Character deformations of the zeros of Selberg's zeta function for $\Gamma(4)$ via transfer operators

(M. Fraczek, D.M., DFG-project*)

- Spectral properties of $\mathbb{Z}$-automorphic Laplace-Beltrami operators

- Selberg's zeta function and the transfer operator for $\Gamma(4)$ with Selberg's character $\chi_a$

- Some numerical results

* "Transfer operator approach to the Phillips-Sarnak Conjecture"

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Spectral properties of \( \Gamma \)-automorphic functions.

\( \Gamma \) is a cofinite Fuchsian group, \( \Gamma \subseteq \text{PSL}(2, \mathbb{R}) \).

\[ g \tau = \frac{az + b}{cz + d}, \quad g \in \Gamma, \quad g = (a \, b \mid c \, d), \quad \tau \in \mathbb{H} = \{ x + iy : y > 0 \} \]

\[ ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad d\mu(\tau) = \frac{dx \, dy}{y^2} \]

\[ \Delta \llbracket \tau \rrbracket = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \]

\( F_\tau \) is a fundamental domain, \( \mu(F_\tau) < \infty \).

\( \chi : \Gamma \to \mathbb{C} \) is a character (1-dim. unit. repr.).

Hilbert space of \( \Gamma \)-automorphic fcts:

\[ \mathcal{H}_\chi = \left\{ f : \mathbb{H} \to \mathbb{C} : f(g \tau) = \chi(g) f(\tau) \quad \forall g \in \Gamma \right\} \]

\[ \int_{\mathcal{F}} |f(\tau)|^2 \frac{d\mu(\tau)}{\mu(\mathcal{F})} < \infty \]

\( \sigma(\Delta \chi) \) is the spectrum of \( \Delta \) on \( \mathcal{H}_\chi \).

\( \zeta_1, \ldots, \zeta_k \) are inequivalent cusps of \( \Gamma \).

\( T_i \zeta_i = \zeta_i \), \( T_i \) is a primitive parabolic element in \( \Gamma \).
Def: \( \gamma \) singular in cusp \( z_i \) \( \iff \gamma(T_i) = 1 \)

\( k(\gamma) = \# \{ i : \gamma(T_i) = 1 \} \) degree of singularity

\( \gamma \) non-singular in \( z_i \) \( \iff \gamma(T_i) \neq 1 \)

("cusp \( z_i \) is closed")

If \( \gamma(T_i) \neq 1 \) \( \forall 1 \leq i \leq k \) \( \Rightarrow \Delta \gamma \) has spectrum like cocompact case, only eigenvalues obeying Bégly's law:

\[
N(\lambda, \Delta) = \# \{ \lambda_i < \lambda \} \sim \frac{M(C)}{4\pi^2} \lambda^2 \quad \lambda \to \infty
\]

(Selberg)

If \( k(\gamma) > 0 \) \( \Rightarrow \Delta \gamma \) has continuous spectrum \([1, \infty)\) of multiplicity \( k(\gamma) \)

generalized eigenforms are the analytically continued Eisenstein-Maass series

\( E_\ell(\beta, z, \Gamma) \) for \( \beta = \frac{1}{2} + it \), \( \ell = 1, \ldots, k(\gamma) \)

Problem: do there exist eigenvalues and cusp forms for general cofinite \( \Gamma \)?

- yes for congruence subgroups \( \Gamma \leq SL(2, \mathbb{Z}) \)
- and \( \chi = 1 \) (Selberg), Bégly-Law
Samuel-Phillips conjecture: (trivial character)
Weyl law valid only for arithmetic groups
(besides certain arithmetic non-congruence subgroups)

**Singular character:**

studied only for special groups like congruence subgroups and special characters.
Kernel of character again congruence subgroup.

Samuel-Phillips: non-singular perturbation in Teichmüller-space

Wolpert: singular perturbation in T-space

Samuel-Phillips: singular character deformation for \( P(2) \)

Balster-Vekhtov: singular and nonsingular character deformation for several congruence subgroups

**Method:** perturbation theory of automorphic Laplacian

"Fermi's golden rule"
The Selberg zeta funct. via the transfer operator

$$Z_{r,t}(s) = \prod_{g \in \Gamma} \prod_{k=0}^{\infty} (1 - q^{-(r+\frac{1}{2}+k)} g)$$

$r$ primitive closed orbit of geodesic flow on $S^1 \times \mathbb{H}$
$c_l(g)$ period, $g \in \Gamma$ hyperbolic elem. with $g \frac{1}{2} = t$.

Zero’s of $Z_{r,t}(s)$:

- trivial zeros (or poles) at $s = 0, -1, -2, \ldots$
- non-trivial zeros on $\Re \beta = \frac{1}{2}$ with $T = \beta(1-\beta)$
  EV of $-\Delta_g$ corr. to cusp form
- non-trivial zeros in $\Re \beta < \frac{1}{2}$
  resonances = complex poles of scattering determinant
- finitely many zeros in $(\frac{1}{2}, 1)$
  small eigenvalues of $-\Delta_g$, residues of poles of
  Eisenstein series in $(\frac{1}{2}, 1)$
- poles at $\beta = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots$ order $\mathbb{C}(g)$. 
\[ \Gamma_0(4) = \{ g = (\begin{array}{cc} a & b \\ c & d \end{array}) \in \text{SL}(2, \mathbb{Z}) : c = 0 \mod 4 \} \]

\[ [\text{SL}(2, \mathbb{Z}) : \Gamma_0(4)] = 6, \quad \Gamma_0(4) \sim \Gamma(2) \text{ principal subgroup} \]

generator of \( \Gamma_0(4) \): parabolic

\[ A = (1 1), \quad B = (1 0), \quad S = (1 -1) \]

relation: \( A B S = 1 \)

inequivalent cusps: \( z_1 = \infty, \quad z_2 = \frac{1}{2}, \quad z_3 = 0 \)

\( A z_1 = z_1, \quad B z_2 = z_2, \quad S z_3 = z_3 \)

\( \Gamma_0(4) \) freely generated by \( A, B \)

\( \Gamma_0(4) \ni g = A^{n_1} B^{m_1} \ldots A^{n_k} B^{m_k}, \quad n_i, m_i \in \mathbb{Z} \nabla \)

Selberg's character \( \chi_\alpha \) for \( \Gamma_0(4) \), \( 0 \leq \alpha \leq 1 \)

\[ \chi_\alpha(g) = e^{2 \pi i \alpha P_A(g)} \]

\[ P_A(g) = \sum_{i=1}^{k} n_i \]

\[ \chi_\alpha(A) = e^{2 \pi i \alpha}, \quad \chi_\alpha(B) = 1, \quad \chi_\alpha(S) = e^{-2 \pi i \alpha} \]

\( \Rightarrow \chi(\chi_\alpha) = 3 \rightarrow \chi(\chi_\alpha) = 1 \text{ for } 0 < \alpha < 1 \)

\( \alpha = 0 \rightarrow \alpha > 0 \)

multiple of cusp spectrum: \( 3 \rightarrow 1 \)
Selby: • zeros of $Z_{\Gamma_0(4)}(\alpha)$ with $\text{Re} \beta < \frac{1}{2}$ accumulate at $\text{Re} \beta = \frac{1}{2}$ (resonances accumulate at cont. spectrum) for $\alpha \to 0$!
• multiplicity changes from 3 to 1: Selby zeros should appear in pairs.

• For $\alpha \to \frac{1}{2} - \varepsilon$ some zeros tend to $-\infty$.

On the other hand (Phillips-Pamuk, Balasubramanyam,)

for $x_j = \frac{1}{4}, j = 0, 1, \ldots, 7$ is $X_{x_j}$ arithmetic.

All nontrivial zeros of $Z_{\Gamma_0(4)}, X_{x_j}$ are on $\text{Re} \beta = \frac{1}{2}, \text{Re} \beta = \frac{1}{4}$ (Riemann)
and $\text{Re} \beta = 0$.

\[ Z_{\Gamma_0(4)}(\beta) = \frac{1}{11} \sum_{i=0}^{7} Z_{(\Gamma_0(4), X_{x_i})}(\beta) \]

$\Gamma_8 \subset \Gamma_0(4)$ congruence subgroup of $SL(2, \mathbb{Z})$

normal in $\Gamma_0(4)$ with $[\Gamma_0(4) : \Gamma_8] = 8$.

$\Gamma_8 = \{ g \in \Gamma_0(4) : 8 \mid P_8(g) \}$

$\Gamma_0(4)/\Gamma_8 = \{ 1, A, A^2, \ldots, A^8 \}$ finite, cyclic group.
The transfer operator for $P_0(4)$ with character $\chi^\alpha$

$Z_{P_0(4),\chi^\alpha}(\gamma) = Z_{SL(2,Z),\chi^\text{ind}}(\gamma)$

$\chi^\text{ind}$ is repres. of $SL(2,Z)$ induced from character $\chi^\alpha$ of $P_0(4)$

$[SL(2,Z):P_0(4)] = 6$

$\chi^\text{ind}: SL(2,Z) \to \text{end} (\mathbb{C}^6)$

$SL(2,Z) = \bigsqcup_{i=1}^6 P_0(4) g_i$

$(U_{\chi}(g))_{ij} = \delta_{P_0(4)}(g g_i g_j^{-1}) \chi^\alpha(g g_i g_j^{-1})$, $1 \leq i,j \leq 6$

$\chi = 0$ permutation matrix

$\chi > 0$ monomial matrix

$L_{\beta,\chi} = L_{\beta,\chi^\text{ind}} = \left( \begin{array}{cc} 0 & \chi^{(++)} \\ \chi^{(-)} & 0 \end{array} \right)$

$L_{\beta,\chi} f(x) \xrightarrow{m \to \infty} \sum_{k=0}^\infty \frac{C_{\beta,\chi}(m\epsilon)}{k!} f^{(k)}(0)$

$\sum_{m=0}^{\infty} U_{\epsilon}(m, \beta, 2\beta + k, \frac{z+m}{\beta}) f^{(k)}(0) \frac{1}{k!}$
with \( \zeta_M(\Lambda, \beta, z) = \sum_{n=0}^{\infty} \Lambda^n (z+n)^{-\beta} \), \( \Lambda \) unitary, matrix valued Lerch zeta fct. (\( \text{Re} \beta > 1 \)) has meromorphic contiuity to entire \( \beta \)-plane with pole at \( \beta = 1 \).

\( \Rightarrow \) when defined on appropriate \( \beta \)-space of vector valued holomorphic fcts in \( L^p(\mathbb{S}) \) meromorphic, nuclear operator in \( \beta \)-plane

\[
E(\Gamma_0(4), \chi_\alpha)(\beta) = \det (1 - T_{\beta, \alpha})
\]

Since \( E(\Gamma_0(4), \chi_\alpha) = E(\Gamma_0(4), \chi_{\alpha-1}) \)

restricted to \( 0 \leq \alpha \leq \frac{1}{2} \)
Numerical calculation of $Z_{\Gamma_6(4)}(\beta)\,\chi_\alpha$ for $\beta$'s with $\text{Im}\,\beta$ small

Approximation of transfer operator $L_{\beta,\alpha}$:
Truncate Taylor expansion at $N$-th term
leads to $2\mu N \times 2\mu N$ matrix, whose EV have to be determined: $(\mu=6)$

$$
\left( M_{\beta,\alpha}^{(s)} \right)_{ij} = (-1)^s \frac{s-1}{s!} \binom{s}{2\beta + \lambda + p} 4^{-(2\beta + \lambda + s)} - \sum_{m=1}^{4} \sum_{\lambda \in \Sigma_L (\mathcal{F}(\mathcal{T}_{\beta,\alpha}^{m})), \lambda \in \Sigma_L (\mathcal{F}(\mathcal{T}_{\beta,\alpha}^{m}))} \chi_\alpha \left( \Gamma_6 \left( \mathcal{T}_{\beta,\alpha}^{m} \right), 2\beta + \lambda + s, \frac{m}{4} \right) 1 \leq \beta \leq N, 1 \leq i, j \leq 6 = [\mathcal{SL}(2,\mathbb{Z}) : \Gamma_6(4)]
$$

with

$$
\Sigma_L (\mathcal{F}(\mathcal{T}_{\beta,\alpha}^{m})) = \sum_{n=0}^{\infty} e^{2\pi i \mathbf{v}_n \cdot \mathbf{a}} \left( \frac{1}{\mathbf{a} + n} \right)^s
$$

$L_{\beta,\alpha} \sim \begin{pmatrix} 0 & M_{\beta,\alpha}^{(s)} \\ M_{\beta,\alpha}^{(-s)} & 0 \end{pmatrix}$

$Z_{\Gamma_6(4)}(\beta)\,\chi_\alpha(\beta) \sim \det \left( 1 - \left( \begin{array}{cc} 0 & M_{\beta,\alpha}^{(s)} \\ M_{\beta,\alpha}^{(-s)} & 0 \end{array} \right) \right)$