Hurwitz-Nakada C-F and the transfer operator for Hecke triangle groups

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- Transfer operator and Selberg fct. for modular groups
- Hurwitz-Nakada-CF and Hecke groups
- The generating maps f_q and f_q^*
- A Poincaré section for geodesic flow on Hecke surfaces
- Transfer operator $\mathcal{L}_\beta$ for Hecke groups

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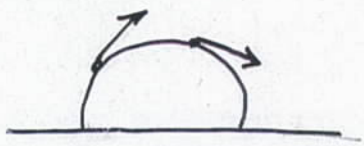
Transfer operator and Selberg's zeta fct. for modular groups

$\Gamma \subset PSL(2,\mathbb{Z})$ finite index subgroup

$M_\Gamma := \Gamma \backslash \mathbb{H}$ modular surface

$\Gamma \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \; gz = \dfrac{az+b}{cz+d}$

$\Phi_t : SM_\Gamma \to SM_\Gamma$ geodesic flow



$$Z(\beta) = \prod_{k=0}^{\infty} \prod_{\gamma} \left(1 - e^{-(\beta+k)\,l(\gamma)}\right) \quad \mathrm{Re}\,\beta > 1$$

γ primitive periodic orbit, $l(\gamma)$ period

- classical approach: Selberg's trace formula
- dynamical approach: transfer operator

$P : \Sigma \to \Sigma$ Poincaré section Σ

$r : \Sigma \to \mathbb{R}_+$ recurrence time function

Lemma (D. Ruelle)

$$Z(\beta) = \prod_{k=0}^{\infty} \exp - \sum_{n=1}^{\infty} \frac{Z_n(\beta+k)}{n}, \quad \mathrm{Re}\,\beta > 1$$

$$Z_n(\beta) = \sum_{x \in \mathrm{Fix}\, P^n} \exp -\beta \sum_{k=0}^{n-1} r(P^k x)$$

thermodynamic formalism:

look for an operator $\mathcal{L}_\beta$ with $Z_n(\beta) = \mathrm{trace}\, \mathcal{L}_\beta^n$

$$\Longrightarrow \boxed{Z(\beta) = \det(1-\mathcal{L}_\beta)}$$

- meromorphic continuation to entire β-plane
- zero's of $Z(\beta) \Longleftrightarrow \beta$-values with $1 \in \sigma(\mathcal{L}_\beta)$
- If $\mathcal{L}_\beta f = f$, $\mathrm{Re}\,\beta = \frac{1}{2} \Longleftrightarrow \exists$ Maass wave form φ with $\Delta\varphi = \beta(1-\beta)\varphi \Longleftrightarrow f$ is period fct.

worked out for modular groups Γ

more general Fuchsian groups $\Gamma \subset PSL(2,\mathbb{R})$?

non-arithmetic ones? (Phillips-Sarnak Conj.)

Problem: find "good" Poincare-section for geodesic flow on $\Gamma\backslash\mathbb{H}$

Hecke triangle groups G_q, arithmetic only for $q = 3, 4, 6$

The H-N CF's and Hecke triangle groups G_q

$$G_q = \langle S, T_q \rangle, \quad Sz = -\frac{1}{z}, \quad T_q z = z + \lambda_q$$

$$\lambda_q = 2\cos\frac{\pi}{q}, \quad q = 3, 4, \dots, \quad G_3 = SL(2,\mathbb{Z})$$

$$(ST_q)^q = 1, \quad S^2 = 1 \quad (\text{in } PSL(2,\mathbb{R}))$$

$$I_q := \left[-\frac{\lambda_q}{2}, \frac{\lambda_q}{2}\right] \ni x$$

$$x = \cfrac{-1}{a_1\lambda_q - \cfrac{1}{a_2\lambda_q - \cfrac{1}{a_3\lambda_q - \cdots}}} := [0; a_1, a_2, \dots], \quad a_i \in \begin{cases} \mathbb{Z}\setminus\{0\} \\ \mathbb{Z}\setminus\{0,\pm 1\} \quad (q=3) \end{cases}$$

$q = 3$: Hurwitz

q even: Nakada

H-N CF

generating map $f_q : I_q \to I_q$

$$f_q(x) = \begin{cases} -\frac{1}{x} - \left\lfloor -\frac{1}{x\lambda_q} + \frac{1}{2} \right\rfloor \lambda_q & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$x_0 := x, \quad x_i := f_q(x_{i-1}) = -\frac{1}{x_{i-1}} - a_i\lambda_q, \quad x_{i-1} \neq 0$$

$$\Rightarrow f_q([0; a_1, a_2, \dots]) = [0; a_2, a_3, \dots]$$

$$[0; a_1, a_2, \dots] = \lim_{n\to\infty} ST_q^{a_1} ST_q^{a_2} \cdots ST_q^{a_n} 0$$

$$f_q(-x) = -f_q(x)$$

dual H-N CF:

$$r_q = \begin{cases} [0;\overline{3}] & q=3 \\ [0;\underbrace{\overline{1^{h_q-1},2}]} & q=2h_q+2 \\ [0;1^{h_q},2,\overline{1^{h_q-1},2}] & q=2h_q+3 \end{cases}$$

$R_q := r_q + \lambda_q \quad (R_q = 1 \text{ for } q \text{ even})$

$I_{R_q} = [-R_q, R_q] \ni y$

$$y = \cfrac{-1}{b_1\lambda_q - \cfrac{1}{b_2\lambda_q - \cfrac{1}{b_3\lambda_q - \cdots}}} := [0; b_1, b_2, \dots]^*, \quad b_i \in \begin{cases} \mathbb{Z}\setminus\{0\} \\ \mathbb{Z}\setminus\{0,\pm1\} \quad q=3 \end{cases}$$

$q=3$ Hurwitz

dual H-N CF

generating map $f_q^* : I_{R_q} \to I_{R_q}$

$$f_q^*(y) = \begin{cases} -\frac{1}{y} - \left\lfloor \frac{-1}{y\lambda_q} - \frac{r_q}{\lambda_q} \right\rfloor \lambda_q & y \geq 0 \\ -\frac{1}{y} - \left\lfloor \frac{-1}{y\lambda_q} + \frac{R_q}{\lambda_q} \right\rfloor \lambda_q & y < 0 \\ 0 & y = 0 \end{cases}$$

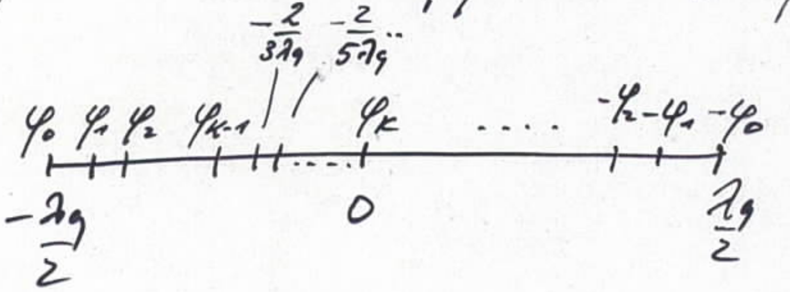
$y_0 := y \quad y_i := f_q^*(y_{i-1}) = -\frac{1}{y_{i-1}} - b_i\lambda_q, \quad y_{i-1} \neq 0$

$f_q^*(-y) = -f_q^*(y)$

Lemma. The maps $f_q : I_q \to I_q$ and $f_q^* : I_{R_q} \to I_{R_q}$ are locally analytic, expanding Markov-maps and hence conjugate to subshifts over infinite alphabets F resp. F^*.

Markov partitions $\mathcal{M}(f_q)$ and $\mathcal{M}(f_q^*)$ (q even)

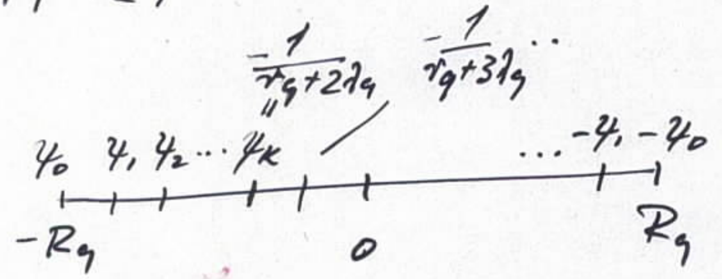
f_q:



$\mathcal{M}(f_q) = \{ \mathcal{J}_i : i \in F \}$

$\varphi_i = f_q^i\left(-\frac{\lambda_q}{2}\right)$

f_q^*:



$\mathcal{M}(f_q^*) = \{ \mathcal{J}_i^* : i \in F^* \}$

$\psi_i = (f_q^*)^i(-R_q)$

$F = \{ \pm 1_i,\ 1 \le i \le k_q \} \cup \{ \pm m,\ m \ge 2 \}$ alphabet

$A = (A_{ij})_{i,j \in F}$ $\quad A_{ij} = \begin{cases} 1 & f_q(\mathcal{J}_i) \supset \mathcal{J}_j \\ 0 & \text{otherwise} \end{cases}$

transition matrix

$F^* = \{ \pm 1_j,\ 1 \le j \le k_q \} \cup \{ \pm m,\ m \ge 2 \}$

$A^* = (A^*_{ij})_{i,j \in F^*}$ $\quad A^*_{ij} = \begin{cases} 1 & f_q^*(\mathcal{J}_i^*) \supset \mathcal{J}_j^* \\ 0 & \text{otherwise} \end{cases}$

$F_A^{\mathbb{N}} = \{ \underset{>}{\xi} = (\xi_i)_{i\in\mathbb{N}},\ \xi_i \in F,\ A_{\xi_i \xi_{i+1}} = 1\ \forall i \in \mathbb{N} \}$

$\tau_> : F_A^{\mathbb{N}} \to F_A^{\mathbb{N}} \quad (\tau_> \underset{>}{\xi})_i = \xi_{i+1}$

relation $\tau : F_A^{\mathbb{N}} \circlearrowleft$, $\tau^* : F_A^{*\mathbb{N}} \circlearrowleft$ to H-N CF resp. dual H-N CF ?

Consider sofic systems: replace all the symbols $\pm 1_i$ in F resp. F^* by symbol ± 1.

Lemma The sequence $\underline{a} = (a_i)_{i\in\mathbb{N}}$, $a_i \in \mathbb{Z}\setminus\{0\}$ (resp. $a_i \in \mathbb{Z}\setminus\{0, \pm 1\}$, $q=3$) defines a H-N CF iff $(a_i)_{i\in\mathbb{N}}$ does not contain any subsequence $(\pm 1^{h_q+1})$ and $(\pm 1^{h_q}, \pm m)$, $m = 2, 3, \ldots$ (respectively $(\pm 2, \pm m)$, $m \geq 2$).
The sequence $\underline{b} = (b_j)_{j\in\mathbb{N}}$, $b_j \in \mathbb{Z}\setminus\{0\}$ (resp. $b_j \in \mathbb{Z}\setminus\{0, \pm 1\}$ for $q=3$) defines a dual H-N CF $\iff \forall N \geq 1$ the sequence $(b_N, b_{N-1}, \ldots, b_1)$ defines a H-N CF.

Denote by $\mathcal{A}_>$ and $\mathcal{A}_>^*$ the set of sequences $\underline{a}_>$ respectively $\underline{b}_>$ in above lemma

$\tau_> : \mathcal{A}_> \to \mathcal{A}_>$, $\tau_>^* : \mathcal{A}_>^* \to \mathcal{A}_>^*$ shifts of sofic systems

A Poincaré section for $\phi_t : SM_{G_q} \to SM_{G_q}$

Natural extension $F_q : \Omega_q \to \Omega_q$ of $f_q : I_q \to I_q$ should be conjugate to natural extension $\tau : \mathcal{A} \to \mathcal{A}$ of $\tau_> : \mathcal{A}_> \to \mathcal{A}_>$.

$$\mathcal{A} = \{ \underline{a} = (a_i)_{i\in\mathbb{Z}} : \forall N \ \underline{a}_N^> := (a_{N+i})_{i\in\mathbb{N}} \in \mathcal{A}_> \}$$

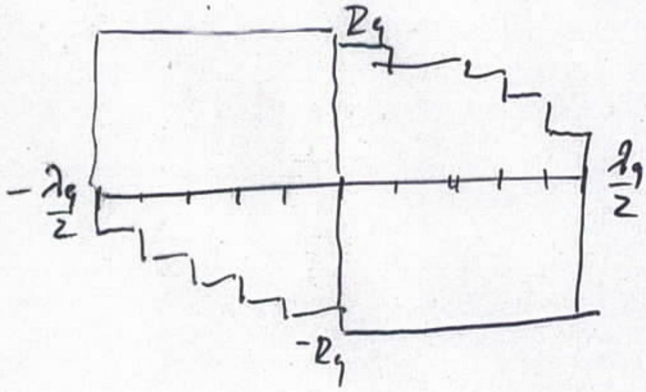
$(\tau \underline{a})_i = a_{i+1}$, $i \in \mathbb{Z}$ 2-sided shift on $\mathcal{A}$

but $\underline{a} \in \mathcal{A} \iff \forall N \ \underline{b}_N^> = (a_{N-i})_{i\in\mathbb{N}} \in \mathcal{A}_>^*$

$$\Omega_q := \{ (x,y),\ x = [0; a_1, a_2, \dots],\ y = [0; b_1, b_2, \dots]^* : (\dots b_{-2}, b_{-1}; a_1, a_2, \dots) \in \mathcal{A} \} \subset I_q \times I_{R_q}$$

if one allows also finite H-N CF's

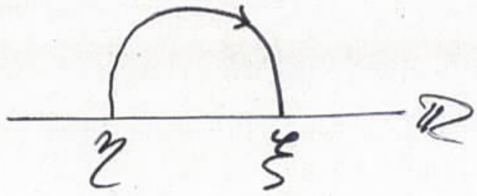
Lemma: $\Omega_q \stackrel{+}{=} \bigcup_{i=1}^{\kappa_q} [\varphi_{i-1}, \varphi_i] \times [\psi_{\kappa_q - i+1}, R_q]$



$F_q(x,y) = (f_q(x), ST_q^{a_1} y)$ $\quad x = [0; a_1, \dots]$

$F_q^{-1}(x,y) = (ST^{b_1} x, f_q^*(y))$ $\quad y = [0; b_1, b_2, \dots]^*$

Def: γ geodesic $\gamma = \gamma(\xi, \eta)$

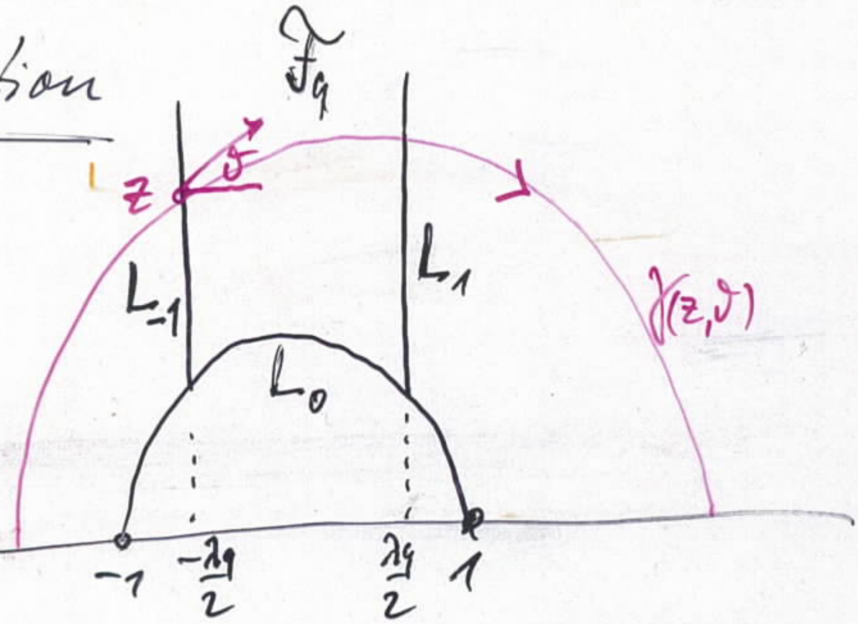


γ is reduced geodesic $\iff (S\xi, -\eta) \in \mathcal{D}_q$ and

$|\xi| > \frac{3\lambda_q}{2}$ or $\xi \cdot \eta < 0$

Lemma: Given any geodesic γ there exist $g \in G_q$ with $g\gamma$ is reduced.

Poincaré section



$$\Sigma = \{(z,\vartheta) : z \in L_1 \cup L_{-1} \cup L_0,\ \gamma(z,\vartheta) \text{ is reduced}\}$$

$\pi^* : SH \to SM_{G_q}$ projection map

Lemma: $\pi^*(\Sigma) \subset SM_{G_q}$ is a Poincaré section for geodesic flow $\Phi_t : SM_{G_q} \to SM_{G_q}$

<u>Proposition</u>: For $(z,\vartheta)\in\Sigma$ with $\gamma_{z,\vartheta}=\gamma(\xi,\eta)$ reduced the "Poincaré" map $\tilde{P}:\Sigma\to\Sigma$ is given by the map $P(z,\vartheta)=(z',\vartheta')$ with $\gamma_{z',\vartheta'}=\gamma(\xi',\eta')$ and $(\xi',\eta')=\tilde{S}F_q^{-k}\tilde{S}(\xi,\eta)$, where k is determined by $\xi=[a_0;\varepsilon^{k-1},a_k,\dots]$, $\varepsilon=\operatorname{sign}a_0$ and $\tilde{S}(\xi,\eta)=(S\xi,-\eta)$

The transfer operator for G_q

Obviously is the map $f_q:I_q\to I_q$ the restriction of F_q to its unstable direction. It acts as a Möbius transformation in G_q.
The recurrence time fct. is therefore

$$r(x)=\ln f_q'(x)=-\ln x^2$$

generalized Perron–Frobenius operator for f_q:

$$\mathcal{L}_\beta f(x)=\sum_{y\in f_q^{-1}(x)}\exp-\beta r(y)\,f(y)$$

inverse branches of f_q: $\quad \psi_n(x)=\dfrac{-1}{x+n\lambda_q}\quad n\neq 0$

<u>Lemma</u>: For $x \in [\varphi_{i-1}, \varphi_i]$, $1 \le i \le k_q$ the preimage $f_q^{-1}(\{x\})$ is given by $f_q^{-1}(\{x\}) = \{ y \in I_q : y = \eta_n(x), n \in \mathcal{N}_i \}$ with $\mathcal{N}_i = \bigcup_{j=1}^{k_q} \mathcal{N}_{ij}$ and $\mathcal{N}_{ij} = \{ n \in \mathbb{Z} \setminus \{0\} : \eta_n(I_i) \subset I_j \}$

Then one gets for $x \in [\varphi_{i-1}, \varphi_i]$

$$(\mathcal{L}_\beta f)(x) = \sum_{n \in \mathcal{N}_i} |\eta_n'(x)|^\beta f(\eta_n(x))$$

or if $f_i := f|_{[\varphi_{i-1}, \varphi_i]}$

$$\boxed{(\mathcal{L}_\beta f)_i(x) = \sum_{j=1}^{k_q} \sum_{n \in \mathcal{N}_{ij}} |\eta_n'(x)|^\beta f_j(\eta_n(x))}$$

Since the maps η_n are all contracting and real analytic there exist discs $D_i \subset \mathbb{C}$, $I_q \subset D_i$ with $\eta_n(D_i) \subset D_j \ \forall n \in \mathcal{N}_{ij}$

Set $\mathcal{B} := \bigoplus_{i=1}^{k_q} \mathcal{B}(D_i)$

<u>Prop.</u> $\mathcal{L}_\beta : \mathcal{B} \to \mathcal{B}$ is meromorphic family of trace class operators in entire complex β-plane. with poles only at $\beta_k = \frac{1-k}{2}$, $k = 0, 1, 2, \ldots$

Lemma: If $\tilde{Z}_n(\beta) = \sum_{x \in Fix f_g^n} \exp -\beta \sum_{k=0}^{n-1} r(f_g^k x)$

then $\tilde{Z}_n(\beta) = \text{trace } \mathcal{L}_\beta^n - \text{trace } \mathcal{L}_{\beta+1}^n$

But $x \in Fix f_g^n$ determines uniquely $(x,y) \in Fix F_g^n$ and $r(x,y) = r(x) \Rightarrow$

$$\tilde{Z}_n(\beta) = \sum_{z \in Fix F_g^n} \exp -\beta \sum_{k=0}^{n-1} r(F_g^k z)$$

Poincaré map for $\phi_t : SM_{G_g} \to SM_{G_g}$ is given by

$$\begin{array}{ccc} \tilde{P} : \Sigma & \to & \Sigma \\ \pi^* \downarrow & & \downarrow \pi^* \\ P : \pi^*(\Sigma) & \to & \pi^*(\Sigma) \end{array}$$

Need $Fix\, P^n$ and not $Fix\, \tilde{P}^n$

Lemma: There exist exactly one pair of periodic points (x_1, x_2) of $\tilde{P}$ which are G_g-equivalent and hence determine the same periodic orbit of $\phi_t : SM_{G_g} \to SM_{G_g}$: $x_1 = r_g$, $x_2 = -r_g$

$\Rightarrow$ Need to subtract from $\tilde{Z}_n$ the contribution of r_g for all n with k_g / n !

$$Z_n^{(r_q)}(\beta) := \begin{cases} 0 & h_q \nmid n \\ \exp -\beta \sum_{l=0}^{n-1} r\left(f_q^{l}(r_q)\right) & h_q \mid n \end{cases}$$

$$K_\beta := \left.\begin{pmatrix} 0 & \mathcal{L}_\beta^{(1)} & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ 0 & & & \ddots & \mathcal{L}_\beta^{(1)} \\ \mathcal{L}_\beta^{(2)} & 0 & \cdots & \cdots & 0 \end{pmatrix}\right\} h_q \qquad \mathcal{L}_\beta^{(i)} f(z) = \left(\frac{1}{i\lambda_q + z}\right)^{2\beta} f\left(\frac{-1}{i\lambda_q + z}\right)$$

$$Z_n^{(r_q)}(\beta) = \operatorname{trace} K_\beta^n - \operatorname{trace} K_{\beta+1}^n$$

<u>Proposition</u> $\quad Z(\beta) = \dfrac{\det(1-\mathcal{L}_\beta)}{\det(1-K_\beta)}$,

$$\det(1-K_\beta) = \det\left(1-\left(\mathcal{L}_\beta^{(1)}\right)^{h_q-1}\mathcal{L}_\beta^{(2)}\right) \quad q \text{ even}$$

$$\det(1-K_\beta) = \det\left(1-\left(\mathcal{L}_\beta^{(1)}\right)^{h_q}\mathcal{L}_\beta^{(2)}\left(\mathcal{L}_\beta^{(1)}\right)^{h_q-1}\mathcal{L}_\beta^{(2)}\right) \quad q \text{ odd}$$

$$r_q = [0; \overline{1^{h_q-1}, 2}] \quad q \text{ even}$$

$$r_q = [0; \overline{1^{h_q}, 2, 1^{h_q-1}, 2}] \quad q \text{ odd}$$