blocks $[(\pm1)^{h_q}, \pm2, (\pm1)^{h_q}, \pm m]$ and for $m \in \mathbb{Z}_{\geq 1}$. The ambiguity of the regular λ_q-CF's in Lemma 2.2.2 implies Property (5).

To finish the proof of Lemma 6.1.3 consider q even and the finite regular λ_q-CF $[\![a_0; a_1, \ldots, a_n, (1)^{h_q}]\!]$. Using Equation (6.1.3) we rewrite it as the Rosen λ-fraction

$$[a_0; (-\operatorname{sign}(a_1) : |a_1|), \ldots$$
$$\ldots, (-\operatorname{sign}(a_{n-1})\operatorname{sign}(a_n) : |a_n|), (-\operatorname{sign}(a_n) : 1), (-1 : 1)^{h_q-1}].$$

Since by the equation below (4.2) in [21] and by [1, (4)] the identity $[0; (1 : 1), (-1 : 1)^{h-1}] = \frac{\lambda_q}{2}$ holds, we are in the situation of Property (5). If $a_n < 0$ we choose the "+"-sign in Property (5). If $a_n > 0$ we use Lemma 2.2.2 to rewrite the finite regular λ_q-CF such that its tail ends in $[a_n - 1, (-1)^{h_q}]$ with $\operatorname{sign}(a_n - 1) = \operatorname{sign}(a_n)$. Using Equation (6.1.3) we arrive at the Rosen λ-fraction

$$[a_0; (-\operatorname{sign}(a_1) : |a_1|), \ldots$$
$$\ldots, (-\operatorname{sign}(a_{n-1})\operatorname{sign}(a_n - 1) : |a_n - 1|), (\operatorname{sign}(a_n - 1) : 1), (-1 : 1)^{h_q-1}].$$

with the correct tail.

The case q odd is analogous to q even, with the only difference that the reduced Rosen λ-fraction $\frac{\lambda_q}{2} = [0; (1 : 1), (-1 : 1)^{h_q-1}, (-1 : 2), (-1, 1)^{h_q}]$ as given in [1, (4)] has the corresponding tail $[(1)^{h_q}, 2, (1)^{h_q}]$. □

R 6.1.4. Consider the λ_q-CF of $\pm r_q$ in (2.3.2). Their corresponding Rosen λ-fractions according to formula (6.1.2) are

$$r_q = \begin{cases} [0; (-1 : 1)^{h_q-1}, \overline{(-1 : 2), (-1, 1)^{h_q-1}}] & \text{for } q \text{ even and} \\ [0; (-1, 1), \overline{(-1 : 1)^{h_q-1}, (-1 : 2), (-1 : 1)^{h_q-1}, (-1 : 2), (-1, 1)}] & \\ & \text{for } q \text{ odd} \end{cases}$$

and

$$-r_q = \begin{cases} [0; (1 : 2), \overline{(-1 : 2)}] & \text{for } q = 4, \\ [0; (1 : 1), (-1 : 1)^{h_q-2}, \overline{(-1 : 2), (-1, 1)^{h_q-1}}] & \text{for even } q \geq 6, \\ [0; (1, 1), \overline{(-1 : 1)^{h_q-1}, (-1 : 2), (-1 : 1)^{h_q-1}, (-1 : 2), (-1, 1)}] & \\ & \text{for } q \text{ odd}, \end{cases}$$

where $(-1 : 1)^0$ means that the digit $(-1 : 1)$ is absent. The Rosen λ-fractions hence have the same tail.

R 6.1.5. The generating map $f_q^\star$ for the dual regular λ_q-CF and the generating map f_q^{R} for the Rosen λ-fractions in [1] satisfy

$$f_q^\star(-x) = \frac{1}{x} - \lambda_q \left\lfloor \frac{1}{x\lambda_q + \frac{R_q}{2}} \right\rfloor = \frac{1}{x} - \lambda_q \left\lfloor \frac{1}{x\lambda_q + 1 + \frac{r_q}{2}} \right\rfloor = f_q^{\mathrm{R}}(x)$$

for all $x \in \left(0, \frac{\lambda_q}{2}\right)$.

Formally, we find also

$$f_q^\star(-x) = \frac{1}{x} - \lambda_q \left\lfloor \frac{1}{x\lambda_q + \frac{R_q}{2}} \right\rfloor = \frac{1}{x} - \lambda_q \left\lfloor \frac{1}{x\lambda_q + 1 + \frac{r_q}{2}} \right\rfloor = T_{-\frac{r}{\lambda_q}}(x)$$

for all $x \in \left(0, \frac{\lambda_q}{2}\right)$ where T_α is the generating map of the α-Rosen fractions discussed in [5]. However, the parameter $\alpha = -\frac{r_q}{\lambda_q}$ lies outside the range $\alpha \in \left[\frac{1}{2}, \frac{1}{\lambda_q}\right]$ discussed in [5], since by (2.3.2) and (2.3.8) $-r_q = \lambda_q - R_q \in \left(0, \frac{\lambda_q}{2}\right)$.

6.2. Regular λ_q-CF's and convergents. We define the n^{th} convergent in the following way:

D 6.2.1. Given a regular λ_q-CF $[\![a_0; a_1, \ldots, a_n, \ldots]\!]$ of length at least n we define its n^{th} *convergent* as the fraction $\frac{p_n}{q_n}$ where the numerator p_n and denominator q_n are given as entries in the vector

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = T_q^{a_0} S T_q^{a_1} S T_q^{a_2} \cdots S T_q^{a_n} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6.2.1}$$

The convergents then satisfy the recursion relation

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{pmatrix} S T_q^{a_n} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_n\lambda_q\, p_{n-1} - p_{n-2} \\ a_n\lambda_q\, q_{n-1} - q_{n-2} \end{pmatrix} \tag{6.2.2}$$

which holds also for $n = 0$ and $n = 1$ if we define $p_{-1} = 1$, $p_{-2} = 0$, $q_{-1} = 0$ and $q_{-2} = -1$.

R 6.2.2. In the case $q = 3$ Definition 6.2.1 of the n^{th} convergent coincides with the usual definition as the ratio $\frac{p_n}{q_n} = [\![a_0; a_1, a_2, \ldots, a_n]\!]$, since

$$[\![a_0; a_1, a_2, \ldots, a_n]\!] = \begin{pmatrix} * & p_n \\ * & q_n \end{pmatrix} 0$$

where the last expression is to be understood as a Möbius transformation.

The following lemma will show that regular λ_q-CF's are indeed well defined and determine real numbers. This obviously is true for finite regular λ_q-CF's.

L 6.2.3. *Let $[a_0; a_1, a_2, \ldots]$ be an infinite regular λ_q-CF and denote by $\frac{p_n}{q_n}$ it's n^{th} convergent. Then for $q \geq 4$ the fraction $\frac{\operatorname{sign}(q_n)\, p_n}{|q_n|}$ is the n^{th} convergent of the corresponding reduced Rosen λ-fraction as defined in* [21, Definition 3]*; for $q = 3$ the fraction $\frac{p_n}{q_n}$ is a "Näherungsbruch" in the sense of Hurwitz* [8, §2].

P . The case $q = 3$ has been shown in [8].

Hence assume $q \geq 4$. Since the regular λ_q-CF is infinite, we don't have the ambiguities in Lemma 6.1.3. which shows that the corresponding Rosen λ-fraction

$$[a_0; (-\operatorname{sign}(a_1) : |a_1|), (-\operatorname{sign}(a_1)\operatorname{sign}(a_2) : |a_2|), \ldots]$$

is reduced. The n^{th} convergent $\frac{P_n}{Q_n}$ of the reduced Rosen λ-expansion is well defined and satisfies $Q_n \geq 1$ by [21, Lemma 4]. We have

$$\begin{aligned}\frac{P_n}{Q_n} &= [a_0; (-\operatorname{sign}(a_1) : |a_1|), (-\operatorname{sign}(a_1)\operatorname{sign}(a_2) : |a_2|), \dots, \\ &\qquad \dots, (-\operatorname{sign}(a_{n-1})\operatorname{sign}(a_n) : |a_n|)] \\ &= [a_0; a_1, a_2, \dots, a_n] = \frac{p_n}{q_n}\end{aligned}$$

with p_n and q_n satisfying (6.2.1). Hence we find indeed $P_n = \operatorname{sign}(q_n)\, p_n$ and $Q_n = |q_n|$. □

This lemma shows that the results on convergents in [21] hold also for the regular λ_q-CF's. We collect the relevant results in [21] and [8] in the following

L 6.2.4. *The convergents $\frac{p_n}{q_n}$ of an infinite regular λ_q-CF satisfy:*

- *$q_n \neq 0$ and $|q_n| \geq |q_{n-1}|$. For $q = 3$ we have $|q_n| > |q_{n-1}|$.*
- *$\lim_{n\to\infty} |q_n| \to \infty$.*
- *The sequence $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence.*

P . This follows from Lemma 4, Lemma 5, Theorem 4 and the proof of Theorem 5 in [21] for $q \geq 4$ and for $q = 3$ from §2 and §3 in [8]. □

Now we can (re-)define infinite regular λ_q-CF's in the following way: Let $[\![a_0; a_1, a_2, \dots]\!]$ be a regular λ_q-CF. We assign the value x to the regular λ_q-CF expansion and write $x = [\![a_0; a_1, a_2, \dots]\!]$ where x is the limit of the sequence of convergents (see Definition 6.1.4 and Lemma 6.2.4) of the corresponding Rosen λ-fraction.

Then the following estimate for the approximation of x by the convergents holds:

L 6.2.5. *Let $[\![a_0; a_1, a_2, \dots]\!]$ be an infinite regular λ_q-CF and denote its n^{th} convergents by $\frac{p_n}{q_n}$. There exists a constant $\kappa_q > 0$, independent of x, such that*

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{\kappa_q q_n^2}$$

holds for all n.

P . The lemma follows for $q \geq 4$ from Theorem 4.6 in [1] and for $q = 3$ from Satz on page 383 in [8]. □

R 6.2.6. Obviously, Lemma 6.2.5 implies that infinite regular λ_q-CF's converge. This gives another proof of part of Proposition 2.2.1.

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References

[1] R.M. Burton, C. Kraaikamp and T.A. Schmidt, *Natural extensions for the Rosen fractions*, Transactions of the American Mathematical Society **352** (1999), 1277-1298.
doi:10.1090/S0002-9947-99-02442-3

[2] R.W. Bruggeman, J. Lewis and D. Zagier, *Period functions for Maass wave forms. II: Cohomology*, in preparation.

[3] R.W. Bruggeman and T. Mühlenbruch, *Eigenfunctions of transfer operators and cohomology*, J. of Number Theory **129** (2009), 158–181.
doi:10.1016/j.jnt.2008.08.003

[4] C.H. Chang and D. Mayer, *The transfer operator approach to Selberg's zeta function and modular Maass wave forms for* PSL(2, $\mathbb{Z}$), In Emerging Applications of Number Theory, eds. D. Hejhal, M. Gutzwiller et al., IMA Volumes **109** (1999), 72–142, Springer-Verlag New York, 1999.

[5] K. Dajani, C. Kraaikamp and W. Steiner *Metrical theory for α-Rosen fractions*, To appear in J. of the European Math. Soc. (2009).
arXiv:math/0702516v2 [math.NT]

[6] E. Hecke, *Mathematische Werke*, Vandenhoeck & Ruprecht, Göttingen, 1959.

[7] D.A. Hejhal, *The Selberg trace formula for* PSL(2, R), Vol. 2, Lecture Notes in Mathematics **1001**, Springer-Verlag, Berlin, 1983.

[8] A. Hurwitz, *Über eine besondere Art der Kettenbruch-Entwicklung reeller Grössen*, Acta Mathematica **12** (1889), 367–405.
doi:10.1007/BF02391885

[9] S. Katok,and I. Ugarcovic, *Symbolic dynamics for the modular surface and beyond*, Bulletin of the American Mathematical Society **44** (2007), 87–132.
doi:10.1090/S0273-0979-06-01115-3

[10] J. Lewis and D. Zagier, *Period functions for Maass wave forms. I*, Annals of Mathematics **153** (2001), 191–258.
doi:10.2307/2661374

[11] J. Marklof, *Selberg's trace formula: an introduction*, Proceedings of the International School "Quantum Chaos on Hyperbolic Manifolds" (Schloss Reisensburg, Günzburg, Germany, 4-11 October 2003), To appear in *Lecture Notes in Physics*, Springer-Verlag.
http://arxiv.org/abs/math/0407288

[12] D. Mayer, *The thermodynamic formalism approach to Selberg's zeta function for* PSL(2, $\mathbb{Z}$), Bull. Amer. Math. Soc. (N.S.) **25** (1991), 55–60.
doi:10.1090/S0273-0979-1991-16023-4

[13] D. Mayer, *Transfer operators, the Selberg-zeta function and Lewis-Zagier theory of period functions*, Lecture notes of a course given in Günzburg, Germany, 4-11 October 2003. To appear in *Lecture Notes in Physics*, Springer-Verlag.
http://www.dynamik.tu-clausthal.de/research/preprints/Mayer.001.ps

[14] D. Mayer and F. Strömberg, *Symbolic dynamics for the Geodesic flow on Hecke surfaces*, Journal of Modern Dynamics **2** (2008), 581–627.
doi:10.3934/jmd.2008.2.581

[15] H. Nakada, *Continued fractions, geodesic flows and Ford circles*, in *Algorithms, Fractals, and Dynamics*, Edited by T. Takahashi, Plenum Press, New York, 1995.
MR1402490

[16] Z. Rudnick, *What is... quantum chaos?*, Notices Amer. Math. Soc. **55** (2008), no. 1, 32–34.
http://www.ams.org/notices/200801/tx080100032p.pdf

[17] D. Ruelle, *Zeta Functions for Piecewise Monotone Maps of the Interval*, CRM Monograph Series, **vol. 4** , AMS, Providence, Rhode Island, 1994.
[18] D. Ruelle, *Thermodynamic Formalism*, Cambridge Mathematical Library Cmabridge University Press, Cambridge CB2 2RU, UK, 2004.
[19] P. Sarnak, *Arithmetic quantum chaos*, The Schur lectures (1992) (Tel Aviv), 183–236, Israel Math. Conf. Proc. **8**, Bar-Ilan Univ., Ramat Gan, 1995. http://www.math.princeton.edu/sarnak/Arithmetic Quantum Chaos.pdf
[20] T.A. Schmidt and M. Sheingorn, *Length spectra of the Hecke triangle groups*, Mathematische Zeitschrift **220** (1995), 369–397. doi:10.1007/BF02572621
[21] D. Rosen, *A class of continued fractions associated with certain properly discontinuous groups*, Duke Mathematical Journal **21** (1954), 549–563. doi:10.1215/S0012-7094-54-02154-7

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