The transfer operator for Rosen-type λ-expansions and the period functions for Hecke triangle groups

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Quantum chaos: quantum vs. classical behavior

- spectral statistics (trace formulas, random matrix
- eigenstates (transfer operator?)

Geodesic flows on surfaces of constant neg. curv.

- modular surfaces $\Gamma \backslash \mathbb{H}$, $\Gamma \subset SL(2,\mathbb{Z})$
- Lewis-Zagier theory of period functions for Γ modular
- Gauss continued fractions and their transfer operators (Artin, ...)
- extension of this theory to non-arithmetic groups/surfaces: Hecke triangle groups G_q
- Rosen-type λ-continued fractions

Hecke's triangle groups G_q, $q = 3, 4, \dots$

$G_q = \langle S, T_{\lambda_q} \rangle$, $Sz = -\frac{1}{z}$, $T_{\lambda_q} z = z + \lambda_q$, $z \in \mathbb{H}$

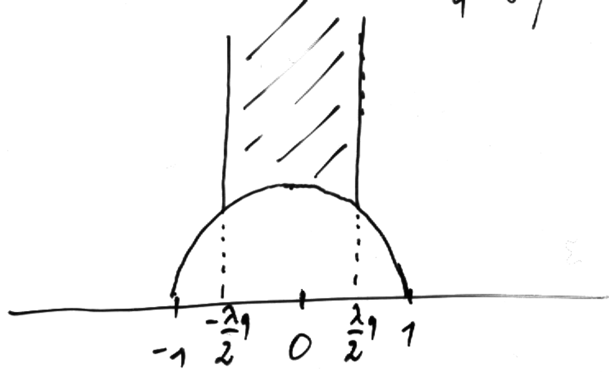
$\lambda_q = 2\cos\frac{\pi}{q}$, $\lambda_3 = 1$, $\lambda_4 = \sqrt{2}$, $\lambda_5 = \frac{\sqrt{5}+1}{2}$, $\lambda_6 = \sqrt{3}, \dots$

relations: $S^2 = (S T_{\lambda_q})^q = \text{identity}$

$q = 3$: $G_3 = SL(2, \mathbb{Z})$

Leutbecher: G_q arithemetic $\iff q = 3, 4, 6$

fundamental domain $\mathcal{F}_q$, finite area



Rosen-type λ expansions

$\lambda = \lambda_q$, q fixed

$I_\lambda = [-\frac{\lambda}{2}, \frac{\lambda}{2}]$

$f_\lambda^R : I_\lambda \to I_\lambda$, $f_\lambda^N : I_\lambda \to I_\lambda$ (R = Rosen) (N = Nakada)

$$f_\lambda^R(x) = \frac{1}{|x|} - \left\lfloor \frac{1}{|x|\lambda} + \frac{1}{2} \right\rfloor \lambda, \quad x \neq 0 \qquad \lfloor x \rfloor = \max\{n : n \leq x\}$$

$$f_\lambda^N(x) = -\frac{1}{x} - \left\lfloor -\frac{1}{x\lambda} + \frac{1}{2} \right\rfloor \lambda, \quad x \neq 0$$

Properties of f_λ^R resp. f_λ^N

$$f_\lambda^R(-x) = f_\lambda^R(x) \qquad f_\lambda^N(-x) = -f_\lambda^N(x)$$

$$f_\lambda^N(x) = f_\lambda^R(x) \quad \text{for } x \leq 0$$

$$f_\lambda^N(x) = -f_\lambda^R(x) \quad \text{for } x \geq 0$$

Rosen, Schmidt, Sheingorn, Kraaikamp, Nakada, ...

$$\phi_k^\pm := \left(f_q^N\right)^k\left(\pm\frac{\lambda}{2}\right), \quad k = 0, 1, 2, \ldots$$

$$\phi_k^- = \left(f_q^R\right)^k\left(\pm\frac{\lambda}{2}\right), \quad \phi_k^+ = -\phi_k^-$$

$$q = 2p, \quad p = 2, 3, \ldots$$

$$-\frac{\lambda}{2} = \phi_0^- < \phi_1^- < \cdots < \phi_{p-2}^- < -\frac{2}{3\lambda} < \phi_{p-1}^- = 0$$

$$\frac{\lambda}{2} = \phi_0^+ > \phi_1^+ > \cdots > \phi_{p-2}^+ > \frac{2}{3\lambda} > \phi_{p-1}^+ = 0$$

$q = 2p+3 \quad , p = 0, 1, 2, \cdots$

$$-\frac{\lambda}{2} = \phi_0^- < \phi_{p+1}^- < \phi_1^- < \phi_{p+2}^- < \cdots < \phi_{p-1}^- < \phi_{2p}^- < -\frac{2}{3\lambda} < \phi_p^- < -\frac{2}{5\lambda} < \phi_{2p+1}^- = 0$$

$$\frac{\lambda}{2} = \phi_0^+ > \phi_{p+1}^+ > \phi_1^+ > \phi_{p+2}^+ > \cdots > \phi_{2p}^+ > \frac{2}{3\lambda} > \phi_p^+ > \frac{2}{5\lambda} > \phi_{2p+1}^+ = 0$$

$q = 2p$:

<u>partition $\mathcal{A}$ for f_λ^N</u>

$$\mathcal{J}_{-k}^- := [\phi_k^-, \phi_{k+1}^-] \quad , k = 0, 1, \cdots, p-3$$

$$\mathcal{J}_{-(p-2)}^- := [\phi_{p-2}^-, -\frac{2}{3\lambda}]$$

$$\mathcal{J}_n^- := \left[\frac{-2}{(2n-1)\lambda}, \frac{-2}{(2n+1)\lambda}\right] \quad , n \geq 2$$

$$\mathcal{J}_{-k}^+ := [\phi_{k+1}^+, \phi_k^+] = -\mathcal{J}_{-k}^- \quad , k = 0, 1, \cdots, p-3$$

$$\mathcal{J}_{-(p-2)}^+ := [\frac{2}{3\lambda}, \phi_{p-2}^+] = -\mathcal{J}_{-(p-2)}^-$$

$$\mathcal{J}_n^+ := \left[\frac{2}{(2n+1)\lambda}, \frac{2}{(2n-1)\lambda}\right] = -\mathcal{J}_n^- \quad , n \geq 2$$

$$I_\lambda \subset \bigcup_{\substack{\varepsilon = \pm \\ \infty > m \geq -(p-2)}} \mathcal{J}_m^\varepsilon \quad ; \quad \mathcal{A} = \{ \mathcal{J}_m^\varepsilon : \varepsilon = \pm, -(p-2) \leq m < \infty \}$$

$$\text{if } \partial\mathcal{A} = \{ \partial\mathcal{J}_m^\varepsilon \} \Rightarrow f_\lambda^N(\partial\mathcal{A}) \subset \partial\mathcal{A}$$

$\Rightarrow$ $\mathcal{A}$ is a Markov-partition for f_λ^N

$\Rightarrow$ f_λ^N has Markov property

$$F = \{ (\varepsilon, m) , \varepsilon = \pm , -(p-2) \le m < \infty \}$$

transition matrix $\mathbb{A} = \mathbb{A}_{(\varepsilon,m),(\varepsilon',m')}$

$$\mathbb{A}_{(\varepsilon,m),(\varepsilon',m')} = \begin{cases} 1 & \text{if } f_\lambda^N(\mathcal{J}_m^\varepsilon) \supset \mathcal{J}_{m'}^{\varepsilon'} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{ccc} I_\lambda & \xrightarrow{f_\lambda^N} & I_\lambda \\ \pi \downarrow & & \downarrow \pi \\ F_{\mathbb{A}}^{\mathbb{N}} & \xrightarrow{\tau} & F_{\mathbb{A}}^{\mathbb{N}} \end{array}$$

1-sided subshift over F.

$$\pi(x) = (\{(\varepsilon_i, m_i)\}) , \ (f_\lambda^N)^i x \in \mathcal{J}_{m_i}^{\varepsilon_i} \quad i = 0,1,2,\dots$$

Identify all symbols $(-,-k)$ with $(-,1)$ and all symbols $(+,-k)$ with $(+,1)$

$\Rightarrow$ sofic system with symbols $F = \mathbb{Z} \setminus \{0\}$

$$x = \pi^{-1}(\{n_i\}) = \cfrac{-1}{n_1\lambda - \cfrac{1}{n_2\lambda - \cfrac{1}{n_3\lambda - \dots}}} , \quad n_i \in \mathbb{Z} \setminus \{0\}$$

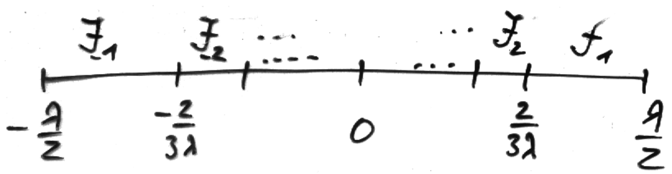
continued fraction expansion to nearest λ-integer

Expansion related to f_λ^R:

$$x = \cfrac{\varepsilon_1}{n_1\lambda + \cfrac{\varepsilon_2}{n_2\lambda + \cfrac{\varepsilon_3}{n_3\lambda + \cdots}}} \qquad n_i \geq 1;\ \varepsilon_i = \pm 1$$

<u>Transfer operator for f_λ^N</u>

$q = 4 \Leftrightarrow p = 2$



$$f_\lambda^N(\mathcal{J}_1) = [-\tfrac{\lambda}{2}, 0], \quad f_\lambda^N(\mathcal{J}_{-1}) = [0, \tfrac{\lambda}{2}]$$

$$f_\lambda^N(\mathcal{J}_{\pm n}) = I_\lambda, \quad n \geq 2$$

$$f_\lambda^N\big|_{\mathcal{J}_{\pm n}}(x) = -\frac{1}{x} \pm n\lambda, \quad f_\lambda^{N\prime}(x) = \frac{1}{x^2} > 1$$

$$\psi_{\pm n}(x) = \left(f_\lambda^N \big| \mathcal{J}_{\pm n}\right)^{-1}(x) = \frac{1}{\pm n\lambda - x}$$

Perron-Frobenius Operator:

$$(\mathcal{L}g)(x) = \sum_{y \in f_\lambda^{-1}} \frac{1}{|f_\lambda'(y)|} g(y), \quad g: I_\lambda \to \mathbb{C}$$

$$g(x) = \begin{cases} g_1(x) & x \leq 0 \\ g_2(x) & x \geq 0 \end{cases} \qquad g_i \text{ continuous}$$

$$(\mathcal{L}g)_1(x) = \sum_{n=2}^{\infty} \left(\frac{1}{n\lambda + x}\right)^2 g_1\left(\frac{-1}{n\lambda + x}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{n\lambda - x}\right)^2 g_2\left(\frac{1}{n\lambda - x}\right)$$

$$(\mathcal{L}g)_2(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n\lambda + x}\right)^2 g_1\left(\frac{-1}{n\lambda + x}\right) + \sum_{n=2}^{\infty} \left(\frac{1}{n\lambda - x}\right)^2 g_2\left(\frac{1}{n\lambda - x}\right)$$

If $D_1 = \left\{ z : \left| z - \frac{\lambda - 2}{4} \right| < \frac{\lambda}{4} + \frac{1}{2} \right\} \supset \left[-1, \frac{\lambda}{2}\right]$

$D_2 = \left\{ z : \left| z - \frac{2 - \lambda}{4} \right| < \frac{\lambda}{4} + \frac{1}{2} \right\} \supset \left[-\frac{\lambda}{2}, 1\right]$

then

- $I_\lambda \subset \overline{D_1 \cap D_2}$
- $\overline{\psi_{-n}(D_1)} \subset D_1 \quad n \geq 2$

  $\overline{\psi_{+n}(D_1)} \subset D_2 \quad n \geq 1$
- $\overline{\psi_{-n}(D_2)} \subset D_1 \quad n \geq 1$

  $\overline{\psi_{+n}(D_2)} \subset D_2 \quad n \geq 2$

$B = B(D_1) \oplus B(D_2)$

$B(D_i)$ B-space of holom. fcts on D_i

transfer operator $\mathcal{L}_\beta : \mathcal{B} \to \mathcal{B}$

$$(\mathcal{L}_\beta \underline{g})(z) = \begin{pmatrix} \mathcal{L}_\beta^{1,1} & \mathcal{L}_\beta^{1,2} \\ \mathcal{L}_\beta^{2,1} & \mathcal{L}_\beta^{2,2} \end{pmatrix} \underline{g}\,(z)$$

$$\mathcal{L}_\beta^{1,1} g(z) = \sum_{n=2}^{\infty} \left(\frac{1}{n\lambda + z}\right)^{2\beta} g\left(-\frac{1}{n\lambda+z}\right)$$

$$\mathcal{L}_\beta^{1,2} g(z) = \sum_{n=1}^{\infty} \left(\frac{1}{n\lambda - z}\right)^{2\beta} g\left(\frac{1}{n\lambda-z}\right)$$

$$\mathcal{L}_\beta^{2,1} g(z) = \sum_{n=1}^{\infty} \left(\frac{1}{n\lambda + z}\right)^{2\beta} g\left(-\frac{1}{n\lambda+z}\right)$$

$$\mathcal{L}_\beta^{2,2} g(z) = \sum_{n=2}^{\infty} \left(\frac{1}{n\lambda - z}\right)^{2\beta} g\left(\frac{1}{n\lambda-z}\right)$$

- $\mathcal{L}_\beta$ is trace class for $\mathrm{Re}\,\beta > \frac{1}{2}$
- $\mathcal{L}_\beta$ has meromorphic extension into entire β-plane with poles at $\beta_k = \frac{1-k}{2}$, $k = 0, 1, \ldots$
- $\det(1 - \mathcal{L}_\beta)$ is meromorphic in entire β-plane
- $\mathcal{L}_1$ has leading EV $\rho = 1$ with spectral gap
  $\Rightarrow f_\lambda^N$ is exact, exponential mixing, ...

<u>Remark</u>: true for all λ_q, $q = 3, 4, 5, \ldots$

Period fcts. for G_q

$h : \mathbb{H} \to \mathbb{C}$ modular cusp form of weight $k \Leftrightarrow$

- $h : \mathbb{H} \to \mathbb{C}$ holomorphic
- $h(gz) = \mathcal{J}_k(g,z)\, h(z) \quad \forall g \in G_q,\ z \in \mathbb{H}$

  $\mathcal{J}_k(g,z) = (cz+d)^k \quad , g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- $h(z) = O(y^C)$ for $y = \operatorname{Im} z \to \infty \quad \forall C \in \mathbb{R}$

u is a Maass cusp form $\Leftrightarrow$

- $u : \mathbb{H} \to \mathbb{C}$ real analytic
- $u(gz) = u(z) \quad \forall g \in G_q,\ z \in \mathbb{H}$
- $-\Delta_{LB}\, u = \beta(1-\beta)\, u \quad , \quad \Delta_{LB} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$
- $u(z) = O(y^C) \quad , y \to \infty \ , \ \forall C \in \mathbb{R}$

Eichler - Shimura - Manin theory of period polynom.

h modular cusp form for G_q of weight k

$$P(x) = \int_0^{i\infty} h(z)\,(z-x)^{k-2}\, dz$$

$P_{\pm}(x)$ even/odd part of $P(x)$ fulfill
Eichler's cocycle relations

$$g(x) + z^{2-k} g(Sx) = 0$$

$$g(x) + j_{2-k}(U,x) g(Ux) + j_{2-k}(U^2,x) g(U^2 x) = 0 \quad (q=3)$$

polynomial solutions: <u>period polynomials</u>

Eichler: 2 − 1 relation between period polyn. and modular cusp forms of weight k

<u>Lewis-Zagier</u>: Theory of period. fcts. for $SL(2,\mathbb{Z})$

u Maass cusp form $\quad \Delta u = \beta(1-\beta)u$

$$R_\zeta(z) = \frac{y}{(x-\zeta)^2 + y^2} \quad \text{Poisson kernel}$$

$$\eta(u,v) = (v\,\partial_y u - u\,\partial_y v)\,dx + (u\,\partial_x v - v\,\partial_x u)\,dy$$

$u, v : \mathbb{C} \to \mathbb{C}$ smooth, $\Delta u = \beta(1-\beta)u$
$\Delta v = \beta(1-\beta)v \implies \eta$ closed 1-form

$$\Phi(\zeta) := \int_0^{i\infty} \eta(u, R_\zeta^\beta)(z) \qquad (v = R_\zeta^\beta(z))$$

$$\implies \Phi(z) - \Phi(z+1) - (z+1)^{-2\beta}\Phi\left(\frac{z}{z+1}\right) = 0 \quad L$$

and vice versa!

Relation to transfer operator for geodesic flow?

case $SL(2,\mathbb{Z})$:

symbolic dynamics of ϕ_t related to Gauss map

$$T_G x = \frac{1}{x} \bmod 1 \quad \text{resp.} \quad x = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}}$$

corresponding transfer operator

$$\mathcal{L}_\beta f(z) = \sum_{n=1}^{\infty} \left(\frac{1}{z+n}\right)^{2\beta} f\left(\frac{1}{z+n}\right)$$

if $\mathcal{L}_\beta f = \pm f \Rightarrow \phi(z) = f(z-1)$ fulfills

Lewis fct.-equation (L) and hence is related

to a Maass wave form for $SL(2,\mathbb{Z})$:

$$\psi_n(z) = \frac{1}{z+n} = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} z \,, \quad \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} \in GL(2,\mathbb{Z})$$

$\Rightarrow$ transfer operator for $SL(2,\mathbb{Z})$ is $\mathcal{L}_\beta^2$!

since

$$\mathcal{L}_\beta^2 f(z) = \sum_{n,m} \left(\frac{1}{z+n}\right)^{2\beta} \left(\frac{1}{m+\frac{1}{z+n}}\right)^{2\beta} f\left(\frac{1}{m+\frac{1}{n+z}}\right)$$

$$\psi_{n,m}(z) = \frac{n+z}{1+mn+mz} = \begin{pmatrix} 1 & n \\ m & 1+mn \end{pmatrix} z$$

hyperbolic $\qquad \cap SL(2,\mathbb{Z})$

Hecke triangle group G_4 :

$$\mathcal{L}_\beta = \begin{pmatrix} \mathcal{L}_\beta^{11} & \mathcal{L}_\beta^{12} \\ \mathcal{L}_\beta^{21} & \mathcal{L}_\beta^{22} \end{pmatrix}$$

$$q_{\pm n}(z) = \frac{1}{\pm n\lambda - z} = \begin{pmatrix} 0 & 1 \\ -1 & \pm n\lambda \end{pmatrix} z = S T_\lambda^{\mp n} z$$

for $n \geq 2$ is $q_{\pm n}$ hyperbolic in G_4

for $n = 1$ is $q_{\pm 1}$ elliptic in G_4

But in the diagonal of $\mathcal{L}_\beta, \mathcal{L}_\beta^2, \mathcal{L}_\beta^3 \ldots$ appear only composition operators $g \mapsto g \circ q(z)$ with q hyperbolic in G_4

<u>Conjecture</u>

The Selberg zeta function

$$Z_S(\beta) = \prod_{k=0}^{\infty} \prod_{\gamma} 1 - e^{-(\beta+k)\ell(\gamma)}$$

(γ periodic orbit of geodesic flow with period $\ell(\gamma)$) for G_4 can be written as

$$Z_S(\beta) = \det(1 - \mathcal{L}_\beta)$$

$\Longrightarrow$ if $\mathcal{L}_\beta g = g \Rightarrow Z_S(\beta) = 0$

$\Longleftrightarrow \rho = \beta(1-\beta)$ is EV of $-\Delta$ for G_4

<u>numerical calculation</u> (F. Strömberg)

$q = 4$, $q = 5$

approximate $\mathcal{L}_\beta$ by matrices

finds eigenvalue 1 for $\mathcal{L}_\beta$ exactly at β-values on $\mathrm{Re}\,\beta = \frac{1}{2}$, for which $\rho = \beta(1-\beta)$ is eigenvalue of Laplacian

- prove conjecture

- relate solutions of Lewis equation for G_4 to Eichler-cocycle relations

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- prove conjecture

- relate solutions of Lewis equation for G_4 to Eichler-cocycle relations

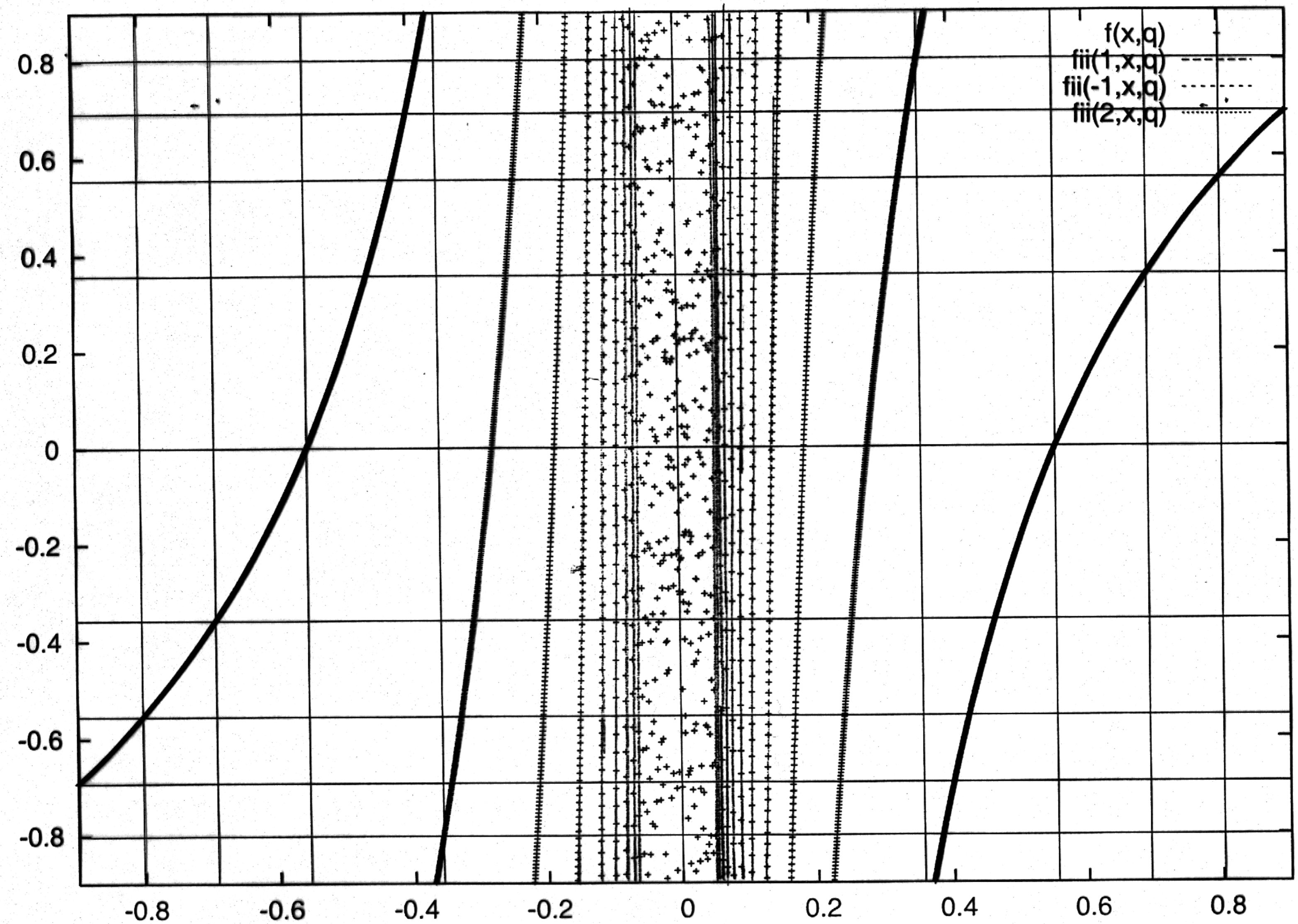
For $q = 4$

$\beta = \frac{1}{2} + iR$	Eigenvalue	N	Sign. on $\Gamma_0(2)$
7.22087197595801289	1.0000037438383 + 0.0000012012564477751235 I	30	Odd/Even
8.92287648699171676	0.9999921534363095 + 0.0000003145741302146341 I	30	Even/Even
9.53369526135354128	1.000000147768505 - 0.0000028706023654 4082 I,	35	Oldform
"	- 0.9999995249668023 - 0.0000022112618554 88584 I	"	"
10.92039200293940837	1.000024923668428 + 0.0000842580618379 4591 I	35	Even/Even
11.31767970146653468	1.0001259531974 + 0.0000893833132343 9804 I	35	Even/Even
12.17300832467966920	0.9999911474172795 + 0.0000022516282078 17527 I	40	Oldform
"	- 0.9999522127629927 + 0.0000440073197684186 I	40	Oldform

Also, for eigenvalues corresponding to a subgroup of G_4 :

$\beta = \frac{1}{2} + iR$	Eigenvalue	N	Signature on $\Gamma_0(2)$
5.41733480684440	- 0.9999999999403617 - 0.000000000108036614720199 I	35	Odd/Odd
8.27366588958613	- 0.9999997605021699 - 0.000000021073714909 40865 I	25	Odd/Odd
10.71270690069776	- 1.00003003761223 + 0.000060564445051672 22 I	30	Odd/Odd
12.09299487507860	- 1.000054628094959 - 0.0000253379602955 1303 I	35	Even/Odd

$f_7^N : I_7 \to I_7$



$q = 7$