From the transfer operator for geodesic flows on modular surfaces to the Hecke operators on period functions of $\Gamma_0(n)$

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Abstract. Recently it has been shown how from certain special eigenfunctions of the transfer operators for the congruence subgroups $\Gamma_0\,(nm)$ a family of Hecke like operators can be constructed acting on the eigenfunctions of this operator for $\Gamma_0\,(n)$. In the case $n = 1$ these operators are closely related to the Hecke operators acting on the space of period functions for the full modular group. In a different approach to Hecke operators on period functions Mühlenbruch extended an integral transformation used by Lewis and Zagier relating Maass forms of the full modular group to their period functions to the congruence subgroups $\Gamma_0\,(n)$. The well known Hecke operators on the Maass forms thereby get transformed into operators acting on vector valued period functions. In the present paper we analyze the exact relation between the two kinds of operators and show that for $\gcd(n, m) = 1$ they are indeed identical. Our discussion leads also to a better understanding of the construction of these eigenfunctions of the transfer operators for $\Gamma_0\,(nm)$ in terms of the so called old Maass forms for these groups.

Introduction

There is a close connection between certain eigenfunctions of the transfer operator for the geodesic flow on the modular surface and the automorphic functions for the modular group $\mathrm{SL}(2,\mathbb{Z})$. Both the period polynomials respectively the rational period functions of the holomorphic modular forms and also the period functions of Lewis and Zagier [**LZ01**] of the real analytic Maass forms for $\mathrm{SL}(2,\mathbb{Z})$ give rise to eigenfunctions with eigenvalue $\lambda = 1$ of the transfer operator for the geodesic flow on the modular surface $\mathbb{H}\backslash\mathrm{SL}(2,\mathbb{Z})$. Of special interest is the connection of Maass forms with the period functions since the former are eigenfunctions of the hyperbolic Laplace-Beltrami operator and hence can be interpreted as the eigenstates or scattering states of a quantized particle moving freely on this surface (see [**Ma03**]). On the other hand, the period functions are eigenfunctions of the transfer

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operator which is constructed basically from datas describing the classical geodesic flow, namely the Poincaré map and the classical length spectrum. In this case the classical system determines completely the properties of the quantum system especially its eigenstates and the scattering states. At least for this system the classical transfer operator offers hence a possible new approach to the study of quantum chaos.

Different aspects of this theory are understood in the mean time also for general subgroups of $\mathrm{SL}(2,\mathbb{Z})$ with finite index, the so called modular groups. Typical examples of such modular groups are the Hecke congruence subgroups $\Gamma_0(n)$, which play an important role in quite different areas of number theory. Their transfer operators have been investigated in [**CM00**], [**CM01**] and the relation between their eigenfunctions with eigenvalue $\lambda = 1$ and the Maass forms has been clarified quite recently by Deitmar and Hilgert [**DH04**].

Here we address another aspect of the theory of period functions for these groups. The Hecke operators play an important role in the theory of automorphic functions and L-functions for arithmetic groups [**Sa90**]. Their action on the Maass forms has a nice physical interpretation. Since these operators commute with the Laplace-Beltrami operator and hence the Hamiltonian for a quantized free particle on the corresponding surface they describe internal symmetries of this system.

In the case $\mathrm{SL}(2,\mathbb{Z})$ one knows explicit expressions for the Hecke operators when acting on the period polynomials respectively rational period functions of the holomorphic modular forms [**M73**], [**Z90**], [**Z93**]. These results have been extended to the period functions of the Maass forms for the same group by Mühlenbruch [**Mü04a**]. All these authors use the Eichler-Shimura respectively Lewis-Zagier correspondence relating the modular forms by an explicit integral transformation to the period polynomials respectively period functions. The well known Hecke operators on the modular forms then induce operators on the period functions via this correspondence. The form of these operators can be calculated explicitly.

A completely different approach to the Hecke operators on period functions for $\mathrm{SL}(2,\mathbb{Z})$ has been given in [**HMM03**]. These authors construct to any eigenfunction of the transfer operator for $\mathrm{SL}(2,\mathbb{Z})$ with eigenvalue $\lambda = 1$ a nontrivial vector valued eigenfunction of this operator for any of the groups $\Gamma_0(m)$. It then turns out that the sum of the components of these solutions are exactly the Hecke operators for m prime, and are closely related to the Hecke operators for arbitrary m.

The same approach can be applied in the case of the congruence subgroups $\Gamma_0(n)$ [**HMM03**]. The transfer operator for these groups involves the representation of $\mathrm{SL}(2,\mathbb{Z})$ induced from the trivial representation of $\Gamma_0(n)$ and hence its eigenfunctions with eigenvalue $\lambda = 1$, the so called vector valued period functions, belong to this representation space. As eigenfunctions they fulfill a three-term functional equation as in the case of the full modular group $\mathrm{SL}(2,\mathbb{Z})$. The authors of [**HMM03**] then succeed in constructing to any eigenfunction of the transfer operator for $\Gamma_0(n)$ a nontrivial eigenfunction of the corresponding operator for $\Gamma_0(nm)$, $m = 2, 3, \ldots$. Sums of certain components of these eigenfunctions again determine a new eigenfunction of the transfer operator for $\Gamma_0(n)$ depending linearly on the original eigenfunction. In [**HMM03**] these linear operators are called "Hecke like" operators.

Recently in [**Mü04b**] Mühlenbruch worked out the classical approach via an integral transformation to vector valued period functions for the group $\Gamma_0\,(n)$. Indeed, most of his results are valid for any modular group $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ of finite index. Thereby he generalizes the integral transformation of Lewis and Zagier to vector valued Maass forms transforming under an induced representation of $\mathrm{SL}(2,\mathbb{Z})$ which leads to vector valued period functions fulfilling also a three-term functional equation closely related to the one derived from the transfer operator for the eigenfunctions. This allows him also in this case to determine explicit expressions for the Hecke operators acting on these vector valued period functions.

In the present paper we analyze the exact relation between Mühlenbruch's Hecke operators and the Hecke like operators in [**HMM03**] for the group $\Gamma_0\,(n)$. Even if the form of these operators seems to be rather different on a first glance, it turns out that for m prime and $\gcd(n,m) = 1$ the operators are indeed conjugate to each other. On the way to this result we find a new interpretation of the eigenfunctions $\vec{F}$ for $\Gamma_0\,(nm)$ constructed in [**HMM03**] for any eigenfunction $\vec{f}$ for $\Gamma_0\,(n)$. They are the vector valued period functions corresponding to the old Maass form $u(z) = v(mz)$ for $\Gamma_0\,(nm)$ if the eigenfunction $\vec{f}$ corresponds to the Maass form $v(z)$ for $\Gamma_0\,(n)$. This shows that the eigenfunctions $\vec{F}$ for $\Gamma_0\,(nm)$ are indeed "old eigenfunctions" in the sense of Atkin-Lehner.

In detail the paper is organized as follows: In the first Chapter we recall the main steps in the derivation in [**HMM03**] of the Hecke like operators for $\Gamma_0\,(n)$ from the eigenfunctions of the transfer operator for the groups $\Gamma_0\,(nm)$. In the second Chapter we recall the Hecke operators acting on the vector valued period functions of $\Gamma_0\,(n)$ induced by their action on vector valued Maass forms and determined by Mühlenbruch via an integral transformation in [**Mü04b**]. The main part of this paper is the third Chapter. There we show that the eigenfunctions of the transfer operators for the groups $\Gamma_0\,(nm)$ determined in Chapter 1 are indeed the vector valued period functions arizing from the old Maass forms $v(mz)$ for $v(z)$ a Maass form for $\Gamma_0\,(n)$. We discuss the exact relation between the Hecke like operators in [**HMM03**] and the Hecke operators in [**Mü04b**]. We prove that for prime m with $\gcd(n,m) = 1$ the operators are unitarily equivalent.

1. The "Hecke like" operators of Hilgert, Mayer and Movasati

Let us fix some notations which we will use throughout this paper. We denote by $\mathbb{H} = \{x + iy : y > 0\}$ the hyperbolic plane with the hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. A group $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ of finite index $\mu = [\mathrm{SL}(2,\mathbb{Z}) : \Gamma]$ is called a modular subgroup of the full modular group $\mathrm{SL}(2,\mathbb{Z})$. The surfaces $M_\Gamma = \Gamma\backslash\mathbb{H}$ with Γ a modular group are accordingly called modular surfaces. Obviously they are covering surfaces of $M = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$. The geodesic flow on these modular surfaces is denoted by $\Phi_t^\Gamma : SM_\Gamma \to SM_\Gamma$, where SM_Γ is the unit tangent bundle of M_Γ. This flow can be described symbolically as the special flow over a Poincaré map $\mathcal{P}_\Gamma : \Sigma_\Gamma \to \Sigma_\Gamma$, where Σ_Γ is a Poincaré section which can be chosen as follows [**Se85**], [**CM00**]:

$$\Sigma_\Gamma = I \times I \times \mathbb{Z}_2 \times \Gamma\backslash\mathrm{SL}(2,\mathbb{Z}) \tag{1.1}$$

where I denotes the unit interval $I = [0,1]$ and $\mathbb{Z}_2 = \{+1,-1\}$. With respect to this Poincaré section the Poincaré map $\mathcal{P}_\Gamma$ has the following explicit form [**CM00**]:

$$\mathcal{P}_\Gamma(x_1,x_2,\varepsilon,[g]) = \left(\frac{1}{x_1} \mod 1, \frac{1}{x_2+n}, -\varepsilon, [gT^{n\varepsilon}S]\right)$$

with $n = \left[\frac{1}{x_1}\right]$ and T respectively S denoting the generators

$$T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \qquad S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \tag{1.2}$$

of $\mathrm{SL}(2,\mathbb{Z})$. Later on we also need the matrices

$$M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \qquad T' = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \tag{1.3}$$

It is then a standard procedure to write down the Ruelle transfer operator $\bar{\mathcal{L}}_\beta^\Gamma$ for this map $\mathcal{P}_\Gamma$ [**CM00**]

$$\left(\bar{\mathcal{L}}_\beta^\Gamma \vec{f}\right)(x_1,\varepsilon) = \sum_{n=1}^\infty \left(\frac{1}{x_1+n}\right)^{2\beta} \chi_\Gamma(ST^{n\varepsilon})\, \vec{f}\left(\frac{1}{x_1+n}, -\varepsilon\right) \tag{1.4}$$

with $\vec{f} : I \times \mathbb{Z}_2 \to \mathbb{C}^\mu$ some vector valued observable on the unstable directions of $\mathcal{P}_\Gamma$ and $\chi_\Gamma : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}^\mu)$ the right regular representation of $\mathrm{SL}(2,\mathbb{Z})$ induced from the trivial representation of Γ. The operator $\bar{\mathcal{L}}_\beta^\Gamma$ is well defined on the following Banach space $B(D)$ of holomorphic functions:

$$B(D) = \left\{\vec{f} : D \to \mathbb{C}^\mu, f \text{ holomorphic in } D \text{ and continuous on } \bar{D}\right\}$$

with $D = \{z : |z-1| < \frac{3}{2}\}$. On this B-space the operator $\bar{\mathcal{L}}_\beta^\Gamma$ has rather nice properties (see for instance [**Ma91**]):

- $\bar{\mathcal{L}}_\beta^\Gamma$ is nuclear for $\mathrm{Re}(\beta) > \frac{1}{2}$.
- $\bar{\mathcal{L}}_\beta^\Gamma$ has a meromorphic extension as a nuclear operator to the entire complex β-plane with possible poles at $\beta_k = \frac{1-k}{2}$, $k = 0,1,2,\ldots$.
- $Z^\Gamma(\beta) = \det(1 - \bar{\mathcal{L}}_\beta^\Gamma)$ with $Z^\Gamma(\beta)$ denoting the Selberg zeta function for the modular group Γ, that means

$$Z^\Gamma(\beta) = \prod_{k=0}^\infty \prod_\gamma \left(1 - e^{-(\beta+k)l(\gamma)}\right) \tag{1.5}$$

  where the second product is over the closed orbits γ of Φ_t^Γ with prime period $l(\gamma)$.

In [**MM02**] Manin and Marcolli introduced another transfer operator $\mathcal{L}_\beta^\Gamma$ which leads to a factorization of the Selberg function for Γ well known for the full modular group. Assume that the modular group Γ has the property

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Gamma\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Gamma.$$

For such Γ's consider then the group $\widetilde{\Gamma} := \Gamma \cup \Gamma\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and its extension $\overline{\Gamma} := \widetilde{\Gamma} \cup \widetilde{\Gamma}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\mathrm{GL}(2,\mathbb{Z})$. Then obviously $\overline{\Gamma}\backslash\mathrm{GL}(2,\mathbb{Z}) \cong \Gamma\backslash\mathrm{SL}(2,\mathbb{Z})$. In [**MM02**] the following transfer operator $\mathcal{L}_\beta^\Gamma$ has been introduced

$$\left(\mathcal{L}_\beta^\Gamma \vec{f}\right)(z) = \sum_{n=1}^\infty \left(\frac{1}{z+n}\right)^{2\beta} \chi_{\overline{\Gamma}}\left(\begin{smallmatrix} -n & 1 \\ 1 & 0 \end{smallmatrix}\right) \vec{f}\left(\frac{1}{z+n}\right) \tag{1.6}$$

acting on functions $\vec{f} : D \to \mathbb{C}^\mu$ holomorphic in D, where $\chi_{\overline{\Gamma}}$ is the right regular representation of $\mathrm{GL}(2,\mathbb{Z})$ induced from the trivial representation of $\overline{\Gamma}$.

It turns out that the Selberg zeta function for the group Γ can be expressed also in terms of the operators $\mathcal{L}_\beta^\Gamma$, indeed [**MM02**] one finds

$$Z^\Gamma(\beta) = \det(1 - \mathcal{L}_\beta^\Gamma)\,\det(1 + \mathcal{L}_\beta^\Gamma) \tag{1.7}$$

and hence the zeros of this function are determined by those values of β for which $\mathcal{L}_\beta^\Gamma$ has the eigenvalues $+1$ or -1.

Remark 1.1. In the case $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ the transfer operator $\mathcal{L}_\beta^\Gamma$ has the form $\left(\mathcal{L}_\beta^\Gamma f\right)(z) = \sum_{n=1}^\infty \left(\frac{1}{z+n}\right)^{2\beta} f\left(\frac{1}{z+n}\right)$ and hence coincides with the well known transfer operator of the Gauss map $T_G x = \frac{1}{x} \mod 1$.

We call the function $\vec{\phi} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}^\mu$ with $\vec{\phi}(z) = \vec{f}(z-1)$ and $\vec{f} = \vec{f}(z)$ an eigenfunction of $\mathcal{L}_\beta^\Gamma$ with eigenvalue $\lambda = +1$ or $\lambda = -1$ a *vector valued period function* for Γ.

Obviously these vector valued period functions have the following properties:

- $\vec{\phi}$ is holomorphic in the complex z plane cut along the line $(-\infty, 0]$.
- $\vec{\phi}$ fulfills the three-term functional equation

$$\vec{\phi}(z) - \chi_{\overline{\Gamma}}(T^{-1})\,\vec{\phi}(z+1) - \lambda\, z^{-2\beta}\,\chi_{\overline{\Gamma}}(T^{-1}M)\,\vec{\phi}\left(\frac{z+1}{z}\right) = 0 \tag{1.8}$$

  the so called *generalized Lewis equation*.
- The components ϕ_i of $\vec{\phi}$ fulfill certain growth properties on the real axis for $z \to 0$ and $z \to \infty$ depending on β.

Remark 1.2. In the special case of the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ the period functions fulfill the original Lewis equation

$$\phi(z) - \phi(z+1) - \lambda\, z\, \phi\left(\frac{z+1}{z}\right) = 0. \tag{1.9}$$

The vector valued period function $\vec{\phi} = \vec{\phi}(z)$ can obviously be interpreted also as a function on $\mathbb{C} \setminus (-\infty, 0] \times \overline{\Gamma}\backslash\mathrm{GL}(2,\mathbb{Z})$ with $\phi(z, \overline{\Gamma} g_i) := \left(\vec{\phi}(z)\right)_i$ where i is just an abbreviation for the rest class $\overline{\Gamma} g_i$ with g_i, $i = 1, \ldots, \mu$ representatives of the rest classes in $\overline{\Gamma}\backslash\mathrm{GL}(2,\mathbb{Z})$. There is a generalized slash action of the group $\mathrm{GL}(2,\mathbb{Z})$ on these functions given formally by

$$\phi\big|_\beta h(z, \overline{\Gamma} g) := (cz+d)^{-2\beta}\,\phi(hz, \overline{\Gamma} g T^{-1} h T) \tag{1.10}$$

for $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{Z})$ and $hz = \frac{az+b}{cz+d}$. For details see [**HMM03**]. The generalized Lewis equation (1.8) can then be written in the simple form

$$\phi - \phi\big|_\beta T - \lambda \phi\big|_\beta TM = 0. \tag{1.11}$$

We do not know how to solve this equation in general, but it is possible to describe special solutions. For this, consider the ring $\mathcal{R} = \mathbb{Z}[\mathrm{Mat}_*(2,\mathbb{Z})]$ of finite linear combinations of nonsingular 2×2 integer matrices and the right $\mathcal{R}$-ideal $\mathcal{J}^\lambda$ with $\mathcal{J}^\lambda := (\mathbf{1} - T - \lambda TM)\mathcal{R}$ and $\mathbf{1}$ the unit matrix. The slash action in (1.10) extends obviously linearly to an action of $\mathcal{R}$. Consider next the following set of equations in the ring $\mathcal{R}$:

$$\psi_i - \psi_{iT^{-1}}T - \lambda\psi_{iT^{-1}M}TM \equiv 0 \mod \mathcal{J}^\lambda \tag{1.12}$$

with i denoting again the rest class $\overline{\Gamma}g_i$ in $\overline{\Gamma}\backslash\mathrm{GL}(2,\mathbb{Z})$ on which the group $\mathrm{GL}(2,\mathbb{Z})$ acts in a natural way. Then one has the obvious

LEMMA 1.3. *Given any solution $\phi = \phi(z)$ of the Lewis equation (1.9) for* $\mathrm{SL}(2,\mathbb{Z})$ *the functions*

$$\phi_i = \phi_i(z) := \phi\big|_\beta \psi_i(z), \quad i \in \overline{\Gamma}\backslash\mathrm{GL}(2,Z)$$

solve the Lewis equation (1.8) for the modular group Γ if the matrices $\psi_i \in \mathcal{R}$ solve equation (1.12).

There is a trivial solution of equation (1.12) namely $\psi_i = \mathbf{1}$, $i \in \overline{\Gamma}\backslash\mathrm{GL}(2,\mathbb{Z})$. This leads to the special but trivial solution $\phi_i(z) = \phi(z)$, $i \in \overline{\Gamma}\backslash\mathrm{GL}(2,\mathbb{Z})$, of equation (1.8). That such a solution exists is not surprising since we know in the case of the full modular group from the work of Lewis and Zagier [**LZ01**] and for general modular groups from the work of Deitmar and Hilgert [**DH04**] that there is a 1-1 correspondence between the period functions and the Maass forms for these groups. But every Maass form for $\mathrm{SL}(2,\mathbb{Z})$ is trivially a Maass form for any modular subgroup Γ and hence also a period function for $\mathrm{SL}(2,\mathbb{Z})$ should give rise to such a function for every of its modular subgroups. Consistently with the terminology for automorphic forms we should hence call the above solution of the Lewis equation for the modular subgroup Γ an "old solution". It is not difficult to show that if $\phi = \phi(z)$ is a period function for $\mathrm{SL}(2,\mathbb{Z})$ then $\vec{\phi} = \vec{\phi}(z)$ with $\phi_i(z) = \phi(z)$ for all $i \in \overline{\Gamma}\backslash\mathrm{GL}(2,\mathbb{Z})$ is indeed a vector valued period function for Γ with the correct asymptotic behavior for $z \to 0$ and $z \to \infty$.

In the following we will restrict our discussion to the construction of other nontrivial solutions of equations (1.8) respectively (1.12) for the Hecke congruence subgroups $\Gamma_0(n)$. For this we need a certain special characterization of the index set $\overline{\Gamma_0(n)}\backslash\mathrm{GL}(2,\mathbb{Z})$ as given in [**HMM03**]. Consider on $\mathbb{Z}\times\mathbb{Z}$ namely the following equivalence relation $\sim_n$ defined as

$$\begin{aligned}(x,y) \quad \sim_n \quad (x',y') \quad \text{iff} \quad & \exists k \in \mathbb{Z}, \gcd(k,n)=1 \text{ such that} \\ & x' \equiv kx \mod n, y' \equiv ky \mod n\end{aligned}$$

and the natural right action of $\mathrm{GL}(2,\mathbb{Z})$

$$(x,y)\begin{pmatrix} a & b \\ c & d\end{pmatrix} = (xa+yc, xb+yd) \tag{1.13}$$

which is obviously compatible with $\sim_n$. Hence $\mathrm{GL}(2,\mathbb{Z})$ acts also on $(\mathbb{Z}\times\mathbb{Z})_n := (\mathbb{Z}\times\mathbb{Z})/\sim_n$. Denote the elements of $(\mathbb{Z}\times\mathbb{Z})_n$ by $[x:y]_n$. It is easy to see that the stabilizer in $\mathrm{GL}(2,\mathbb{Z})$ of the element $[0:1]_n \in (\mathbb{Z}\times\mathbb{Z})_n$ is just the subgroup $\overline{\Gamma_0(n)}$. Hence the following map $\overline{\pi}_n : \overline{\Gamma_0(n)}\backslash\mathrm{GL}(2,\mathbb{Z}) \to (\mathbb{Z}\times\mathbb{Z})_n$ is well defined and injective:

$$\overline{\pi}_n\left(\overline{\Gamma_0(n)}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) := [0:1]_n\begin{pmatrix} a & b \\ c & d\end{pmatrix} = [c:d]_n. \tag{1.14}$$

Denote by I_n the image of $\overline{\pi}_n$, that is

$$I_n := \overline{\pi}_n\left(\overline{\Gamma_0(n)}\backslash\mathrm{GL}(2,\mathbb{Z})\right). \tag{1.15}$$

It is then not very difficult to show [**HMM03**] that

$$I_n = \{[x:y]_n \in (\mathbb{Z}\times\mathbb{Z})_n :\, \gcd(x,y,n)=1\}\,. \tag{1.16}$$

Consider next the subset $P_n \subset \mathbb{Z}\times\mathbb{Z}$ defined as

$$P_n = \left\{(c,b)\in\mathbb{Z}\times\mathbb{Z} :\, c\geq 1, c|n, 0\leq b\leq \frac{n}{c}-1, \gcd(c,b,\frac{n}{c})=1\right\}. \tag{1.17}$$

Then there is a bijection between I_n and P_n given by the map [**HMM03**]

$$P_n \ni (c,b) \longmapsto [c : d_n(c,b)]_n \in I_n \tag{1.18}$$

with

$$d_n(c,b) = \min_{0\leq k\leq c-1}\left\{c+b+k\frac{n}{c} : \gcd(c, b+k\frac{n}{c})=1\right\}. \tag{1.19}$$

For simplicity we denote in the following the elements of I_n also by i. Relation (1.18) allows us to identify every element $i \in I_n$ uniquely with a matrix $A_i \in \mathrm{Mat}_n(2,\mathbb{Z})$ with

$$A_i = \begin{pmatrix} c & b \\ 0 & \frac{n}{c}\end{pmatrix}. \tag{1.20}$$

where $\mathrm{Mat}_n(2,\mathbb{Z})$ denotes the 2×2 matrices with integer entries and determinant n. For the following discussion we need some sets of 2×2 matrices with nonnegative integer entries:

$$S_n = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{Mat}_n(2,\mathbb{Z}) :\, a>c\geq 0, d>b\geq 0\right\}, \tag{1.21}$$

$$X_n = \left\{\begin{pmatrix} c & b \\ 0 & \frac{n}{c}\end{pmatrix} \in S_n\right\}, \qquad Y_n = \left\{\begin{pmatrix} \frac{n}{c} & 0 \\ b & c\end{pmatrix} \in S_n\right\} \text{ and} \tag{1.22}$$

$$X_n^\star = \left\{\begin{pmatrix} c & b \\ 0 & \frac{n}{c}\end{pmatrix} \in X_n, \gcd(c,b,\frac{n}{c})=1\right\}. \tag{1.23}$$

Obviously the matrix A_i in (1.20) belongs to $X_n^\star$ for all $i \in I_n$.

Consider next the map

$$K : S_n \setminus Y_n \longrightarrow S_n \setminus X_n \tag{1.24}$$

defined as

$$K\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) := \begin{pmatrix} -c+\lceil\frac{d}{b}\rceil a & -d+\lceil\frac{d}{b}\rceil b \\ a & b\end{pmatrix} \tag{1.25}$$

with $\lceil r\rceil \in \mathbb{Z}$ determined for $r\in\mathbb{R}$ by $\lceil r\rceil - 1 < r \leq \lceil r\rceil$.

Obviously the map K is well defined and there exists for every $A \in S_n \setminus Y_n$ an integer $k_A > 0$ such that $K^j A \notin Y_n$ for $0 \leq j < k_A$ and $K^{k_A} A \in Y_n$. For $A \in Y_n$ we set $k_A = 0$ so that k_A is well defined for all $A \in S_n$.

The following Theorem has been proven in [**HMM03**].

THEOREM 1.4. *The matrices* $\psi_i := \sum_{j=0}^{k_i} K^j(A_i)$, $i \in I_n$, *with* $k_i = k_{A_i}$ *and* A_i *as in (1.20) solve the Lewis equation (1.12) for the group* $\Gamma_0(n)$.

As an immediate Corollary one then gets

COROLLARY 1.5. *For* $\phi = \phi(z)$ *any solution of the Lewis equation (1.9) for* $\mathrm{SL}(2,\mathbb{Z})$ *the functions* $\phi_i = \phi_i(z) := \phi\big|_\beta \psi_i$, $i \in I_n$ *solve the Lewis equation (1.8) for the group* $\Gamma_0(n)$ *with identical parameter* β.

This result can be generalized in the following way. Since

$$\mathrm{SL}(2,\mathbb{Z}) = \Gamma_0\,(1) \supset \Gamma_0\,(n) \supset \Gamma_0\,(nm)$$

for fixed n and all $m = 1, 2, \dots$ one has a natural projection

$$\sigma_{m,n} : I_{nm} \longrightarrow I_n \tag{1.26}$$

induced from the map $\Gamma_0\,(nm)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \Gamma_0\,(n)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

In [**HMM03**] we proved

LEMMA 1.6. *For any $i \in I_{nm}$ and $0 \le j \le k_{\sigma_{n,m}(i)}$ there exists a unique index $l_{i,j} \in I_n$ such that*

$$A_{l_{i,j}}\big(K^j(A_{\sigma_{n,m}(i)})\big)A_i^{-1} \in \mathrm{SL}(2,\mathbb{Z}).$$

This on the other hand allows one to show [**HMM03**]

THEOREM 1.7. *For $\vec{\psi} = \big(\psi_i\big)_{i\in I_n}$ respectively $\vec{\phi} = \big(\phi_i(z)\big)_{i\in I_n}$ any solution of the Lewis equations (1.12) respectively (1.8) with parameter β for the group $\Gamma_0\,(n)$ the matrices $\vec{\Psi} = \big(\Psi_j\big)_{j\in I_{nm}}$ respectively the functions $\vec{\Phi} = \big(\Phi_j(z)\big)_{j\in I_{nm}}$ with*

$$\Psi_j := \sum_{s=0}^{k_{\sigma_{n,m}(j)}} \psi_{l_{j,s}}\, K^s(A_{\sigma_{n,m}(j)}) \tag{1.27}$$

respectively

$$\Phi_j(z) := \sum_{s=0}^{k_{\sigma_{n,m}(j)}} \phi_{l_{j,s}}\big|_\beta K^s(A_{\sigma_{n,m}(j)})(z) \tag{1.28}$$

solve the corresponding Lewis equations for the group $\Gamma_0\,(nm)$ with the same parameter β.

An immediate Corollary is

COROLLARY 1.8. *For $\vec{\phi} = \big(\phi_i(z)\big)_{i\in I_n}$ a vector valued period function for $\Gamma_0\,(n)$ the function $\vec{\Phi} = \big(\Phi_j(z)\big)_{j\in I_{nm}}$ in (1.28) is a vector valued period function for $\Gamma_0\,(nm)$ with the same parameter β.*

REMARK 1.9. The function $\vec{F} = \big(F_j(z)\big)_{j\in I_{nm}}$ with $F_j(z) = \Phi_j(z+1)$ is an eigenfunction of the transfer operator $\mathcal{L}_\beta^{\Gamma_0(nm)}$ with eigenvalue $\lambda = \pm 1$ if the function $\vec{f} = \big(f_i(z)\big)_{i\in I_n}$ with $f_i(z) = \phi_i(z+1)$ is an eigenfunction of the operator $\mathcal{L}_\beta^{\Gamma_0(n)}$ with the same eigenvalue $\lambda = \pm 1$.

To derive from these special vector valued period functions operators similar to the Hecke operators on the automorphic forms we need another result from [**HMM03**]:

PROPOSITION 1.10. *If the matrices $\vec{\Psi} = \big(\Psi_i\big)_{i\in I_{nm}}$ solve the Lewis equation (1.12) for $\Gamma_0\,(nm)$ then the matrices $\vec{\psi} = \big(\psi_j\big)_{j\in I_n}$ with $\psi_j := \sum_{i\in\sigma_{m,n}^{-1}(j)} \Psi_i$ solve this equation for $\Gamma_0\,(n)$. If on the other hand $\vec{\psi} = \big(\psi_j\big)_{j\in I_n}$ solve equation (1.12) for $\Gamma_0\,(n)$ then $\vec{\Psi} = \big(\Psi_i\big)_{i\in I_{nm}}$ with $\Psi_i := \psi_{\sigma_{m,n}(i)}$ solve this equation for $\Gamma_0\,(nm)$.*

REMARK 1.11. The second part of this Proposition shows that any period function for $\Gamma_0(n)$ determines a "trivial" old period function for $\Gamma_0(nm)$ whose components are just given by the components of the former one.

An immediate consequence in the case $n = 1$, that is for the full modular group $\mathrm{SL}(2,\mathbb{Z})$, is

COROLLARY 1.12. *If $\vec{\Psi} = (\Psi_i)_{i\in I_m}$ solve equation (1.12) for $\Gamma_0(m)$ then the matrix $\psi^{(m)}$ with*

$$\psi^{(m)} := \sum_{i\in I_m} \Psi_i \tag{1.29}$$

solves this equation for the group $\mathrm{SL}(2,\mathbb{Z})$ for the same parameter β.

A straightforward calculation shows that for any m prime one has indeed

$$\psi^{(m)} := \sum_{A\in S_m} A. \tag{1.30}$$

COROLLARY 1.13. *For any period function $\phi = \phi(z)$ for $\mathrm{SL}(2,\mathbb{Z})$ and for any $m \in \mathbb{N}$ the function $\tilde{\phi} = \tilde{\phi}(z) := \phi\big|_\beta \psi^{(m)}(z)$ is again a period function for the group with identical parameter β.*

In complete analogy one derives from Theorem 1.7 and Proposition 1.10

THEOREM 1.14. *For any vector valued period function $\vec{\phi} = (\phi_i)_{i\in I_n}$ for $\Gamma_0(n)$ and any $m \in \mathbb{N}$ the function $\vec{\tilde{\phi}} = \tilde{\mathrm{T}}_{n,m}\vec{\phi}$ with*

$$\left(\tilde{\mathrm{T}}_{n,m}\vec{\phi}\right)_i(z) = \sum_{s\in\sigma_{m,n}^{-1}(i)} \sum_{j=0}^{k_{\sigma_{n,m}(s)}} \phi_{l_{s,j}}\big|_\beta K^j(A_{\sigma_{n,m}(s)})(z) \tag{1.31}$$

is again a period function for $\Gamma_0(n)$ with identical parameter β.

Hence we have constructed this way linear operators $\tilde{\mathrm{T}}_{n,m}$ mapping the space of vector valued period functions for $\Gamma_0(n)$ with parameter β into itself. In the case $n = 1$ and m prime the operator $\tilde{\mathrm{T}}_{1,m}$ reduces to the form

$$\left(\tilde{\mathrm{T}}_{1,m}\phi\right)(z) = \sum_{A\in S_m} \phi\big|_\beta A(z) \tag{1.32}$$

and hence coincides exactly with the m^{th} Hecke operator $\tilde{\mathrm{H}}_m$ in the form derived by Mühlenbruch in [**Mü04a**] for period functions of Maass forms for $\mathrm{SL}(2,\mathbb{Z})$. For modular symbols this form of the Hecke operator was also derived earlier by Merel in [**Me94**].

Indeed in [**HMM03**] it was shown how the operators $\tilde{\mathrm{T}}_{1,m}$ are related to the Hecke operators $\tilde{\mathrm{H}}_m$ for arbitrary $m \in \mathbb{N}$.

To understand the relation between the operators $\tilde{\mathrm{T}}_{n,m}$ and the Hecke operators $\tilde{\mathrm{H}}_{n,m}$ for an arbitrary subgroup $\Gamma_0(n)$ we have to recall briefly in the next chapter the recent approach by Mühlenbruch to the operators $\mathrm{H}_{n,m}$ on vector valued Maass forms.

2. The Hecke operators on vector valued period functions for $\Gamma_0(n)$

The approach by Mühlenbruch to the vector valued period functions for the group $\Gamma_0(n)$ works indeed for any modular subgroup $\Gamma \in \mathrm{SL}(2,\mathbb{Z})$ of finite index and also for the vector valued period polynomials for these groups. For simplicity we restrict however our discussion to the group $\Gamma_0(n)$ and their Maass wave forms. Denote by $S(n,\beta)$ the space of Maass forms u for $\Gamma_0(n)$ with spectral parameter β where

- $u : \mathbb{H} \to \mathbb{C}$ is real analytic,
- $u(gz) = u(z)$ for all $g \in \Gamma_0(n)$,
- $\Delta u = \beta(1-\beta)u$ with $\Delta = -y^2\left(\partial_x^2 + \partial_y^2\right)$ and
- u is of rapid decay at the cusps of $\Gamma_0(n)$.

On the space $S(n,\beta)$ one has an action of the Hecke operators T_p which for p prime are defined as follows [**AL70**]: if $\gcd(p,n)=1$ set

$$T(p) = \sum_{\substack{ad=p \\ 0\le b<d}} \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), \tag{2.1}$$

if on the other hand $p \mid n$ set

$$T(p) = \sum_{0\le b<p} \left(\begin{smallmatrix} 1 & b \\ 0 & p \end{smallmatrix}\right). \tag{2.2}$$

Then the p^{th} Hecke operator T_p is given by

$$\mathrm{T}_p u = u\big|_0 T(p). \tag{2.3}$$

For arbitrary $p \in \mathbb{N}$ we refer to [**Mi89**] for a suitable description of $T(p)$ in terms of matrices in X_p.

Consider next the representation $\rho = \rho_{\Gamma_0(n)}$ of $\mathrm{SL}(2,\mathbb{Z})$ induced from the trivial 1-dimensional representation of $\Gamma_0(n)$ defined as

$$\rho_{\Gamma_0(n)}(g) := \left(\delta_{\Gamma_0(n)}(g_i\, g\, g_j^{-1})\right)_{1\le i,j\le\mu} \tag{2.4}$$

where

$$\delta_{\Gamma_0(n)}(g) = \begin{cases} 1 & \text{if } g \in \Gamma_0(n) \text{ and} \\ 0 & \text{if } g \notin \Gamma_0(n), \end{cases}$$

μ denotes the index of $\Gamma_0(n)$ in $\mathrm{SL}(2,\mathbb{Z})$ and the g_i are representatives of the rest classes of $\Gamma_0(n)$ in $\mathrm{SL}(2,\mathbb{Z})$ such that

$$\Gamma_0(n)\backslash\mathrm{SL}(2,\mathbb{Z}) = \{\Gamma_0(n)g_i : 1 \le i \le \mu\}.$$

Definition 2.1. Denote by $S_{\text{ind}}(n,\beta)$ the space of vector valued Maass forms $\vec{u} = \left(u_i\right)_{1\le i\le\mu}$, $\mu = [\mathrm{SL}(2,\mathbb{Z}) : \Gamma_0(n)]$, such that for all $1 \le i \le \mu$ one has

- $u_i : \mathbb{H} \to \mathbb{C}$ is real analytic,
- $\vec{u}(gz) = \rho(g)\,\vec{u}(z)$ for all $g \in \mathrm{SL}(2,\mathbb{Z})$,
- $\Delta u_i = \beta(1-\beta)\,u_i$ and
- $u_i(z) = \mathrm{O}\left(y^C\right)$ as $y = \mathrm{Re}(z) \to \infty$ for all $C \in \mathbb{R}$.

Remark 2.2. Every Maass form $u \in S(n,\beta)$ determines a vector valued form $\vec{u} \in S_{\text{ind}}(n,\beta)$ with $\vec{u} = \left(u_i\right)_{1\le i\le\mu}$ where $u_i = u\big|_0 g_i$.

On the other hand given any $\vec{u} \in S_{\rm ind}(n, \beta)$ consider the element g_j such that $\Gamma_0\,(n) = \Gamma_0\,(n)g_j$. One then derives from the properties of the right regular representation ρ that

$$\begin{aligned} u_j &= u_j\big|_0 g && \text{for all } g \in \Gamma_0\,(n) \text{ and} \\ u_i &= u_j\big|_0 g_i && \text{for all } 1 \le i \le \mu. \end{aligned}$$

This shows that the map $\Pi = \Pi_n : S(n, \beta) \to S_{\rm ind}(n, \beta)$ with

$$\big(\Pi(u)\big)_i = u\big|_0 g_i, \qquad 1 \le i \le \mu, \tag{2.5}$$

is bijective.

According to (2.3) the Hecke operators T_p act on the space $S(n, \beta)$ as

$$\mathrm{T}_p u = u\big|_0 \sum_{A \in \mathcal{A}_p} A \tag{2.6}$$

with $\mathcal{A}_p = X_p$ for $\gcd(n, p) = 1$ and $\mathcal{A}_p = X_p \setminus \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $p|n$. Consider then the corresponding vector valued Maass form $\Pi(\mathrm{T}_p u) = \Pi\big(u\big|_0 \sum_{A \in \mathcal{A}_p} A\big)$. Its $i^{\rm th}$ component is given as

$$(\Pi(\mathrm{T}_p u))_i = \big(u\big|_0 \sum_{A \in \mathcal{A}_p} A\big)\big|_0 g_i = u\big|_0 \sum_{A \in \mathcal{A}_p} A g_i. \tag{2.7}$$

To write this in terms of the slash action of certain matrices on the components $u_k = u\big|_0 g_k$ of the Maass form $\vec{u} = \Pi(u)$ we need some properties of the matrices $A \in X_p$ from [**Mü04b**].

Lemma 2.3. *For $m \in \mathbb{N}$ and $g \in \mathrm{SL}(2, \mathbb{Z})$ there exists a unique bijective map $\sigma_g : X_m \to X_m$ with*

$$A g \big(\sigma_g(A)\big)^{-1} \in \mathrm{SL}(2, \mathbb{Z}) \qquad \textit{for all } A \in X_m. \tag{2.8}$$

This allows us to define for every $A \in X_m$ and $\mu_n = [\mathrm{SL}(2, \mathbb{Z}) : \Gamma_0(n)]$ a unique map

$$\Phi_A = \Phi_{A,n} : \{1, \ldots, \mu_n\} \to \{1, \ldots, \mu_n\} \tag{2.9}$$

such that for the representatives $g_1, \ldots, g_{\mu_n} \in \mathrm{SL}(2, \mathbb{Z})$ of the right cosets of $\Gamma_0\,(n)$ in $\mathrm{SL}(2, \mathbb{Z})$ the relation

$$A g_i \in \Gamma_0\,(n)\, g_{\Phi_A(i)}\, \sigma_{g_i}(A) \tag{2.10}$$

holds for all $1 \le i \le \mu_n$. In [**Mü04b**] Mühlenbruch then defines for p prime the $p^{\rm th}$ Hecke operator $\mathrm{T}_{n,p}$ on the space $S_{\rm ind}(n, \beta)$ of vector valued Maass forms $\vec{u} = \big(u_i\big)_{1 \le i \le \mu_n}$ as

$$\big(\mathrm{T}_{n,p} \vec{u}\big)_i = \sum_{A \in \mathcal{A}_p} u_{\Phi_A(i)}\big|_0 \sigma_{g_i}(A), \qquad 1 \le i \le \mu_n. \tag{2.11}$$

Applying (2.10) and the fact that $S_{\rm ind}(n, \beta) = \Pi\big(S(n, \beta)\big)$ one gets for p prime [**Mü04b**]

Proposition 2.4. *If $\mathrm{T}_p : S(n, \beta) \to S(n, \beta)$ is the $p^{\rm th}$ Hecke operator on Maass forms for $\Gamma_0\,(n)$ and $\mathrm{T}_{n,p} : S_{\rm ind}(n, \beta) \to S_{\rm ind}(n, \beta)$ the induced $p^{\rm th}$ Hecke operator on vector valued Maass forms for $\Gamma_0\,(n)$ then*

$$\Pi\big(\mathrm{T}_p u\big) = \mathrm{T}_{n,p}\big(\Pi u\big) \tag{2.12}$$

for all $u \in S(n, \beta)$.

This shows that the operator $\mathrm{T}_{n,p}$ is exactly the ordinary Hecke operator acting on the vector valued version of the Maass forms. In the first chapter of our paper we introduced the space of vector valued period functions with parameter β for $\Gamma_0\,(n)$ as the linear span of all functions $\vec{\phi} = \vec{\phi}(z) = \vec{f}(z-1)$ with $\vec{f}$ an eigenfunction with eigenvalue $\lambda = \pm 1$ of the transfer operator $\mathcal{L}_\beta^{\Gamma_0(n)}$ or equivalently as the linear span of the holomorphic solutions $\vec{\phi} = \vec{\phi}(z)$ of the Lewis equation (1.8) for $\lambda = \pm 1$ with certain asymptotic behavior at $z = 0$ and $z = \infty$.

Mühlenbruch uses in [**Mü04b**] this last characterization of the vector valued period functions for deriving the action of the Hecke operators on these functions for $\Gamma_0\,(n)$. In more detail he characterizes these functions as follows: Let $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ be any modular subgroup with finite index $\mu = [\mathrm{SL}(2,\mathbb{Z}) : \Gamma]$. Denote by FE_Γ^* the space of so called period like functions $\vec{\varphi} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}^\mu$ for Γ which are holomorphic and fulfill the three-term equation

$$\vec{\varphi} = \rho_\Gamma(T^{-1})\, \vec{\varphi}\big|_\beta T + \rho_\Gamma({T'}^{-1})\, \vec{\varphi}\big|_\beta T' \tag{2.13}$$

where T' denotes the matrix $T' = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Thereby $\rho_\Gamma : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}^\mu)$ denotes the right regular representation of $\mathrm{SL}(2,\mathbb{Z})$ induced by the trivial representation of Γ in analogy to (2.4). If such a period like function $\vec{\varphi} = \big(\varphi_i\big)_{1 \leq i \leq \mu_n}$ satisfies the growth conditions

$$\varphi_i(z) = \begin{cases} \mathrm{O}\big(z^{\max\left(0, -2\mathrm{Re}(\beta)\right)}\big) & \text{as } z \to 0 \text{ and} \\ \mathrm{O}\big(z^{\min\left(0, -2\mathrm{Re}(\beta)\right)}\big) & \text{as } z \to \infty \end{cases} \tag{2.14}$$

for all i on the positive real axis then $\vec{\varphi}$ is called a vector valued period function of $\Gamma_0\,(n)$ with parameter β. The space of such functions is denoted by $\mathrm{FE}_\Gamma(\beta)$. For $\Gamma = \Gamma_0\,(n)$ we set $\mathrm{FE}(n,\beta) := \mathrm{FE}_{\Gamma_0(n)}(\beta)$.

Remark 2.5. In [**LZ01**] Lewis and Zagier showed that for $\Gamma_0\,(1) = \mathrm{SL}(2,\mathbb{Z})$ there is a bijection between $S(1,\beta)$ and $\mathrm{FE}(1,\beta)$. It follows from the recent work of Deitmar and Hilgert [**DH04**] that this is true also for $S(n,\beta)$, $S_{\mathrm{ind}}(n,\beta)$ and $\mathrm{FE}(n,\beta)$.

There is obviously a close relation between the solutions of $\vec{\phi}$ of the Lewis equation (1.8) derived from the transfer operator and solutions $\vec{\varphi}$ of the three-term equation (2.13). A simple calculation [**HMM03**] shows that any solution $\vec{\phi}$ of equation (1.8) for $\lambda = \pm 1$ fulfills the equation $\lambda\, \chi_{\overline{\Gamma_0(n)}}(T^{-1}MT)\, \vec{\phi}\big|_\beta M = \vec{\phi}$ or written explicitly

$$\lambda\, z^{-2\beta}\, \chi_{\overline{\Gamma_0(n)}}(T^{-1}MT)\, \vec{\phi}\left(\frac{1}{z}\right) = \vec{\phi}(z). \tag{2.15}$$

Hence any such $\vec{\phi}$ fulfills also the equation

$$\vec{\phi} = \chi_{\overline{\Gamma_0(n)}}(T^{-1})\, \vec{\phi}\big|_\beta T + \chi_{\overline{\Gamma_0(n)}}(T^{-1}{T'}^{-1}T)\, \vec{\phi}\big|_\beta T'$$

where we used the identity $T^{-1}M = M{T'}^{-1}$. Hence the function

$$\vec{\phi}' = \vec{\phi}'(z) = \chi_{\overline{\Gamma_0(n)}}(T)\, \vec{\phi}(z) \tag{2.16}$$

fulfills the two equations

$$\vec{\phi}' = \chi_{\overline{\Gamma_0(n)}}(T^{-1})\, \vec{\phi}'\big|_\beta T + \chi_{\overline{\Gamma_0(n)}}({T'}^{-1})\, \vec{\phi}'\big|_\beta T' \tag{2.17}$$

and

$$\vec{\phi}' = \lambda\, \chi_{\overline{\Gamma_0(n)}}(M)\, \vec{\phi}'\big|_\beta M. \tag{2.18}$$

On the other hand for any given solution $\vec{\phi}'$ of equation (2.17) the function

$$\chi_{\overline{\Gamma_0(n)}}(M)\, \vec{\phi}'\big|_\beta M$$

fulfills this equation and hence ϕ' can be written as

$$\vec{\phi}' = \frac{1}{2}\left(\vec{\phi}' + \chi_{\overline{\Gamma_0(n)}}(M)\, \vec{\phi}'\big|_\beta M\right) + \frac{1}{2}\left(\vec{\phi}' - \chi_{\overline{\Gamma_0(n)}}(M)\, \vec{\phi}'\big|_\beta M\right).$$

But $\vec{\phi}_\pm := \vec{\phi}' \pm \chi_{\overline{\Gamma_0(n)}}(M)\, \vec{\phi}'\big|_\beta M$ then fulfills also equation (2.17) and furthermore one finds

$$\chi_{\overline{\Gamma_0(n)}}(M)\, \vec{\phi}_\pm\big|_\beta M = \chi_{\overline{\Gamma_0(n)}}(M)\, \vec{\phi}'\big|_\beta M \pm \vec{\phi}' = \pm\vec{\phi}_\pm.$$

Hence any solution $\vec{\phi}'$ of equation (2.17) is a linear combination of solutions of the equation

$$\vec{\phi}' = \chi_{\overline{\Gamma_0(n)}}(T^{-1})\, \vec{\phi}'\big|_\beta T + \lambda \chi_{\overline{\Gamma_0(n)}}({T'}^{-1}M)\, \vec{\phi}'\big|_\beta MT'. \tag{2.19}$$

But with (2.16) equation (2.19) is equivalent to the equation

$$\vec{\phi} = \chi_{\overline{\Gamma_0(n)}}(T^{-1})\, \vec{\phi}\big|_\beta T + \lambda\, \chi_{\overline{\Gamma_0(n)}}(T^{-1}{T'}^{-1}MT)\, \vec{\phi}\big|_\beta MT'. \tag{2.20}$$

But $T^{-1}{T'}^{-1}MT = T^{-1}M$ and hence any solution of (2.17) is a linear combination of solutions of (1.8). The space of solutions of this equation therefore coincides with the span of solutions of equation (1.8).

Equation (2.17) almost coincides with the three-term equation (2.13) where the right regular representation $\chi_{\overline{\Gamma_0(n)}}$ of $\mathrm{GL}(2,\mathbb{Z})$ is replaced by the representation $\rho_{\Gamma_0(n)}$ of $\mathrm{SL}(2,\mathbb{Z})$. But the restriction of $\chi_{\overline{\Gamma_0(n)}}$ to the subgroup $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{GL}(2,\mathbb{Z})$ obviously is equivalent to $\rho_{\Gamma_0(n)}$ where equivalence is induced by the use of different but unitarily equivalent index sets I_n. Hence $\chi_{\overline{\Gamma_0(n)}}(g) = U\, \rho_{\Gamma_0(n)}(g)\, U^{-1}$ for all $g \in \mathrm{SL}(2,\mathbb{Z})$ where U is some permutation matrix. Therefore (1.8) implies

$$\vec{\phi}' = U\rho_{\Gamma_0(n)}(T^{-1})U^{-1}\, \vec{\phi}'\big|_\beta T + U\rho_{\Gamma_0(n)}({T'}^{-1})U^{-1}\, \vec{\phi}'\big|_\beta T'.$$

Defining $\vec{\phi} := U^{-1}\, \vec{\phi}'$ we see that equation (2.17) is equivalent to the equation

$$\vec{\phi} = \rho_{\Gamma_0(n)}(T^{-1})\, \vec{\phi}\big|_\beta T + \rho_{\Gamma_0(n)}({T'}^{-1})\, \vec{\phi}\big|_\beta T' \tag{2.21}$$

which indeed coincides with equation (2.13). We have therefore shown that the space of solutions of equation (2.13) and the one of equation (1.8) can be related to each other by a permutation matrix and hence these solutions coincide up to a permutation of their components.

In a next step Mühlenbruch constructs vector valued period functions for any modular subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ with finite index μ. The method he uses is a straightforward extension of one applied by Lewis and Zagier in the case $\mathrm{SL}(2,\mathbb{Z})$ in [**LZ01**] and by Martin in his work on modular forms for subgroups of $\mathrm{SL}(2,\mathbb{Z})$ in [**Mr01**].

Consider namely the function $R_\zeta(z)$ with

$$R_\zeta(z) = \frac{y}{(x-\zeta)^2 + y^2} \tag{2.22}$$

which is well defined for $z \in \mathbb{H}$, and $\zeta \neq z$ and the 1-form

$$\eta(u,v) := (v\partial_y u - u\partial_y v)dx + (u\partial_x v - v\partial_x u)dy \tag{2.23}$$

on $\mathbb{H}$ with $u, v : \mathbb{H} \to \mathbb{C}$ smooth functions. The function R_ζ satisfies $R_{g\zeta}(gz) = (c\zeta + d)^2 R_\zeta(z)$ for all $g = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ and $\Delta R_\zeta^\beta = \beta(1-\beta) R_\zeta^\beta$.

The following Lemma states some properties of the 1-form η [**LZ01**].

Lemma 2.6. *For u, v eigenfunctions of the hyperbolic Laplacian Δ on $\mathbb{H}$ with identical eigenvalue the 1-form $\eta(u,v)$ is closed and transforms under any holomorphic change h of variables as $\eta(u \circ h, v \circ h) = \eta(u,v) \circ h$.*

For $\vec{u} \in S_{\mathrm{ind}}(n,\beta)$ one considers then the integral transform $\mathrm{P}\vec{u}$ with

$$\left(\mathrm{P}\vec{u}\right)_i(\zeta) = \int_0^{i\infty} \eta(u_i, R_\zeta^\beta). \tag{2.24}$$

Since R_ζ^β and u_i are eigenfunctions of Δ on $\mathbb{H}$ with the same eigenvalue $\beta(1-\beta)$ the 1-form is closed and $\mathrm{P}\vec{u}$ does not depend on the path $L_{0,\infty}$ of integration from 0 to $i\infty$. Using the transformation properties both of $\vec{u} \in S_{\mathrm{ind}}(n,\beta)$ and of R_ζ under the action of $g \in \mathrm{SL}(2,\mathbb{Z})$ one finds [**Mü04b**]

Lemma 2.7. *For $\zeta > 0$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z}) \cap \mathrm{Mat}_1^+(2,\mathbb{Z})$ the function $\mathrm{P}\vec{u}$ with $\vec{u} \in S_{\mathrm{ind}}(n,\beta)$ fulfills the relation*

$$(c\zeta + d)^{-2\beta} \rho_{\Gamma_0(n)}(\gamma^{-1}) \, \mathrm{P}\vec{u}(\gamma\zeta) = \int_{L_{\gamma^{-1}0,\gamma^{-1}\infty}} \eta(\vec{u}, R_\zeta^\beta) \tag{2.25}$$

and the growth condition

$$\left(\mathrm{P}\vec{u}\right)_i(\zeta) = \begin{cases} \mathrm{O}\left(\zeta^{\max(0,-2\mathrm{Re}(s))}\right) & \text{as } \zeta \downarrow 0 \text{and} \\ \mathrm{O}\left(\zeta^{\min(0,-2\mathrm{Re}(s))}\right) & \text{as } \zeta \to \infty. \end{cases} \tag{2.26}$$

where $L_{\gamma^{-1}0,\gamma^{-1}\infty}$ denotes the geodesic line connecting $\gamma^{-1}0$ and $\gamma^{-1}\infty$.

$\mathrm{Mat}_1^+(2,\mathbb{Z})$ denotes the set of 2×2 matrices with nonnegative entries and determinant 1. From this it is not difficult to derive [**Mü04b**]

Proposition 2.8. *For $\vec{u} \in S_{\mathrm{ind}}(n,\beta)$ the function $\mathrm{P}\vec{u}(\zeta)$ is a vector valued period function for $\Gamma_0\,(0)$.*

Proof. Obviously the geodesic $L_{0,\infty}$ connecting 0 and ∞ is homotopic to the geodesic lines $L_{T'^{-1}0,T'^{-1}\infty} \cup L_{T^{-1}0,T^{-1}\infty}$ where both the matrices T and T' have only nonnegative entries. By Lemma 2.7 we then get for $\zeta > 0$

$$\begin{aligned} \mathrm{P}\vec{u}(\zeta) &= \int_{L_{0,\infty}} \eta(\vec{u}, R_\zeta^\beta) \\ &= \int_{L_{T^{-1}0,T^{-1}\infty}} \eta(\vec{u}, R_\zeta^\beta) + \int_{L_{T'^{-1}0,T'^{-1}\infty}} \eta(\vec{u}, R_\zeta^\beta) \\ &= \rho_{\Gamma_0(n)}(T^{-1}) \, \mathrm{P}\vec{u}(Tz) + \rho_{\Gamma_0(n)}({T'}^{-1}) \, (\zeta+1)^{-2s} \, \mathrm{P}\vec{u}(T'z). \end{aligned} \tag{2.27}$$

Together with the growth condition in Lemma 2.7 this and a bootstrap argument similar as the one in [**LZ01**] for extending the function to the cut z-plane shows that $\mathrm{P}\vec{u}$ is a period function for $\Gamma_0\,(n)$. ∎

REMARK 2.9. One can show that the map $\mathrm{P} : S_{\mathrm{ind}}(n, \beta) \to \mathrm{FE}(n, \beta)$ is injective and hence also surjective, since $S_{\mathrm{ind}}(n, \beta)$ and $\mathrm{FE}(n, \beta)$ can be mapped bijectively onto each other [**DH04**].

To determine finally the form of the Hecke operators acting on the vector valued period functions $\vec{\varphi} \in \mathrm{FE}(n, \beta)$ of $\Gamma_0\,(n)$ we follow again [**Mü04b**].

In a first step one defines for any rational $q \in [0, 1)$ an element $M(q) \in \mathcal{R}_1 = \mathbb{Z}[\mathrm{SL}(2, \mathbb{Z})]$ as follows: for q there exists a unique sequence $(y_0, y_1, \ldots, y_{L(q)})$ of rational numbers $y_l = \frac{a_l}{b_l}$, $\gcd(a_l, b_l) = 1$, $b_l \geq 0$ with $y_0 = \frac{-1}{0} = -\infty$, $y_1 = \frac{0}{1} = 0$, $y_{L(q)} = q$, $0 < y_l < 1$ for $l = 2, \ldots, L(q)$ such that $y_0 < y_1 < \ldots < y_{L(q)}$ and $\det \begin{pmatrix} a_{l-1} & a_l \\ b_{l-1} & b_l \end{pmatrix} = -1$ for $l = 1, \ldots, L(q)$.

The element $M(q) \in \mathcal{R}_1$ is then defined as

$$M(q) = \sum_{l=1}^{L(q)} m_l \tag{2.28}$$

with $m_l = \begin{pmatrix} b_l & -a_l \\ b_{l-1} & -a_{l-1} \end{pmatrix}$ for $l = 1, \ldots, L(q)$. Obviously one has $m_l \in \mathrm{SL}(2, \mathbb{Z})$ for all $l = 1, \ldots, L(q)$ and $m_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

An easy calculation [**Mü04b**] shows that the action of the operator K as defined in (1.25) when acting on any $A \in X_m$ can be expressed through $M(q)$ where q is given by $q = A0$: if $M(q) = \sum_{l=1}^{L(q)} m_l$ then

$$K^l(A) = m_{l+1}\, A \qquad \text{for } l = 0, 1, \ldots, L(q) - 1. \tag{2.29}$$

We have in particular that $k_A = L(A0) - 1$.

Consider first the case of period functions $\varphi \in \mathrm{FE}(1, \beta)$ for $\mathrm{SL}(2, \mathbb{Z})$. Then one has [**Mü04b**]:

LEMMA 2.10. *For $u \in S(1, \beta)$ a Maass form for $\mathrm{SL}(2, \mathbb{Z})$ and $\varphi = \mathrm{P}u$ its period function the following identity holds for any $A \in X_m$:*

$$\int_{L_{0,\infty}} \eta\big(u\big|_0 A, R_\zeta^\beta\big) = \varphi\big|_\beta \big(M(A0)A\big)(\zeta). \tag{2.30}$$

Using relation (2.29) one can rewrite identity (2.30) as follows

$$\int_{L_{0,\infty}} \eta\big(u\big|_0 A, R_\zeta^\beta\big) = \sum_{l=0}^{L(A0)-1} \big(\varphi\big|_\beta K^l(A)\big)(\zeta). \tag{2.31}$$

Since the m^{th} Hecke operator $\mathrm{T}_m : S(1, \beta) \to S(1, \beta)$ is given as

$$\mathrm{T}_m u = \sum_{A \in X_m} u\big|_0 A$$

a straightforward application of relation (2.31) then gives for the Hecke operator $\tilde{\mathrm{H}}_m : \mathrm{FE}(1, \beta) \to \mathrm{FE}(1, \beta)$ induced from $\mathrm{T}_m : S(1, \beta) \to S(1, \beta)$:

$$\left(\tilde{\mathrm{H}}_m \varphi\right)(\zeta) = \int_{L_{0,\infty}} \eta(\mathrm{H}_m u, R_\zeta^\beta) = \sum_{A \in X_m} \sum_{l=0}^{L(A0)-1} \left(\varphi\big|_\beta K^l(A)\right)(\zeta). \tag{2.32}$$

But the set of matrices $\{K^l(A) : A \in X_m, l = 0, \dots, L(A0) - 1\}$ coincides with the set S_m in (1.21) and hence the operator $\tilde{\mathrm{H}}_m$ can be written also as

$$\tilde{\mathrm{H}}_m \varphi(\zeta) = \sum_{A \in S_m} \big(\varphi\big|_\beta A\big)(\zeta). \tag{2.33}$$

This shows that for m prime the operators $\tilde{\mathrm{T}}_{1,m}$ in [**HMM03**] and $\tilde{\mathrm{H}}_m$ in [**Mü04b**] coincide.

The main step in [**Mü04b**] for deriving the Hecke operators acting on the vector valued period functions $\vec{\varphi} \in \mathrm{FE}(n,\beta)$ of the groups $\Gamma_0\,(n)$ is the following Lemma:

Lemma 2.11. *Let $g_1, \dots, g_\mu$ be representatives of the right cosets of $\Gamma_0\,(n)$ in $\mathrm{SL}(2,\mathbb{Z})$ and let $\mathrm{P}\vec{u}$ be the vector valued period function of $\vec{u} \in S_{\mathrm{ind}}(n,\beta)$. For $A \in X_m$ set*

$$M\big(\sigma_{g_j}(A)0\big) = \sum_{l=1}^{L_{A,j}} m_l^{(A,j)} \in \mathcal{R}_1 \text{ for all } j = 1, \dots, \mu \tag{2.34}$$

with $L_{A,j} = L\big(\sigma_{g_j}(A)0\big)$ as in (2.28). The following identity holds for $1 \le j \le \mu$:

$$\begin{aligned} &\int_{L_{0,\infty}} \eta\big(u_{\Phi_A(j)}\big|_0 \sigma_{g_j}(A), R_\zeta^\beta\big) \\ &\qquad = \sum_{l=1}^{L_{A,j}} \Big[\rho_{\Gamma_0(n)}((m_l^{(A,j)})^{-1})\, P\vec{u}\Big]_{\Phi_A(j)} \Big|_\beta \Big(m_l^{(A,j)}\, \sigma_{g_j}(A)\Big)(\zeta). \end{aligned} \tag{2.35}$$

Since any $\vec{u} \in S_{\mathrm{ind}}(n,\beta)$ is given by $\vec{u} = \Pi(u)$ with $u \in S(n,\beta)$ in (2.5) and, according to relation (2.10), $A\, g_i \in \Gamma_0\,(n)\, g_{\Phi_A(j)}\, \sigma_{g_j}(A)$ for all $A \in X_m$ one has for $u \in S(n,\beta)$

$$u\big|_0 A g_j = u\big|_0 g_{\Phi_A(j)} \sigma_{g_j}(A) \tag{2.36}$$

and hence identity (2.30) can be rewritten as

$$\begin{aligned} &\int_{L_{0,\infty}} \eta\left(\Pi\big(u\big|_0 A\big), R_\zeta^\beta\right) \\ &\qquad = \sum_{l=1}^{L_{A,j}} \Big[\rho_{\Gamma_0(n)}((m_l^{(A,j)})^{-1})\, \mathrm{P}\Pi(u)\Big]_{\Phi_A(j)} \big|_\beta \big(m_l^{(A,j)}\, \sigma_{g_j}(A)\big)(\zeta). \end{aligned}$$

where $L_{A,j}$ and $m_l^{(A,j)}$ depend on A according to (2.34). From this an explicit representation for the Hecke operator $\tilde{\mathrm{H}}_{n,p}$ on $\mathrm{FE}(n,\beta)$ can be derived. For p prime $\tilde{\mathrm{H}}_{n,p}$ has the form as given by

Proposition 2.12. *For p prime, $g_1, \dots, g_\mu$ representatives of the right cosets of $\Gamma_0\,(n)$ in $\mathrm{SL}(2,\mathbb{Z})$ and $\vec{\varphi} \in \mathrm{FE}(n,\beta)$ the Hecke operator $\tilde{\mathrm{H}}_{n,p}$ has the form*

$$\left(\tilde{\mathrm{H}}_{n,p}\vec{\varphi}\right)_j = \sum_{A \in \mathcal{A}_p} \sum_{l=1}^{L_{A,j}} \Big[\rho_{\Gamma_0(n)}\left(\left(m_l^{(A,j)}\right)^{-1}\right)\, \vec{\varphi}\Big]_{\Phi_A(j)} \big|_\beta m_l^{(A,j)}\, \sigma_{g_j}(A) \tag{2.37}$$

with $M\big(\sigma_{g_j}(A)0\big) = \sum_{l=1}^{L_{A,j}} m_l^{(A,j)} \in \mathcal{R}_1$.

3. The operators $\tilde{\mathrm{T}}_{n,m}$ and $\tilde{\mathrm{H}}_{n,m}$

Before discussing the exact relation between the operators $\tilde{\mathrm{H}}_{n,m}$ in (2.37) and the operators $\tilde{\mathrm{T}}_{n,m}$ in (1.31) we want to give another derivation of the special solutions in Theorem 1.7. For this consider any Maass form $u \in S(n,\beta)$ of $\Gamma_0\,(n)$. Since $\Gamma_0\,(n) \supset \Gamma_0\,(nm)$ for all $m \in \mathbb{N}$ one obviously has $u \in S(mn,\beta)$. This u determines a vector valued form $\vec{u} = \Pi_{nm}(u) \in S_{\mathrm{ind}}(nm,\beta)$ given by

$$u_j = (\Pi_{nm} u)_j = u\big|_0 g_j, \qquad j = 1,\dots,\mu_{nm}$$

if $\Gamma_0\,(nm)\backslash\mathrm{SL}(2,\mathbb{Z}) = \{\Gamma_0\,(nm) g_j,\, j = 1,\dots,\mu_{nm}\}$. Here μ_{nm} denotes the index of $[\mathrm{SL}(2,\mathbb{Z}) : \Gamma_0\,(nm)]$.

The vector valued period function $\vec{\varphi}$ corresponding to this "old form" $\Pi_{nm} u \in S_{\mathrm{ind}}(nm,\beta)$ then has the form

$$\big(\vec{\varphi}\big)_j = \int_{L_{0,\infty}} \eta\big(u\big|_0 g_j, R_\zeta^\beta\big) = \big(\mathrm{P_{nm}}\vec{u}\big)_j$$

for all $j = 1,\dots,\mu_{nm}$. If h_i, $1 \le i \le \mu_n$, are representatives of the cosets of $\Gamma_0\,(n)$ in $\mathrm{SL}(2,\mathbb{Z})$ and $\tilde{h}_k$, $k = 1,\dots,\frac{\mu_{nm}}{\mu_n}$, are representatives of the right cosets in $\Gamma_0\,(nm)\backslash\Gamma_0\,(n)$ then $g_{k,i} := \tilde{h}_k\, h_i$, $1 \le i \le \mu_n$, $1 \le k \le \frac{\mu_{nm}}{\mu_n}$ are obviously representatives of the rest classes in $\Gamma_0\,(nm)\backslash\mathrm{SL}(2,\mathbb{Z})$. Identifying the index j with the tuple (k,i) we therefore get

(3.1)
$$\big(\vec{\varphi}\big)_j(\zeta) = \big(\vec{\varphi}\big)_{(k,i)}(\zeta) = \int_{L_{0,\infty}} \eta\big(u\big|_0 \tilde{h}_k h_i, R_\zeta^\beta\big) = \int_{L_{0,\infty}} \eta\big(u\big|_0 h_i, R_\zeta^\beta\big) = \big(\vec{\tilde{\varphi}}\big)_i(\zeta)$$

where $\vec{\tilde{\varphi}}$ is the vector valued period function of $\vec{\tilde{u}} = \Pi_n u \in S_{\mathrm{ind}}(n,\beta)$.

But $i = \sigma_{m,n}(j)$ where we identify the index sets I_ν and $\{1,\dots,\mu_\nu\}$ of the right cosets in $\Gamma_0\,(\nu)\backslash\mathrm{SL}(2,\mathbb{Z})$ for all $\nu \in \mathbb{N}$. Hence we have

$$\big(\vec{\varphi}\big)_j(\zeta) = \big(\mathrm{P_{nm}}\vec{u}\big)_j(\zeta) = \big(\mathrm{P_n}(\vec{\tilde{u}})\big)_{\sigma_{m,n}(j)}(\zeta) = \big(\vec{\tilde{\varphi}}\big)_i(\zeta). \tag{3.2}$$

This result coincides obviously with the second part of Proposition 1.10.

On the other hand consider a Maass form $u \in S(nm,\beta)$ with $u(z) = v(mz)$ and $v \in S(n,\beta) \subset S(nm,\beta)$. Such an u is called an old Maass form for $\Gamma_0\,(nm)$. Then one finds for the corresponding vector valued Maass form $\vec{u} \in S_{\mathrm{ind}}(nm,\beta)$

$$\big(\vec{u}\big)_j = \big(\Pi_{nm} u\big)_j = v\big|_0 A_m\, g_j, \qquad j = 1,\dots,\mu_{nm}$$

where $\{g_1,\dots,g_{\mu_{nm}}\}$ are representatives of $\Gamma_0\,(nm)\backslash\mathrm{SL}(2,\mathbb{Z})$ and $A_m = \left(\begin{smallmatrix} m & 0 \\ 0 & 1\end{smallmatrix}\right)$.

To determine the corresponding vector valued period function one has to apply the integral transform $\mathrm{P} = \mathrm{P}_{nm}$ defined in (2.24):

$$\big(\mathrm{P}_{nm}\vec{u}\big)_j(\zeta) = \int_{L_{0,\infty}} \eta\big(v\big|_0 A_m\, g_j, R_\zeta^\beta\big), \qquad j = 1,\dots,\mu_{nm}.$$

But $v\big|_0 A_m\, g_j = v\big|_0 g_{\Phi_{A_m}(j)}\, \sigma_{g_j}(A_m)$ and hence

$$\big(\mathrm{P}_{nm}\vec{u}\big)_j(\zeta) = \int_{L_{0,\infty}} \eta\big(v\big|_0 g_{\Phi_{A_m}(j)}\, \sigma_{g_j}(A_m), R_\zeta^\beta\big), \qquad j = 1,\dots,\mu_{nm}. \tag{3.3}$$

Therefore we get for $\big(\mathrm{P}_{nm}\vec{u}\big)_j$

$$\big(\mathrm{P}_{nm}\vec{u}\big)_j(\zeta) = \int_{L_{0,\infty}} \eta\big(v\big|_0 g_{l(j)}\, \sigma_{g_j}(A_m), R_\zeta^\beta\big), \qquad j = 1,\dots,\mu_{nm}, \tag{3.4}$$

where $l(j) := \phi_{A_m}(j)$. Applying now identity (2.35) of Lemma 2.11 we get
(3.5)

$$\big(\mathrm{P}_{nm}\vec{u}\big)_j(\zeta) = \sum_{r=1}^{L_{A_m,j}} \Big[\rho_{\Gamma_0(nm)}((m_r^{(A_m,j)})^{-1})\,\mathrm{P}_{nm}\vec{v}\Big]_{l(j)}\,\big|_\beta\Big(m_r^{(A_m,j)}\,\sigma_{g_j}(A_m)\Big)(\zeta),$$

$j = 1,\ldots,\mu_{nm}$, where $M(\sigma_{g_j}(A_m)0) = \sum_{r=1}^{L_{A_m,j}} m_r^{(A_m,j)}$. But

$$\Big[\rho_{\Gamma_0(nm)}(m_r^{(A_m,j)})^{-1}\,\mathrm{P}_{nm}\vec{v}\Big]_{l(j)} = \big(P_{nm}\vec{v}\big)_{l(j,r)}$$

for a certain l depending on j and r and

$$m_r^{(A_m,j)}\,\sigma_{g_j}(A_m) = K^{r-1}\,\sigma_{g_j}(A_m)$$

and hence

$$\begin{aligned}\big(\mathrm{P}_{nm}\vec{u}\big)_j(\zeta) &= \sum_{r=1}^{L_{A_m,j}} \big(\mathrm{P}_{nm}\vec{v}\big)_{l(j,r)}\big|_\beta K^{r-1}\,\sigma_{g_j}(A_m) \\ &= \sum_{r=0}^{L_{A_m,j}-1} \big(\mathrm{P}_{nm}\vec{v}\big)_{l(j,r+1)}\big|_\beta K^{r}\,\sigma_{g_j}(A_m).\end{aligned} \tag{3.6}$$

But according to relation (3.2) this can be written as

$$\big(\mathrm{P}_{nm}\vec{u}\big)_j(\zeta) = \sum_{r=0}^{L_{A_m,j}-1} \big(\mathrm{P}_{n}\vec{\tilde{v}}\big)_{\sigma_{m,n}\left(l(j,r+1)\right)}\big|_\beta K^{r}\,\sigma_{g_j}(A_m). \tag{3.7}$$

This calculation indicates that the vector valued period function $\vec{\Phi} = \big(\Phi_j(z)\big)$, $j \in I_{nm}$, of Theorem 1.7 is exactly the vector valued period function corresponding to the vector valued Maass form $\vec{u} \in S_{\mathrm{ind}}(nm,\beta)$ with $\vec{u} = \Pi_{nm}u$ and $u \in S(nm,\beta)$ the old Maass form for $\Gamma_0\,(nm)$ given by some $v \in S(n,\beta)$ such that

$$u(z) = v(mz) = v\big|_0 A_m.$$

One can prove this statement using Lemma 3.3 and Lemma 3.12 to show that $l(j,r)$ and $l_{j,r}$ coincide up to a certain conjugation and a modified version of the proof of Proposition 3.10 for the diagonal matrix A_m.

The construction of the special solution $\vec{\Phi} = (\Phi_j(z))$, $j \in I_{nm}$, of the Lewis equation (1.8) for $\Gamma_0\,(nm)$ relies strongly on the special parameterization of the rest classes $\overline{\Gamma_0\,(n)}\backslash \mathrm{GL}(2,\mathbb{Z})$ given by the set I_n in (1.15). For the following an explicit identification of I_n with $\Gamma_0\,(n)\backslash\mathrm{SL}(2,\mathbb{Z}) \equiv \overline{\Gamma_0\,(n)}\backslash\mathrm{GL}(2,\mathbb{Z})$ is necessary. We take matrices $\alpha_i \in \mathrm{SL}(2,\mathbb{Z})$, $i \in I_n$ satisfying

$$[0:1]_n\,\alpha_i = i. \tag{3.8}$$

Indeed one finds

Lemma 3.1. *The set $R_n = \{\alpha_i : i \in I_n\}$ of matrices satisfying (3.8) for all $i \in I_n$ is a set of representatives of the rest classes of $\Gamma_0\,(n)$ in* $\mathrm{SL}(2,\mathbb{Z})$.

Proof. The matrices α_i are defined up to an element $\gamma \in \Gamma_0\,(n)$ since we have $[0:1]_n\gamma\alpha_i = [0:1]_n\alpha_i$ and hence $\Gamma_0\,(n)\alpha_i \neq \Gamma_0\,(n)\alpha_j$ for $i \neq j$.

On the other hand take any $i \in I_n$. Since $i = [c:d]_n$ with $\gcd(c,d) = 1$ there exists $a,b \in \mathbb{Z}$ with $ad-bc = 1$ and hence $[0:1]_n\alpha_i = [c:d]_n = i$ if $\alpha_i = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. $\square$

We can therefore interpret the map $\pi_n : \Gamma_0\,(n)\backslash \mathrm{SL}(2,\mathbb{Z}) \to I_n$ similar to (1.14) also as a map $\pi_n : R_n \to I_n$. On the other hand I_n can be identified via the map $\omega_n : I_n \to X_n^\star$ with the set $X_n^\star$ of upper triangular matrices with $\omega_n(i) = A_i = \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix}$ as defined in (1.20) for $i = [c:d]_n$. Combining this map ω_n with the map π_n then induces a bijection $R_n \to X_n^\star$ with $\alpha_i \mapsto \omega_n(\pi_n(\alpha_i))$. There is yet another bijection $\kappa_n : R_n \to X_n^\star$ given by

$$\kappa_n(\alpha_{[c:d]_n}) = \begin{pmatrix} c & b \\ 0 & n/c \end{pmatrix}. \tag{3.9}$$

with $b \equiv d \mod \frac{n}{c}$, such that

$$\begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} \alpha_{[c:d]_n} \in \mathrm{SL}(2,\mathbb{Z})\, \kappa_n(\alpha_{[c:d]_n}) \tag{3.10}$$

holds. Indeed, for $\alpha_{[c:d]_n} = \begin{pmatrix} x & y \\ c & d \end{pmatrix}$ the matrix $g = \begin{pmatrix} 1 & k \\ -\frac{nx}{c} & xb-yc \end{pmatrix}$ with k satisfying $d = b + k\frac{n}{c}$ is integer valued and one finds

$$g \begin{pmatrix} c & b \\ 0 & n/c \end{pmatrix} = \begin{pmatrix} c & d \\ -nx & -ny \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} \alpha_{[c:d]_n}. \tag{3.11}$$

Since $\det g = \dfrac{\det \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} \det \alpha_{[c:d]_n}}{\det \kappa_n(\alpha_{[c:d]_n})} = 1$ we get $g \in \mathrm{SL}(2,\mathbb{Z})$.

Combining the maps π_n, κ_n and ω_n we arrive at a bijective map $h_n : I_n \to I_n$ with

$$h_n := \omega_n^{-1} \circ \kappa_n \circ \pi_n^{-1}. \tag{3.12}$$

Then one finds for all $i \in I_n$ that

$$\kappa_n(\alpha_i) = A_{h_n(i)}. \tag{3.13}$$

There is an important relation between the matrices $\alpha_i \in \mathrm{SL}(2,\mathbb{Z})$ and the matrices $A_i \in X_n^\star$ in (1.20).

Lemma 3.2. *For $i, j \in I_n$ and $g \in \mathrm{SL}(2,\mathbb{Z})$ we have*

$$\alpha_i\, g\, \alpha_j^{-1} \in \Gamma_0\,(n) \qquad \text{iff} \qquad A_{h_n(i)}\, g\, A_{h_n(j)}^{-1} \in \mathrm{SL}(2,\mathbb{Z}).$$

Proof. For $\alpha_i\, g \alpha_j^{-1} \in \Gamma_0\,(n)$ conjugating by the matrix $\begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix}$ leads by using relations (3.10) and (3.13) to

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} \alpha_i\, g\, \alpha_j^{-1} \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix}^{-1} &= g'\, A_{h_n(i)}\, g \left(g''\, A_{h_n(j)}\right)^{-1} \\ &\in \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} \Gamma_0\,(n) \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix}^{-1} \end{aligned} \tag{3.14}$$

for some $g', g'' \in \mathrm{SL}(2,\mathbb{Z})$. But $\begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} \Gamma_0\,(n) \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix}^{-1} = \Gamma_0\,(n)$ and hence

$$A_{h_n(i)}\, g\, A_{h_n(j)}^{-1} \in g'^{-1}\, \Gamma_0\,(n)\, g'' \subset \mathrm{SL}(2,\mathbb{Z}).$$

On the other hand assume that for $i, j \in I_n$ one has $A_{h_n(i)}\, g\, A_{h_n(j)}^{-1} \in \mathrm{SL}(2,\mathbb{Z})$. For $i \in I_n$ there exists an index $j' \in I_n$ such that

$$\alpha_i\, g\, \alpha_{j'}^{-1} \in \Gamma_0\,(n)$$

since $\mathrm{SL}(2,\mathbb{Z}) = \bigcup_{\iota\in I_n} \Gamma_0\,(n)\alpha_\iota$. But then $A_{h_n(i)}\, g\, A^{-1}_{h_n(j')} \in \mathrm{SL}(2,\mathbb{Z})$. Lemma 2.3 implies that there exists exactly one upper triangular matrix $\sigma_g(A_{h_n(i)})$ such that

$$A_{h_n(i)}\, g \left[\sigma_g(A_{h_n(i)})\right]^{-1} \in \mathrm{SL}(2,\mathbb{Z})$$

and hence $h_n(j) = h_n(j')$. $\square$

This allows us to introduce another representation $\tilde{\rho}_n$ of the group $\mathrm{SL}(2,\mathbb{Z})$ in analogy to the right regular representation of $\rho_{\Gamma_0(n)}$ induced from the trivial representation of $\Gamma_0\,(n)$ as given in (2.4):

$$\tilde{\rho}_n(g) := \left(\delta_{\mathrm{SL}(2,\mathbb{Z})}(A_i\, g A_j^{-1})\right)_{i,j\in I_n}. \tag{3.15}$$

Then $\tilde{\rho}_n(g\,g') = \tilde{\rho}_n(g)\,\tilde{\rho}_n(g')$ for all $g, g' \in \mathrm{SL}(2,\mathbb{Z})$. Lemma 3.2 shows that

$$\delta_{\Gamma_0(n)}(\alpha_i\, g\, \alpha_j^{-1}) = \delta_{\mathrm{SL}(2,\mathbb{Z})}(A_{h_n(i)}\, g\, A^{-1}_{h_n(j)}).$$

Hence, if H_n denotes the $\mu_n \times \mu_n$ matrix with entries

$$\left(H_n\right)_{i,j} = \begin{cases} 1 & \text{if } h_n(j) = i \text{ and} \\ 0 & \text{otherwise} \end{cases} \tag{3.16}$$

one immediately shows that

$$\rho_{\Gamma_0(n)}(g) = H_n^{-1}\,\tilde{\rho}_n(g)\,H_n \tag{3.17}$$

and hence

Lemma 3.3. *The representations $\rho_{\Gamma_0(n)}$ and $\tilde{\rho}_n$ are unitarily equivalent and one has $\left(\rho_{\Gamma_0(n)}(g)\right)_{i,j} = \left(\tilde{\rho}_n(g)\right)_{h_n(i),h_n(j)}$ for $i,j \in I_n$.*

For the following discussion let us recall the definition of the maps $\Phi_A : I_n \to I_n$ and $\sigma_{g_j} : X_m \to X_m$ in (2.9) and (2.10) where $A \in X_m$ and the $\{g_j\}$ are the representatives of $\Gamma_0\,(n)\backslash\mathrm{SL}(2,\mathbb{Z})$ such that

$$A g_j \in \Gamma_0\,(n)\, g_{\Phi_A(j)}\, \sigma_{g_j}(A) \tag{3.18}$$

Combining the two maps leads to a map $\lambda_{n,m} : X_m \times I_n \to I_n \times X_m$ with

$$\lambda_{n,m}(A,j) := \left(\Phi_A(j), \sigma_{g_j}(A)\right). \tag{3.19}$$

On the other hand Lemma 1.6 allows us to introduce the map $\Lambda_{n,m} : I_{nm} \to I_n \times X^\star_m$ through

$$\Lambda_{n,m}(i) := \left(l_{i,0}, A_{\sigma_{n,m}(i)}\right) \tag{3.20}$$

where according to Lemma 1.6 $l_{i,0} \in I_n$ is the unique index such that

$$A_{l_{i,0}}\, A_{\sigma_{n,m}(i)} \in \mathrm{SL}(2,\mathbb{Z})A_i \tag{3.21}$$

where the matrices $A_{l_{i,0}}$, $A_{\sigma_{n,m}(i)}$ and A_i are associated to the different indices as given in (1.20). Obviously the projection of $\Lambda_{n,m}$ onto its second component is surjective.

Then the following Lemma can be derived

Lemma 3.4. *The map $\Lambda_{n,m}$ is injective. It is surjective iff* $\gcd(n,m) = 1$.

PROOF. For $i, j \in I_{nm}$ with $\Lambda_{n,m}(i) = \Lambda_{n,m}(j)$ equation (3.21) implies

$$\mathrm{SL}(2,\mathbb{Z})\, A_i \ni A_{l_{i,0}}\, A_{\sigma_{n,m}(i)} = A_{l_{j,0}}\, A_{\sigma_{n,m}(j)} \in \mathrm{SL}(2,\mathbb{Z})\, A_j.$$

Assume $\gcd(n,m) = 1$. For $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in X_n^\star$ and $\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \in X_m^\star$ put

$$B := \begin{pmatrix} aa' & ab'+bd' \\ 0 & dd' \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \in \mathrm{Mat}_{nm}(2,\mathbb{Z}).$$

If we show $\gcd(aa', ab' + bd', dd') = 1$ which implies the existence of a $k \in \mathbb{Z}$ such that $B = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} A_i$ for a certain $i \in I_{nm}$ then $B \in \mathrm{SL}(2,\mathbb{Z})\, A_i$ for some $i \in I_{nm}$ and $\Lambda_{n,m}$ is surjective. Therefore we assume that

$$\exists p \text{ prime with } p \mid \gcd(aa', ab' + bd', dd'). \tag{3.22}$$

Then $p^2 \mid aa'dd' = nm$. Since $\gcd(n,m) = 1$ we find either $p^2 \mid n$ or $p^2 \mid m$. Say $p^2 \mid n$. This implies that $p \mid a$ and $p \mid d$ since $p \mid aa'$ and $p \mid dd'$. But $p \not| b$ since $\gcd(a,b,d) = 1$. Hence $p \not| bd'$ and the assumption (3.22) does not hold. If $p^2 \mid m$ then a similar argument holds.

It remains the case $\gcd(n,m) > 1$. Then $\Lambda_{n,m}$ is not surjective since $\mu_{nm} < \mu_m \cdot \mu_n$ and so $\#I_{nm} < \#(I_n \times I_m)$. □

Furthermore one has

LEMMA 3.5. *For each $n, m \in \mathbb{N}$ with $\gcd(n,m) = 1$ there exists for each matrix $A \in X_m^\star$ a unique $A' \in X_m^\star$ such that*

$$A \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} A' \tag{3.23}$$

and hence the map $\Lambda : X_m^\star \to X_m^\star$ with $\Lambda(A) = A'$ is a bijection.

PROOF. Let $A \in X_m^\star$ be given by $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Since $\gcd(n,m) = 1$ and hence also $\gcd(n,d) = 1$ due to $m = ad$ there exists an integer $0 \le k < n$ such that $n \mid (b + kd)$. Then

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} na & b+kd \\ 0 & d \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & \frac{b+kd}{n} \\ 0 & d \end{pmatrix}$$

shows that $A' = \begin{pmatrix} a & \frac{b+kd}{n} \\ 0 & d \end{pmatrix}$ fulfills (3.23). Furthermore, since $\gcd(a,b,d) = 1$, also $\gcd(a, \frac{b+kd}{n}, d) = 1$ and the matrix A' belongs indeed to $X_m^\star$ because $0 \le b + kd < d(1+k) \le nd$.

To show that Λ is bijective, assume $\begin{pmatrix} a' & \frac{b'+k'd'}{n} \\ 0 & d' \end{pmatrix} = \begin{pmatrix} a & \frac{b+kd}{n} \\ 0 & d \end{pmatrix}$. Then $a' = a$ and $d' = d$ as well as $b' + k'd' = b + kd$ and hence $b' = b + (k - k')d$. But $0 \le b' < d$ and therefore $k = k'$ and hence $b = b'$. □

REMARK 3.6. Lemma 3.5 does not hold for $\gcd(n,m) > 1$, a counter example being $n = m = 2$ and $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \in X_2^\star$.

Furthermore one needs

LEMMA 3.7. *For n, m with $\gcd(n,m) = 1$ there exists for $i \in I_{nm}$ a matrix $A \in X_m^\star$ such that*

$$\begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} A\alpha_{h_n^{-1}\left(\sigma_{n,m}(i)\right)} \in \mathrm{SL}(2,\mathbb{Z})\, A_i. \tag{3.24}$$

PROOF. For $i \in I_{nm}$ consider $\Lambda_{m,n}(i) = (l_{i,0}, A_{\sigma_{m,n}(i)}) \in I_m \times X_n^\star$ as defined in (3.20) such that

$$A_{l_{i,0}}\, A_{\sigma_{m,n}(i)} \in \mathrm{SL}(2,\mathbb{Z})\, A_i. \tag{3.25}$$

Applying relation (3.10) to $A = A_{\sigma_{m,n}(i)} \in X_n^\star$ there exists $g \in \mathrm{SL}(2,\mathbb{Z})$ with

$$A_{\sigma_{m,n}(i)} = g \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} \kappa_n^{-1}\big(A_{\sigma_{m,n}(i)}\big) = g\, S^{-1} \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right) \kappa_n^{-1}\big(A_{\sigma_{m,n}(i)}\big). \tag{3.26}$$

Hence relation (3.25) can be written as

$$A_{l_{i,0}}\, gS^{-1} \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right) \kappa_n^{-1}\big(A_{\sigma_{m,n}(i)}\big) \in \mathrm{SL}(2,\mathbb{Z})\, A_i. \tag{3.27}$$

Relation (3.18) then gives

$$A_{l_{i,0}}\, gS^{-1} \in \mathrm{SL}(2,\mathbb{Z})\, \sigma_{gS^{-1}}(A_{l_{i,0}})$$

and hence (3.27) can be written as

$$\sigma_{gS^{-1}}(A_{l_{i,0}}) \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right) \kappa_n^{-1}\big(A_{\sigma_{m,n}(i)}\big) \in \mathrm{SL}(2,\mathbb{Z})\, A_i. \tag{3.28}$$

Lemma 3.5 then implies the existence of an $A \in X_m^\star$ such that (3.28) can be written as

$$\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right) A\, \kappa_n^{-1}\big(A_{\sigma_{m,n}(i)}\big) \in \mathrm{SL}(2,\mathbb{Z})\, A_i. \tag{3.29}$$

But according to equation (3.13)

$$\kappa_n^{-1}\big(A_{\sigma_{m,n}(i)}\big) = \alpha_{h_n^{-1}\left(\sigma_{m,n}(i)\right)}$$

and hence acting by S^{-1} from the left on (3.29) one finds

$$\begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} A\, \alpha_{h_n^{-1}\left(\sigma_{m,n}(i)\right)} \in \mathrm{SL}(2,\mathbb{Z})\, A_i.$$

□

LEMMA 3.8. *If* $\gcd(n,m) = 1$ *there exists for* $j \in I_n$ *and* $A \in X_m^\star$ *a unique* $A' \in X_m^\star$ *such that* $\begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} A'\, \alpha_j \in \mathrm{SL}(2,\mathbb{Z})\, A\, A_{h_n(j)}$. *The map* $A \mapsto A'$ *is bijective.*

PROOF. For $A\, A_{h_n(j)}$ there exists $g \in \mathrm{SL}(2,\mathbb{Z})$ such that

$$A\, A_{h_n(j)} = A\, g \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} \alpha_j = A\, g\, S^{-1} \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right) \alpha_j \tag{3.30}$$

since $A_{h_n(j)} = \kappa_n(\alpha_j) = g \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} \alpha_j$ according to relations (3.13) and (3.10). But $A\, gS^{-1} = g'\, \sigma_{gS^{-1}}(A)$ for some $g' \in \mathrm{SL}(2,\mathbb{Z})$ by the definition of the map σ_g in (2.8) and hence

$$A\, A_{h_n(j)} = g'\, \sigma_{gS^{-1}}(A) \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right) \alpha_j. \tag{3.31}$$

Lemma 3.5 shows that

$$A\, A_{h_n(j)} = g'' \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right) A'\, \alpha_j = g''\, S^{-1} \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} A'\, \alpha_j$$

for some matrix $A' \in X_m^\star$ and for some $g'' \in \mathrm{SL}(2,\mathbb{Z})$. This proves the first part of the Lemma.

Since both the maps $A \mapsto \sigma_{gS^{-1}}(A)$ and $\sigma_{gS^{-1}}(A) \mapsto A'$ are bijective also $A \mapsto A'$ is bijective. □

LEMMA 3.9. *If* $\gcd(n,m)=1$ *then for any* $j \in I_n$ *and* $i \in \sigma_{m,n}^{-1}\big(h_n(j)\big)$ *one has*

$$\big(l_{i,0}, A_{\sigma_{n,m}(i)}\big) = \big(h_n\big(\Phi_B(j)\big), \sigma_{\alpha_j}(B)\big)$$

for some $B \in X_m^\star$. *The map* $\sigma_{m,n}^{-1}\big(h_n(j)\big) \to X_m^\star$ *with* $i \mapsto B$ *is bijective.*

PROOF. Fix an $i \in \sigma_{m,n}^{-1}\big(h_n(j)\big) \subset I_{nm}$. Then there exists an $A \in X_m^\star$ with $A\,A_{h_n(j)} \in \mathrm{SL}(2,\mathbb{Z})\,A_i$ since $\sigma_{m,n}(i) = h_n(j)$, see also Lemma 5.12 and Definition 5.13 in [**HMM03**]. Using Lemma 3.8 we then find

$$\mathrm{SL}(2,\mathbb{Z})\,A_i = \mathrm{SL}(2,\mathbb{Z})\,A\,A_{h_n(j)} \ni \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} B\,\alpha_j \quad \text{for some } B \in X_m^\star.$$

But

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} B\,\alpha_j &\in \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} \Gamma_0(n)\,\alpha_{\Phi_B(j)}\,\sigma_{\alpha_j}(B) \\ &= \Gamma_0(n) \begin{pmatrix} 0 & 1 \\ -n & 0 \end{pmatrix} \alpha_{\Phi_B(j)}\,\sigma_{\alpha_j}(B) \\ &\subset \mathrm{SL}(2,\mathbb{Z})\,A_{h_n\left(\Phi_B(j)\right)}\,\sigma_{\alpha_j}(B) \end{aligned}$$

where relations (3.13) and (3.10) imply the last step. Hence

$$A_i = g\,A_{h_n\left(\Phi_B(j)\right)}\,\sigma_{\alpha_j}(B) \quad \text{with } g \in \mathrm{SL}(2,\mathbb{Z})$$

and this decomposition is unique as shown in the proof of Lemma 3.7. Indeed it coincides with the decomposition

$$A_i = g\,A_{l_{i,0}}\,A_{\sigma_{n,m}(i)}$$

given there and since $\Lambda_{n,m}(i) = \big(l_{i,0}, A_{\sigma_{n,m}(i)}\big)$ we find

$$\Lambda_{n,m}(i) = \big(h_n\big(\Phi_B(j)\big), \sigma_{\alpha_j}(B)\big).$$

The map $\sigma_{m,n}^{-1}\big(h_n(j)\big) \ni i \mapsto B \in X_m^\star$ is bijective because the maps involved in the construction of B are bijective. □

This leads to

PROPOSITION 3.10. *If* $\gcd(n,m)=1$ *then for* $j \in I_n$ *the two sets*

$$\big\{(l_{i,0}, A_{\sigma_{n,m}(i)}) : i \in \sigma_{m,n}^{-1}\big(h_n(j)\big)\big\}$$

and

$$\big\{\big(h_n\big(\Phi_A(j)\big), \sigma_{\alpha_j}(A)\big) : A \in X_m^\star\big\}$$

are identical.

From this follows

COROLLARY 3.11. *If* $\gcd(m,n)=1$ *then for each* $j \in I_n$ *we have*

$$\Lambda_{n,m}\big(\sigma_{m,n}^{-1}\big(h_n(j)\big)\big) = \big\{\big(h_n\big(\Phi_A(j)\big), \sigma_{\alpha_j}(A)\big) : A \in X_m^\star\big\}.$$

PROOF OF COROLLARY. Since $\Lambda_{n,m}^{-1}\big(l_{i,0}, A_{\sigma_{n,m}(i)}\big) = i$ the corollary follows from the proposition. □

PROOF OF PROPOSITION 3.10. The first part of Lemma 3.9 shows that the first set is contained in the second set. The second part of this Lemma shows that the two sets are identical. □

To derive our main result on the relation between the “Hecke like” operators in [**HMM03**] and the Hecke operators in [**Mü04b**] let us reformulate Lemma 1.6 in which the existence of an index $l_{i,j} \in I_n$ for any $i \in I_{nm}$ and any $0 \leq j \leq k_{\sigma_{n,m}(i)}$ with

$$A_{l_{i,j}} \big(K^j(A_{\sigma_{n,m}(i)})\big) A_i^{-1} \in \mathrm{SL}(2,\mathbb{Z}) \tag{3.32}$$

is shown. For this consider the matrix sum $M(A_{\sigma_{n,m}(i)}0) = \sum_{j=1}^{L(A_{\sigma_{n,m}(i)}0)} m_j^{(i)}$. But $k_{\sigma_{n,m}(i)} = L(A_{\sigma_{n,m}(i)}0) - 1$ and according to relation (2.29) $K^j A_{\sigma_{n,m}(i)} = m_{j+1}^{(i)}\, A_{\sigma_{n,m}(i)}$. and hence (3.32) can be rewritten as

$$A_{l_{i,j}}\, m_{j+1}^{(i)}\, A_{\sigma_{n,m}(i)}\, A_i^{-1} \in \mathrm{SL}(2,\mathbb{Z}).$$

But for $j=0$ relation (3.32) reads $A_{\sigma_{n,m}(i)}\, A_i^{-1} \in A_{l_{i,0}}^{-1}\, \mathrm{SL}(2,\mathbb{Z})$ and hence

$$A_{l_{i,j}}\, m_{j+1}^{(i)}\, A_{l_{i,0}}^{-1} \in \mathrm{SL}(2,\mathbb{Z})$$

respectively

$$A_{l_{i,0}}\, \big(m_{j+1}^{(i)}\big)^{-1}\, A_{l_{i,j}}^{-1} \in \mathrm{SL}(2,\mathbb{Z})$$

Using the definition of the presentation $\tilde{\rho}_n$ in (3.15) this then shows :

LEMMA 3.12. *For $i \in I_{nm}$ and $M(A_{\sigma_{n,m}(i)}0) = \sum_{r=1}^{L_i} m_r^{(i)}$ one has*

$$\Big[\tilde{\rho}_n\big((m_r^{(i)})^{-1}\big)\Big]_{l_{i,0},l_{i,j}} = \delta_{r,j+1} = \begin{cases} 1 & \text{if } r = j+1 \text{ and} \\ 0 & \text{if } r \neq j+1. \end{cases}$$

Hence we get

PROPOSITION 3.13. *For $\gcd(n,m) = 1$, m prime the operators $\tilde{\mathrm{T}}_{n,m}$ and $\tilde{\mathrm{H}}_{n,m}$ are conjugate with $H_n^{-1}\, \tilde{\mathrm{T}}_{n,m}\, H_n = \tilde{\mathrm{H}}_{n,m}$.*

PROOF. For $\vec{\phi}$ a vector valued period function for $\Gamma_0\,(n)$ as defined in §1 consider $\tilde{\mathrm{T}}_{n,m}\vec{\phi}$ as given in (1.31). Then using Lemma 3.12 and Equation (2.29) we get

$$\begin{aligned}(3.33)\quad \Big(\tilde{\mathrm{T}}_{n,m}\vec{\phi}\Big)_{h_n(j)} &= \sum_{i \in \sigma_{m,n}^{-1}\left(h_n(j)\right)} \sum_{r=0}^{L_i-1} \Big(H_n\, \rho_{\Gamma_0(n)}\big((m_{r+1}^{(i)})^{-1}\big)\, H_n^{-1}\vec{\phi}\Big)_{l_{i,0}} \Big|_\beta m_{r+1}^{(i)}\, A_{\sigma_{n,m}(i)} \\ &= \sum_{i \in \sigma_{m,n}^{-1}\left(h_n(j)\right)} \sum_{r=1}^{L_i} \Big(\rho_{\Gamma_0(n)}\big((m_r^{(i)})^{-1}\big)\, H_n^{-1}\vec{\phi}\Big)_{h_n^{-1}(l_{i,0})} \Big|_\beta m_r^{(i)}\, A_{\sigma_{n,m}(i)}.\end{aligned}$$

But according to Proposition 3.10 this can be rewritten as

$$\Big(\tilde{\mathrm{T}}_{n,m}\vec{\phi}\Big)_{h_n(j)} = \sum_{A \in X_m^\star} \sum_{r=1}^{L(A0)} \Big(\rho_{\Gamma_0(n)}(m_r^{(A,j)^{-1}})\, H_n^{-1}\,\vec{\phi}\Big)_{\Phi_A(j)} \Big|_\beta m_r^{(A,j)}\, \sigma_{\alpha_j}(A) \tag{3.34}$$

with $M(\sigma_{\alpha_j}(A)0) = \sum_{r=1}^{L(A0)} m_r^{(A,j)}$ and hence $H_n^{-1}\, \tilde{\mathrm{T}}_{n,m} = \tilde{\mathrm{H}}_{n,m}\, H_n^{-1}$. □

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