Transfer operators , the Selberg zeta function and the Lewis-Zagier theory of period functions [1]

Dieter H. Mayer[2]

Abstract

In these lectures we discuss the transfer operator approach to the Selberg zeta function for the geodesic flow on the unit tangent bundle of a modular surface $\Gamma \setminus \mathbb{H}$. Thereby Γ denotes a subgroup of the full modular group $SL(2,\mathbb{Z})$ of finite index and $\mathbb{H}$ is the hyperbolic plane. It turns out that this function can be expressed in terms of the Fredholm determinant of the classical transfer operator $\mathcal{L}_\beta$ of this flow when appropriately extended to the complex "temperature"-plane β.

The work of J. Lewis and D. Zagier respectively our own work for the full modular group $SL(2,\mathbb{Z})$ has shown that the eigenfunctions of this transfer operator at those β-values which belong to the zeroes of the Selberg function are closely related to automorphic forms for this group, both holomorphic and real analytic. For negative integer values of β they coincide with the period polynomials of Eichler, Shimura and Manin which justifies to call them quite generally period functions.

Of special interest are the eigenfunctions of the transfer operator $\mathcal{L}_\beta$ for β values on the critical line $\Re\beta = \frac{1}{2}$. They are related to the Maass wave forms that means the eigenfunctions of the hyperbolic Laplacian on the surface. In the language of quantum mechanics these functions are the eigenstates of the Schroedinger operator for a free particle moving on such a surface with constant negative curvature. It is known that the corresponding classical system namely the geodesic flow on the surface is highly chaotic. The theory of period functions establishes therefore an interesting relation between classical and quantum physics: the eigenfunctions of the classical transfer operator can be used to construct the eigenstates of the quantum system. This correspondence does not make any use of the trace formulas commonly used in the theory of quantum chaos.

It is well known that this quantum system has infinitely many internal symmetries described by the so called Hecke operators. Through the Lewis-Zagier correspondence for $SL(2,\mathbb{Z})$ these operators act also on the eigenfunctions of the transfer operator and hence are expected to describe some new kind of symmetries for the classical system. It turns out that these operators can be derived from the transfer operators for certain modular subgroups of the full moduar group.

1 Introduction

In these lectures we discuss the so called transfer operator approach to Selberg's zeta function for modular groups and its connection to the Lewis-Zagier theory of period functions. Interest in this function stems from the fact that it has both a dynamical interpretation and belongs at the same time to the family of number theoretic zeta functions like the famous Riemann zeta function. Dynamical zeta functions share many of the properties of the arithmetic zeta functions which in special cases like the Selberg or the Artin-Weil zeta function for algebraic varieties are even dynamical functions.

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[2]Institut für Theoretische Physik, TU-Clausthal, D-38678 Clausthal-Zellerfeld.

E-mail: dieter.mayer@tu-clausthal.de,

Obviously the classical methods of number theory cannot be applied in general in the theory of dynamical zeta functions so that zeta functions like the one by Selberg can serve as a testing ground for the applicability and efficiency of new dynamical methods. One of the few new dynamical methods developed recently for general dynamical zeta functions is the transfer operator method. This operator can be regarded as a generalized Perron-Frobenius operator well known from the ergodic theory of dynamical systems. The transfer operator has its origin in statistical mechanics [**?**] and in quantum field theory and is used there to calculate so called partition functions. These are sums of exponential functions, the so called Boltzmann factors, over finite configuration spaces and are of utmost importance in statistical mechanics. It was D. Ruelle who realized that the zeta functions of both discrete and continuous time dynamical systems can be expressed in the so called thermodynamic formalism [**Ru**] in terms of partition functions of 1-dim. lattice spin systems and hence can be investigated by these transfer operators . His method has been used to determine the analytic properties of some of these functions and to characterize their zeros and poles by the spectrum of the transfer operator. Obviously such a spectral interpretation is one of the big problems also for all arithmetic zeta functions: we mention only the recent work of A. Connes [**Co**] on the Riemann zeta function and more general L-functions of number theory.

Another characteristic property of zeta functions are the functional equations. Unfortunately, even for dynamical zeta functions which have their origin in arithmetics like the Selberg function and for which the functional equations are well known it was not possible up to now to derive these equations through the transfer operator. This is obviously a challenging problem and its solution would add to the inherent importance of the tranfer operator method in the general theory of zeta functions.

The dynamical system underlying the Selberg zeta function is the geodesic flow on the unit tangent bundle of a surface of constant negative curvature. In physical terms this is just a particle moving freely on such a surface with constant kinetic energy. The quantized version of this system is described by the free Schroedinger operator which up to some multiplicative constant involving the mass and Planck's constant h is just the Laplace-Beltrami operator for the hyperbolic metric.

In Selberg's classical approach to his zeta function via the trace formula it is shown that this function has a meromorphic continuation to the entire complex plane for surfaces with finite hyperbolic volume like all the modular surfaces. The zeroes of this function fall into two classes: the trivial zeroes s_n corresponding to the zeroes $\zeta_R(2s_n) = 0$ of the Riemann zeta function ζ_R and the zeroes s_n on the line $\Re\ s = \frac{1}{2}$ corresponding to the eigenvalues $\lambda_n = s_n(1 - s_n)$ of the Laplace-Baltrami operator $-\triangle_{LB}$ on the corresponding surface. These eigenvalues, more precisely their statistics, played a central role in recent years in the theory of quantum chaos [**Sa**]. This theory is concerned with the relation between quantum and classical systems with chaotic behaviour.

The special interest in geodesic flows on surfaces of constant negative curvature stems from the fact that they are well understood both classically and quantum mechanically: their quantum mechanics can be directly related to well known problems of analytic number theory like the existence and explicit determination of Maass wave forms or the existence of Hecke operators. Hence deep results in number theory have found new interpretations in quantum chaos and on the other hand numerical results from quantum chaos have lead to new questions and new approaches to open problems in number theory [**Sa**].

For general dynamical systems and their zeta functions however number theoretic methods are obviously not available and one is forced to develop new approaches which obviously have to be of a more dynamical nature. One such dynamical approach is the transfer operator method [**Ba**]. It has been applied with some success to the class of uniformly hyperbolic systems and is up to now the only way to extend the domain of analyticity of their dynamical zeta functions.

In the first lecture we will discuss in some detail this transfer operator approach to the Selberg zeta function for the modular groups, that means subgroups of the full modular group $SL(\mathbb{Z})$ of finite index. It turns out that for these groups there exists a simple symbolic dynamics which allows an explicit expression for the transfer operator and its Fredholm determinant from which many of the properties of the Selberg function can be read off. Knowing this explicit relation between the Selberg zeta function and the Fredholm determinant of the transfer operator one arrives at a new spectral interpretation of the poles and zeros of the Selberg function.

Through the work of Lewis and Zagier [**Le**], [**LZ**], [**LZ1**] for the full modular group it became clear that certain eigenfunctions of the transfer operator are closely related to the holomorphic forms of even weight respectively the real analytic forms of zero weight for the group $SL(2,\mathbb{Z})$. These eigenfunctions are called period functions by these authors since they generalize the period polynomials of Eichler, Shimura and Manin [**Ei**], [**Za2**] and the period functions of Zagier for holomorphic cusp respectively non-cusp forms.
In the second lecture we will discuss this theory of Lewis and Zagier and their main result establishing a $1-1$ correspondence between the eigenfunctions of the transfer operator L_β with eigenvalue $\lambda = \pm 1$ on the line $Re\,\beta = \frac{1}{2}$ and the Maass wave forms for $SL(2,\mathbb{Z})$. The main ingredient for this result is an extension of a converse theorem of Maass for $L-$functions to the period functions.
An important internal symmetry of the Laplace-Beltrami operator for the modular groups manifests itself through the existence of an infinite family of selfadjoint operators commuting among themselves and the Laplace operator. These are the Hecke operators [**AL**]. Through the Lewis-Zagier correspondence these operators can be transfered to the space of period functions. The resulting operators have been determined in the case $SL(2,\mathbb{Z})$ for the period polynomials by Choie and Zagier [**CZ**] and recently for the period functions by T. Muehlenbruch [**Mu**]. By using the transfer operators for the Hecke congruence subgroups $\Gamma_0(n) \subset SL(2,\mathbb{Z})$ and certain of their eigenfunctions we could derive for any of these groups an infinite family of linear operators [**?**] mapping eigenspaces of the transfer operator into themselves. For the group $\Gamma_0(1) = SL(2,\mathbb{Z})$ these operators are closely related to the Hecke operators of Muehlenbruch so that we expect that also our operators for $\Gamma_0(n)$ are more or less the Hecke operators for these groups acting on the space of period functions.

An open problem for these and more general subgroups Γ of the full modular group is a converse theorem establishing for any eigenfunction of the transfer operator for Γ with eigenvalue $\lambda = \pm 1$ on the line $Re\,\beta = \frac{1}{2}$ the existence of exactly one Maass wave form for this group. The main problem here seems to be the corresponding converse theorem for $L-$functions which to our knowledge has been shown only within the Jacquet-Langlands adelic approach. One would rather need a classical form of this result to connect it with the theory of period functions. How one can associate to a Maass wave form a period function for the modular groups on the other hand is more or less well understood.
In the third lecture we will discuss certain eigenfunctions of the transfer operator for $\Gamma_0(n)$ and derive the corresponding Hecke-like operators for this group.
Most of the results discussed in these lectures have their origin in published work by John Lewis and Don Zagier (see References for details) and in our own work with Cheng Chang,

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2 The geodesic flow on a modular surface $\Gamma \setminus \mathbb{H}$

We denote by $\mathbb{H}$ the Poincare upper half plane

$$\mathbb{H} = \{z \in \mathbb{C} : z = x + iy, y > 0\} \tag{1}$$

equipped with the hyperbolic metric

$$ds^2 = y^{-2}\left(dx^2 + dy^2\right) \tag{2}$$

The group of isometries of this space is

$$PSL\,(2,\mathbb{R}) = SL\,(2,\mathbb{R})\,/_{\{+1,-1\}} \tag{3}$$

with $SL\,(2,\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{R},\ \det g = 1 \right\}$.

This group acts on $\mathbb{H}$ through the Moebius transformations

$$gz = \frac{az+b}{cz+d} \quad ,g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL\,(2,\mathbb{R}) \tag{4}$$

The discrete subgroup $SL\,(2,\mathbb{Z}) \subset SL\,(2,\mathbb{R})$ with

$$SL\,(2,\mathbb{Z}) = \{g \in SL\,(2,\mathbb{R})\,, \quad a,b,c,d \in \mathbb{Z}\}$$

is called the full modular group.

A subgroup $\Gamma \subset SL\,(2,\mathbb{Z})$ of finite index $[SL\,(2,\mathbb{Z}) : \Gamma] < \infty$ is called a modular group.

When acting on the hyperbolic 2-space $\mathbb{H}$ we identify the groups Γ and $\mathrm{P}\Gamma = \Gamma/\{+1,-1\}$ if $-1 \in \Gamma$.

The full modular group $SL\,(2,\mathbb{Z})$ has two generators for which we take the two elements Q and T with

$$Qz = -\frac{1}{z} \quad \text{and} \quad Tz = z+1 \quad . \tag{5}$$

They fullfill the relations

$$Q^2 = (QT)^3 = 1 \quad .$$

The orbit space $\Gamma \setminus \mathbb{H}$ of the modular group Γ is called a modular surface where $h_1, h_2 \in \mathbb{H}$ are identified as

$$h_1 \sim h_2 \Leftrightarrow h_2 = gh_1$$

for some $g \in \Gamma$.

Is is well known that these modular surfaces are not compact but have finite hyperbolic area.

A closed region $F_\Gamma \subset \mathbb{H}$ is called a fundamental domain for the group Γ if for all $z \in \mathbb{H}$ there exists $g \in \Gamma$ and $z_0 \in F_\Gamma$ with $z = gz_0$. If $z_1, z_2 \in F_\Gamma$ and $z_1 = gz_2$ for some $g \in \Gamma$, $g \neq id$, then z_1, z_2 are both in the boundary ∂F_Γ of F_Γ .

Obviously the modular surface $\Gamma \setminus \mathbb{H}$ can be identified with the space $F_\Gamma / \sim$ where points in F_Γ equivalent under Γ are identified.

Denote by $\overline{\mathbb{H}} := \mathbb{H} \cup \infty \cup \mathbb{Q}$ the compactified hyperbolic surface where $\mathbb{Q}$ denotes the rational numbers. The points ∞ and $\frac{p}{q} \in \mathbb{Q}$ are called cusps.

Extending the action of the modular group Γ to the real line $\{z = x + iy : y = 0\}$ these cusps decay into several equivalence classes. A cusp of the modular group Γ is just one such equivalence class. Geometrically these are the points in the boundary of the modular surface at infinity.

The geodesic lines in the hyperbolic 2-space are the half circles based on the real axis. For infinite diameter these are the lines parallel to the imaginary axis $\Re z = 0$.

An oriented geodesic γ will be denoted by the two base points $\gamma_{-\infty}, \gamma_{+\infty} \in \mathbb{R} \cup \{\infty\}$ on the real line: we hence write $\gamma = (\gamma_{-\infty}, \gamma_{+\infty})$. Denote next by $\mathcal{A}$ the following set of geodesics

$$(6) \qquad \mathcal{A} := \{\gamma = (\gamma_{-\infty}, \gamma_{+\infty}) : 0 < \mid \gamma_{-\infty} \mid \leq 1 \leq \mid \gamma_{+\infty} \mid, \quad \gamma_{-\infty}\gamma_{+\infty} < 0\}.$$

It is not too difficult to see that for any geodesic γ in $\mathbb{H}$ there exists an element $g \in SL(2, \mathbb{Z})$ with $g\gamma \in \mathcal{A}$.

This shows that any geodesic in $SL(2, \mathbb{Z}) \setminus \mathbb{H}$ has a representative in the set $\mathcal{A}$.

Denote by $\pi^\Gamma : \mathbb{H} \to \Gamma \setminus \mathbb{H}$ the covering map $\pi^\Gamma(z) = \Gamma z$.

Obviously for every geodesic γ in $\mathbb{H}$ $\pi^\Gamma \gamma$ defines a geodesic in $\Gamma \setminus \mathbb{H}$: if $\gamma_1 = g\gamma_2$ for some $g \in \Gamma$ then $\pi^\Gamma \gamma_1 = \pi^\Gamma \gamma_2$. The geodesic flow ϕ_t on $\mathbb{H}$ is the flow along the geodesics with velocity v with $\mid v \mid = 1$.

If $S\mathbb{H}$ denotes the unit tangent bundle of $\mathbb{H}$ then the geodesic flow $\phi_t : S\mathbb{H} \to S\mathbb{H}$ is given as

$$(7) \qquad \phi_t(z, v) = (z_t, v_t)$$

where z_t lies the hyperbolic distance t from z on the geodesic through z with unit tangent vector v and v_t is the unit tangent vector at this geodesic in the point z_t .

Obviously for the projection $\pi^\Gamma_* : S\mathbb{H} \to S(\Gamma \setminus \mathbb{H})$ the induced flow $\pi^\Gamma_* \phi_t (\pi^\Gamma_*)^{-1} : S(\Gamma \setminus \mathbb{H}) \to S(\Gamma \setminus \mathbb{H})$ is the geodesic flow on the unit tangent bundle of the modular surface $\Gamma \setminus \mathbb{H}$. We will denote this geodesic flow by ϕ^Γ_t .

It is well known that the geodesic flow on surfaces of constant negative curvature is uniformly hyperbolic or Anosov and hence allows a description in terms of symbolic dynamics. The surprizing fact for the modular surfaces is that this symbolic dynamics can be described in explicit form, whereas for general Anosov systems or Axiom-A systems only existence of such a description is known.

2.1 Symbolic dynamics of the geodesic flow for modular surfaces

If $\Gamma \subset SL(2, \mathbb{Z})$ has finite index $\mu = [SL(2, \mathbb{Z}) : \Gamma]$ then $M_\Gamma := \Gamma \setminus \mathbb{H}$ is an in general ramified μ-fold covering of the modular surface $M := SL(2, \mathbb{Z}) \setminus \mathbb{H}$. Ddenote then by $\pi_\Gamma : M_\Gamma \to M$ the covering map

$$(8) \qquad \pi_\Gamma(\Gamma h) = SL(2, \mathbb{Z})\, h$$

Hence also $\pi_{\Gamma *}: SM_\Gamma \to SM$ maps the orbits of the geodesic flow on SM_Γ on to the orbits of the geodesic flow on SM. To get the symbolic dynamics of the geodesic flow on SM_Γ one only has to lift the symbolic dynamics of this flow on SM to SM_Γ. The symbolic dynamics for the flow $\phi_t : SM \to SM$ has been determined by Series [**Se**]

To get a Poincare section for this flow consider the set

(9) $$S = \{z = i\varrho : \varrho \geq 1\} \subset \mathbb{H} \quad .$$

To any $z \in S$ attach the tangent vectors

$$C_z = \{v \in T_z\mathbb{H} : |v| = 1 \quad \text{such that either } \gamma(z,v) \text{ or } Q\gamma(z,v) \in A\}$$

where $\gamma(z,v)$ denotes the geodesic determined by z and v .
Then the set

(10) $$X = \bigcup_{\substack{z \in S \\ v \in C(z)}} \pi_*(z,v) \subset SM$$

is a Poincare section for the geodesic flow on SM. Indeed almost all orbits of ϕ_t hit X transversally. The map $\varrho : X \to \mathcal{A}$ defined as

(11) $$\varrho(\pi_*(z,v)) = \gamma(z,v)$$

respectively

(12) $$\varrho(\pi_*(z,v)) = Q\gamma(z,v)$$

is one-to-one up to the two geodesics $\gamma\pm$ with

$$\gamma\pm = (\pm 1, \mp 1)$$

which are mapped onto each other by Q and hence correspond to the same point in X .

An orbit $\phi_t(z,v)$ in SM is periodic iff the geodesic $\gamma(z,v)$ in $\mathbb{H}$ is fixed by an element $g \in SL(2,\mathbb{Z})$, that means for $\gamma = (\gamma_{-\infty}, \gamma_{+\infty})$

(13) $$g\gamma_{\pm\infty} = \gamma_{\pm\infty} \quad .$$

If $\pi_*(z,v) \in X$ is a point on this periodic orbit then $\pi_*(z,v)$ is a periodic point of the Poincare map $P : X \to X$ with

(14) $$Px = \phi_{r(x)}x$$

where $r : X \to \mathbb{R}_+$ denotes the recurrence time for the points in the Poincare section X , that means

(15) $$r(x) = \min\{t > 0 : \phi_t(x) \in X \text{ for } x \in X\} \quad .$$

To determine this Poincare map consider for $x \in X, x = (z,v)$ the orbit $\gamma = (\gamma_{-\infty}, \gamma_{+\infty}) \in \mathcal{A}$ with $\varrho(\pi_+(z,v)) = \gamma$.

Set $\epsilon = 1$ if $\gamma_{-\infty} < 0 < \gamma_{+\infty}$ respectively

$\epsilon = -1$ if $\gamma_{+\infty} < 0 < \gamma_{-\infty}$. Then the oriented orbit $\gamma = (\gamma_{-\infty}, \gamma_{+\infty})$ can be identified with

$$(16) \qquad \gamma = (x_1, x_2, \epsilon) \ , \ (x_1, x_2) \in I_2 = \{0 \le x_1 \le 1; \ 0 \le x_2 \le 1\}, \epsilon \in \{1, -1\},$$

such that $\gamma_{-\infty} = -\epsilon x_2, \ \gamma_{+\infty} = \epsilon x_1^{-1}$.
If $n < \gamma_{+\infty} < n+1$ then the orbit γ' defined as

$$\gamma' = QT^{-n}\gamma$$

belongs to the set $\mathcal{A}$ since $\gamma' = \left(\gamma'_{-\infty}, \gamma'_{+\infty}\right)$ with $0 < \gamma'_{-\infty} < 1$ and $\gamma'_{+\infty} < -1$. This orbit hence determines again a point $x' \in X$ and hence $x' = Px$.
In the parametrization $\gamma = (x_1, x_2, \epsilon)$ of the elements γ in $\mathcal{A}$ one finds for $\gamma' = (x'_1, x'_2, \epsilon')$

$$(17) \qquad \left(x'_1, x'_2, \epsilon'\right) = P\left(x_1, x_2, \epsilon\right) = \left(\frac{1}{x_1} - n, \ \frac{1}{x_2 + n}, \ -\epsilon\right) \text{ with } n = \left[\frac{1}{x_1}\right] \quad .$$

It is not difficult to convince oneself that exactly the same formula holds also in the case $\gamma_{+\infty} < -1$.
If $T_G : I \to I$ denotes the Gauss map

$$(18) \qquad T_G x_1 = \frac{1}{x_1} \text{mod } 1$$

on the unit interval $I = \{x_1 : 0 < x_1 < 1\}$ then the Poincare map for the geodesic flow on SM can be written as

$$(19) \qquad P\left(x_1, x_2, \epsilon\right) = \left(T_G x_1, \frac{1}{x_2 + \left[\frac{1}{x_1}\right]}, -\epsilon\right)$$

Hence P can be considered a natural extension of the Gauss map of the unit interval by the group $\mathbb{Z}_2$. Obviously the closed orbits of $\phi_t : SM \to SM$ can be characterized now by the periodic points of the Poincare map P in (19). Because of the twist in the varible ϵ there exist only periodic points of even period $2k$, hence the coordinate x_1 of a point x in the section X must be a periodic point under the Gauss map of period $2k$: But $T_G^{2k} x_1 = x_1$ iff x_1 has a periodic continued fraction expansion of period $2k$, that means

$$x_1 = [\overline{n_1, \cdots, n_{2k}}] \quad , \quad n_i \in \mathbb{N} \quad 1 \le i \le 2k$$

where $[\overline{n_1, \cdots, n_{2k}}]$ denotes the continued fraction $[n_1, n_2, \cdots]$ with $n_i = n_{i+2k}$ for all $i \in \mathbb{N}$.
A simple calculation then shows that $x = (x_1, x_2, \epsilon)$ is a periodic point of P of period $2k$ iff $x_2 = [\overline{n_{2k}, n_{2k-1}, \cdots, n_1}]$. Hence the periodic points of the Poincare map P of the geodesic flow $\phi_t : SM \to SM$ are in $1-1$ correspondence with the set

$$\{([\overline{n_1, \cdots, n_{2k}}], \epsilon) \ , \ n_i \in \mathbb{N}, \ k \in \mathbb{N}, \ \epsilon = \pm 1\} \quad .$$

To determine the periods of the corresponding closed orbits one has to determine their recurrence times with respect to the Poincare section X . These are given by the geodesic distances between subsequent hits of the Poincare section by the orbit: if $\gamma = (\gamma_{-\infty}, \gamma_{+\infty})$ with $n \le \gamma_{+\infty} < n+1$ then denote by $z_1 = i$ the intersection point of γ with the line

$\Re\, z = 0$ and by $z_2 = n + i\eta$ the intersection of γ with the line $\Re\, z = n$. Then the geodesic distance between these two points is given by [**dMS**]

$$r(x) = \log \frac{|\gamma_{-\infty} - \eta|\, |\gamma_{\infty} - \xi|}{|\gamma_{-\infty} - \xi|\, \gamma_{\infty} - \eta} \tag{20}$$

when γ hits X in the point $x = (z_1, v_1)$ and v_1 is the tangent vector to γ in z_1 .

If the base points $\gamma_{\pm\infty}$ of γ are given as

$$\gamma_{-\infty} = -\,[n_0, n_{-1}, n_{-2}, \cdots]\,,\ \gamma_{+\infty} = n_1 + [n_2, n_3, \cdots] \tag{21}$$

one finds

$$r(x) = -\frac{1}{2} \log\left([n_1, n_2, \cdots]\,[n_0, n_{-1}, n_{-2}, \cdots]\,[n_2, n_3, \cdots]\,[n_1, n_0, n_{-1}, \cdots]\,.\right) \tag{22}$$

For a periodic orbit γ with $\gamma_\infty^{-1} = [\overline{n_1, \ldots, n_{2k}}]$ and $\gamma_{-\infty} = -\,[\overline{n_{2k}, n_{2k-1}, \ldots, n_1}]$ one then derives for the period $l(\gamma)$

$$\begin{aligned} l(\gamma) = -\tfrac{1}{2} \textstyle\sum_{r=1}^{2k} \log \{ & [\overline{n_r, n_{r+1}, \ldots, n_{2k}, n_1, \ldots, n_{r-1}}]\, [\overline{n_{r-1}, n_{r-2}, \ldots, n_1, n_{2k}, \ldots, n_r}] \cdot \\ & \cdot [\overline{n_{r+1}, n_{r+2}, \ldots, n_{2k}, n_1, \ldots, n_r}]\, [\overline{n_r, n_{r-1}, \ldots, n_1, n_{2k}, \ldots n_{r+1}}] \quad . \end{aligned}$$

Using a result by Legendre, namely

$$\prod_{r=1}^{2k} [\overline{n_r, n_{r+1}, \ldots n_{2k}, n_1, \ldots, n_{r-1}}] = \prod_{r=1}^{2k} [\overline{n_r, n_{r-1}, \ldots, n_1, n_{2k}, \ldots, n_{r+1}}]$$

we finally get for the period $l(\gamma)$ of the periodic orbit γ :

$$l(\gamma) = \log \prod_{r=1}^{2k} [\overline{n_r, n_{r+1}, \ldots, n_{2k}, n_1, \ldots, n_{r-1}}] \tag{23}$$

2.2 The Poincare map for the geodesic flow on M_Γ

If $\Gamma \subset SL\,(2,\mathbb{Z})$ has finite index $\mu = [SL\,(2,\mathbb{Z}) : \Gamma]$ consider the coset decomposition

$$SL\,(2,\mathbb{Z}) = \bigcup_{i=1}^{\mu} \Gamma g_i \tag{24}$$

with $g_i \in SL\,(2,\mathbb{Z})$, $1 \leq i \leq \mu$. Define next for Γ the set of oriented geodesics $\mathcal{A}_\Gamma$ as

$$\mathcal{A}_\Gamma := \bigcup_{i=1}^{\mu} g_i\ A, \tag{25}$$

where $\mathcal{A}$ was defined in (6). If $\pi_\Gamma : M_\Gamma \to M$ is the natural projection of M_Γ onto M and $\pi_{\Gamma*}$ its extension to the unit tangent bundles SM_Γ and SM then a Poincare section $X_\Gamma \subset SM_\Gamma$ for the geodesic flow on SM_Γ in obviously given by

$$X_\Gamma = (\pi_{\Gamma*})^{-1}\, X \tag{26}$$

where X is the Poincare section for the geodesic flow on SM as defined in (10).

There is again a $1-1$ correspondence between X_Γ and the set $\mathcal{A}_\Gamma$ in (25) besides a finite number of geodesics in $\mathcal{A}_\Gamma$ corresponding to the same point in X_Γ if Q belongs to Γ . To determine the Poincare map $P_\Gamma : X_\Gamma \to X_\Gamma$ take a point $x \in X_\Gamma$ corresponding to the geodesic $\gamma \in \mathcal{A}_\Gamma$. For $g_i \in SL\,(2,\mathbb{Z})$ with $g_i^{-1}\gamma \in A$ we know that $QT^{-n\epsilon}g_i^{-1}\gamma \in A$ when $g_i^{-1}\gamma = (\gamma_{-\infty}, \gamma_{+\infty}, \epsilon)$ and $\gamma_{+\infty} = \epsilon x_1^{-1}$ with $n = \left[\frac{1}{x_1}\right]$.
In general however $QT^{-n\epsilon}g_i^{-1}$ does not belong to Γ and hence the orbit $QT^{-n\epsilon}g_i^{-1}\gamma$ does not define the same orbit on M_Γ . Since however

$$SL\,(2,\mathbb{Z}) = \bigcup_{i=1}^{\mu} g_i^{-1}\Gamma \tag{27}$$

there exists a unique $g_j \in SL\,(2,\mathbb{Z})$, $1 \le j \le \mu$ such that

$$g_j Q T^{-n\epsilon} g_i^{-1} \in \Gamma \quad . \tag{28}$$

But then $g_j QT^{-n\epsilon}g_i^{-1}\gamma \in \mathcal{A}_\Gamma$ describes the same orbit in SM_Γ . Hence it defines also a unique point $x' \in X_\Gamma$ which is just the image of $x \in X_\Gamma$ under the Poincare map P_Γ .

Any geodesic γ in $\mathcal{A}_\Gamma$ can be characterized as

$$\gamma = (x_1, x_z, \epsilon, \Gamma g_i)$$

if $g_i^{-1}\gamma \in A$ and $g_i^{-1}\gamma = (x_1, x_2, \epsilon)$ as defined in (16). The Poincare map $P_\Gamma : X_\Gamma \to X_\Gamma$ then has the explicit form

$$P_\Gamma\,(x_1, x_2, \epsilon, \Gamma g_i) = \left(T_G x_1, \frac{1}{x_2 + \left[\frac{1}{x_1}\right]}, -\epsilon,\ \Gamma g_j \right) \tag{29}$$

with g_j uniquely determined by

$$\Gamma g_i T^{n\epsilon} Q = \Gamma g_j$$

Obviously $P_\Gamma : X_\Gamma \to X_\Gamma$ can be considered an extension of the map $P : X \to X$ by the set $\Gamma \setminus SL\,(2,\mathbb{Z})$. For $\Gamma = SL\,(2,\mathbb{Z})$ this set consists just of one point and X_Γ can be identified with X .

3 The transfer operator for the geodesic flow on SM_Γ

In this section we will derive the transfer operator for a general subgroup $\Gamma \subset SL(2,\mathbb{Z})$ with finite index $\mu = [SL(2,\mathbb{Z}) : \Gamma]$. For this we recall the general definition of the Ruelle transfer operator $\mathcal{L}_T$ for a map $T : M \to M$ and an observable $A : M \to \mathbb{C}$ [**Ba**] where we do not specify at the moment the space of functions f on which the operator's action is well defined:

$$(\mathcal{L}_T f)(x) = \sum_{y \in T^{-1}x} \exp A(y) f(y). \tag{30}$$

For uniformly hyperbolic systems like Anosov or Axiom A systems it is known from the work of R. Bowen and D. Ruelle (see [**Bo**], [**Ru**]) that the observable

$$A(x) = -log|DT_{ex}(x)| \tag{31}$$

is of special interest for the ergodic theory of these systems, where T_{ex} denotes the restriction of the map T to its expanding directions. In the case of the Poincare map $P_\Gamma : X_\Gamma \to X_\Gamma$ one then finds

$$P_{\Gamma,ex}(x_1, \epsilon, \Gamma g_i) = (T_G x_1, -\epsilon, \Gamma g_j) \tag{32}$$

where g_j is given as in (28). Here we used the fact that the expanding direction of P_Γ consists of 2μ copies of the x_1 coordinate. But then one sees that

$$P^{-1}_{\Gamma,ex}(x_1, \epsilon, \Gamma g) = \left\{ (\frac{1}{x+n}, -\epsilon, \Gamma g Q T^{n\epsilon}), n \in \mathbb{N} \right\}. \tag{33}$$

The generalized transfer operator $\mathcal{L}^\Gamma_\beta$ for the geodesic flow $\Phi^\Gamma_t : SM_\Gamma \to SM_\Gamma$ hence has the form

$$(\mathcal{L}^\Gamma_\beta f)(x, \epsilon, \Gamma g) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^{2\beta}} f(\frac{1}{x+n}, -\epsilon, \Gamma g Q T^{n\epsilon}), \tag{34}$$

since $-\log |DP_{\Gamma,ex}(x, \epsilon, \Gamma g)| = \log x^2$. We have introduced in definition (30) of the transfer operator a complex parameter β which in statistical mechanics describes the inverse temperature. In the space of vector valued functions

$$\underline{f} : I \times \mathbb{Z}_2 \to \mathbb{C}^\mu \tag{35}$$

with $I = [0, 1]$ the unit interval, the operator $\mathcal{L}^\Gamma_\beta$ in (34) induces the operator $\mathcal{L}^{\chi_\Gamma}_\beta$ with

$$\left(\mathcal{L}^{\chi_\Gamma}_\beta \underline{f}\right)(x, \epsilon) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^{2\beta}} \chi_\Gamma(Q T^{n\epsilon})\, \underline{f}(\frac{1}{x+n}, -\epsilon), \tag{36}$$

where $\chi_\Gamma : SL(2,\mathbb{Z}) \to \mathrm{End}\,\mathbb{C}^\mu$ denotes the representation of the group $SL(2,\mathbb{Z})$ induced from the trivial 1-dimensional representation of the group Γ: if $\omega_\Gamma : SL(2,\mathbb{Z}) \to \mathbb{R}$ is the characteristic function of the group $\Gamma \subset SL(2,\mathbb{Z})$ with

$$\omega_\Gamma(g) = \begin{cases} 1 & \text{if } g \in \Gamma \\ 0 & \text{if } g \notin \Gamma \end{cases}$$

then the representation $\chi_\Gamma : SL(2, \mathbb{Z} \to \mathrm{End}\,\mathbb{C}^\mu$ is defined as

$$\chi_\Gamma(g) = \left(\omega_\Gamma(g_i g g_j^{-1})\right)_{1 \le i,j \le \mu}. \tag{37}$$

One realizes immediately that the matrix $\chi_\Gamma(g)$ is for any $g \in SL(2,\mathbb{Z})$ a permutation matrix and hence there exists for any $g \in SL(2,\mathbb{Z})$ a number $r > 0$ with $\chi_\Gamma(g^r) = id$. In general the representation χ_Γ is not irreducible. If

$$\chi_\Gamma = \oplus_i \chi_i$$

is its decomposition into the irreducible components χ_i then also the transfer operator $\mathcal{L}^{\chi_\Gamma}_\beta$ in (36) can be decomposed as

$$\mathcal{L}^{\chi_\Gamma}_\beta = \bigoplus \mathcal{L}^{\Gamma,\chi_i}_\beta. \tag{38}$$

Well understood is the case when Γ is a normal subgroup of $SL(2,\mathbb{Z})$. Then the representation χ_Γ is isomorphic to the right regular representation of the factor group $\Gamma \setminus SL(2,\mathbb{Z})$ and hence $\chi_\Gamma(h) = id$ for all $h \in \Gamma$. Furthermore one has

$$\chi_\Gamma = \oplus_i n_i \chi_i \tag{39}$$

where the sum is over all irreducible unitary representations χ_i of the group $\Gamma \setminus SL(2,\mathbb{Z})$ of dimension n_i. An example of such a normal subgroup is the principal congruence subgroup

$$\Gamma(2) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mod 2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

In this case $\Gamma(2) \setminus SL(2,\mathbb{Z})$ is isomorphic to S_3, the permutation group on three elements. Hence one gets [**CM**]

$$\chi_{\Gamma(2)} = \chi_1 \oplus \chi_{-1} \oplus \chi_2 \oplus \chi_2 \tag{40}$$

where χ_1 and χ_{-1} are the two 1-dimensional representations and χ_2 is the 2-dimensional irreducible representation of S_3. For the Hecke congruence subgroup

$$\Gamma = \Gamma_0(2) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c \mod 2 = 0\}$$

on the other hand one finds [**CM**] $\mu = [SL(2,\mathbb{Z}) : \Gamma_0(2)] = 3$ and

$$\chi_{\Gamma_0(2)} = \chi_1 \oplus \chi_2. \tag{41}$$

Hence in an appropriate basis the operators $\mathcal{L}_\beta^{\Gamma(2)}$ and $\mathcal{L}_\beta^{\Gamma_0(2)}$ have certain eigenfunctions and eigenvalues in common. Indeed all the eigenfunctions and eigenvalues of the operator $\mathcal{L}_\beta^{\Gamma_0(2)}$ show up also for the operator $\mathcal{L}_\beta^{\Gamma(2)}$. This obviously is related to the fact that $\Gamma(2)$ is a subgroup of $\Gamma_0(2)$.

Up to now we did not care about the spaces on which the above transfer operator really are well defined. This we will discuss in the next section.

4 Spectral properties of the transfer operators

For $D = \{z :\mid z-1 \mid < \frac{3}{2}\}$ denote by $B(D)$ the Banach space

$$B(D) = \{f : D \to \mathbb{C} : f \text{ holomorphic and continuous on } \overline{D}\} \tag{42}$$

with the sup norm $\parallel f \parallel = sup_{z\in\overline{D}} \mid f(z) \mid$ and by B the direct sum

$$B = \bigoplus_{i=1}^{2\mu} B(D). \tag{43}$$

Obviously the transfer operator $\mathcal{L}_\beta^\Gamma$ is then well defined on this Banach space for all β-values with $\Re\beta > \frac{1}{2}$ and depends in this domain holomorphically on β. Choose next any $\kappa \in \mathbb{N} \cup \{0\}$. Since there exists to the element $T \in SL(2,\mathbb{Z})$ a number $r > 0$ with $\chi_\Gamma(T^r) = id$ one finds by a simple Taylor expansion

$$\left(\mathcal{L}_\beta^{\chi_\Gamma} \underline{f}\right)(z,\epsilon) = \sum_{n=1}^{\infty}\sum_{m=1}^{r}\left(\frac{1}{z+m+r(n-1)}\right)^{2\beta}\chi_\Gamma(QT^{m\epsilon})$$
$$\cdot \quad \left\{\underline{f}\left(\frac{1}{z+m+r(n-1)},-\epsilon\right) - \sum_{l=0}^{\kappa}\frac{\underline{f}^{(l)}(0,-\epsilon)}{l!}\frac{1}{(z+m+r(n-1))^l}\right\}$$
$$+ \quad \sum_{l=0}^{\kappa}\frac{1}{r^{2\beta+l}}\sum_{m=1}^{r}\chi_\Gamma(QT^{m\epsilon})\frac{\underline{f}^{(l)}(0,-\epsilon)}{l!}\zeta\left(2\beta+l,\frac{z+m}{r}\right),$$

where $\zeta(s,z)$ denotes the Hurwitz zeta function.

$$\zeta(s,z) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^s}. \tag{44}$$

Since $\kappa \in \mathbb{N}\cup\{0\}$ was arbitrary and the Hurwitz zeta function is meromorphic in the entire s-plane with the only pole at $s=1$ one sees from the above expression that the transfer operator $\mathcal{L}_\beta^{\chi_\Gamma}$ has a meromorphic extension to the entire complex β-plane with possible poles only at $\beta=\beta_l=\frac{1-l}{2}$, $l=0,1,2,\ldots.$. From the above expression one also sees that $\mathcal{L}_\beta^{\chi_\Gamma}$ can be written as

$$\mathcal{L}_\beta^{\chi_\Gamma} = \mathcal{L}_\beta^{\chi_\Gamma,\kappa} + \mathcal{A}_\beta^{\chi_\Gamma,\kappa} \tag{45}$$

with $\mathcal{A}_\beta^{\chi_\Gamma,\kappa}$ a finite rank operator mapping onto the linear span of Hurwitz functions. For $\{\underline{u}_i\}_{1\le i\le 2\mu}$ an orthonormal basis of $\mathbb{C}^{2\mu}$ the elements $\underline{\underline{f}}$ in B can be written as

$$\underline{\underline{f}} = \sum_{i=1}^{2\mu}\underline{e}_i\otimes f_i \tag{46}$$

with $f_i \in B(D)$. Then one finds

$$\left(\mathcal{L}_\beta^{\chi_\Gamma,\kappa}\underline{\underline{f}}\right)(z) = \sum_{i=1}^{2\mu}\sum_{m=1}^{r}\chi_m\,\underline{e}_i\otimes\mathcal{L}_{\beta,m}^{\kappa}\,f_i(z) \tag{47}$$

with

$$\chi_m = \begin{pmatrix} 0 & \chi_\Gamma(QT^m) \\ \chi_\Gamma(QT^{-m}) & 0\end{pmatrix} \tag{48}$$

and

(49)
$$(\mathcal{L}_{\beta,m}^{\kappa}f)(z) = \sum_{n=1}^{\infty}\frac{1}{(z+m+r(n-1))^{2\beta}}\left[f\left(\frac{1}{z+m+r(n-1)}\right)-\sum_{l=0}^{\kappa}\frac{f^{(l)}(0)}{l!}\frac{1}{(z+m+r(n-1))^l}\right]$$

Hence we find

$$\mathcal{L}_\beta^{\Gamma,\kappa} = \sum_{m=1}^{r}\chi_m\otimes\mathcal{L}_{\beta,m}^{\kappa}, \tag{50}$$

which shows that the operator $\mathcal{L}_\beta^{\Gamma,\kappa}$ is nuclear of order zero [**Gr**] and hence trace class for all $\beta\neq\beta_l$ since the operators $\mathcal{L}_{\beta,m}^{\kappa}$ have this property (see [**CM3**]). Hence also the transfer operators $\mathcal{L}_\beta^{\chi_\Gamma}$ define a meromorphic family of trace class operators in the complex β-plane.

5 The Selberg zeta function

For the geodesic flow $\Phi_t : SM \to SM$ on the unit tangent bundle of the modular surface $M = SL(2,\mathbb{Z})\setminus\mathbb{H}$ and $\chi : SL(2,\mathbb{Z}) \to End\,(V)$ an arbitrary finite dimensional representation of the full modular group $SL(2,\mathbb{Z})$ the dynamical zeta function $\zeta(\beta,\chi)$ of Ruelle and Smale is defined as [**Ru2**]

$$\zeta(\beta,\chi) = \prod_\gamma \det(1-\chi(\sigma_\gamma)e^{-\beta l(\gamma)})^{-1} \tag{51}$$

where the product is over the periodic orbits γ of Φ_t with prime period $l(\gamma)$ and σ_γ is a representative of the hyperbolic conjugacy class in $SL(2,\mathbb{Z})$ fixing a geodesic lift $\tilde{\gamma}$ of γ in $\mathbb{H}$, that means $\sigma_\gamma\tilde{\gamma} = \tilde{\gamma}$ with $\pi\tilde{\gamma} = \gamma$. For the group $SL(2,\mathbb{Z})$ we have seen that the periodic orbits γ have lifts $\tilde{\gamma}$ with $\tilde{\gamma} = (\tilde{\gamma}_{-\infty},\tilde{\gamma}_\infty) = (x_1,x_2,\epsilon)$ with $x_1 = [\overline{n_1,....,n_{2k}}]$, $x_2 = [\overline{n_{2k},....,n_1}]$ and $\epsilon = \pm 1$ for some $k \in \mathbb{N}$. Obviously the element $\sigma_\gamma = QT^{n_{2k}\epsilon}QT^{-n_{2k-1}\epsilon}\cdot \ldots \cdot QT^{-n_1\epsilon}$ fixes this geodesic $\tilde{\gamma}$ since $\sigma_\gamma\tilde{\gamma}_{\pm\infty} = \tilde{\gamma}_{\pm}\infty$. The period $l(\gamma)$ of the corresponding periodic orbit γ is given in (**??**) as

$$l(\gamma) = \sum_{s=0}^{2k-1} r(P^s(x)) = \log \prod_{s=1}^{2k} [\overline{n_s,n_{s+1},....,n_1,..,n_{s-1}}]. \tag{52}$$

To connect the dynamical zeta function for the geodesic flow with the transfer operator one has to consider the dynamical zeta function for the Poincare map $P : X \to X$. Indeed if $T : M \to M$ is any map of a manifold M with $\mid FixT^m \mid := \mid \{x \in M : T^m x = x\} \mid < \infty$ for all m and $\mathbb{A} : M \to End\,(\mathbb{C}^\mu)$ a map of M into the endomorphisms of $\mathbb{C}^\mu$ the dynamical zeta function $\zeta_R(z,\mathbb{A})$ of Ruelle for the map T is defined as [**Ru1**]

$$\zeta_R(z,\mathbb{A}) := \exp \sum_{n=1}^\infty \frac{z^n}{n} Z_n(T,\mathbb{A}) \tag{53}$$

with $Z_n(T,\mathbb{A})$ the partition functions

$$\mathbb{Z}_n(T,\mathbb{A}) := \sum_{\xi\in FixT^n} trace \prod_{k=0}^{n-1} \mathbb{A}(T^k(\xi))). \tag{54}$$

In [**Ru1**] Ruelle proved the following Lemma

Lemma 5.1. *The zeta function ζ_R in (53) has in its domain of definition the following representation*

$$\zeta_R(z,\mathbb{A}) = \prod_\gamma \det(1 - z^{n(\gamma)} \prod_{k=0}^{n(\gamma)-1} \mathbb{A}(T^k\xi_\gamma))^{-1}, \tag{55}$$

where the product is over the primitive periodic orbits of T in M with prime period $n(\gamma)$ and ξ_γ is an arbitrary point in γ.

In the special case of the Poincare map $P : X \to X$ of the geodesic flow Φ_t with respect to the Poincare section X in (10) and $\mathbb{A}$ given by $\mathbb{A} = \mathbb{A}_{\beta,\chi_\Gamma}(x) = \chi_\Gamma(QT^{N(x)}) \exp -\beta r(x)$

with $x = (x_1, x_2, \epsilon) \in X$, $N(x) = -\epsilon[\frac{1}{x}]$ and $r : X \to \mathbb{R}_+$ the recurrence time function in (22) one finds for $z = 1$

$$\begin{aligned}
\zeta_R(1, \mathbb{A}) &= \prod_{\gamma_P} \det(1 - \prod_{l=0}^{2k-1} (\chi_\Gamma(QT^{N(P^l x_\gamma)}) \exp -\beta r(P^l x_\gamma))^{-1} \\
&= \prod_{\gamma_P} \det(1 - \chi\Gamma(\sigma_\gamma) \exp -\beta l(\gamma))^{-1}
\end{aligned}$$

But this is just the Ruelle-Smale zeta function $\zeta(\beta, \chi_\Gamma)$ in (51) if $n(\gamma_P) = 2k$ and $l(\gamma_P) = \sum_{l=0}^{2k-1} r(P^l x_\gamma)$, since we have seen that there exists a $1-1$ correspondence between the prime periodic orbits γ_P of the Poincare map and the prime periodic orbits γ of the geodesic flow $\Phi_t : SM \to SM$ such that $l(\gamma_P)$ is exactly the period $l(\gamma)$ of the periodic orbit γ and $\sigma_\gamma \in SL(2, \mathbb{Z})$ is the hyperbolic element fixing the lift $\tilde{\gamma}$ of γ.

The Selberg zeta function Z_S^χ for the geodesic flow on the unit tangent bundle SM of the modular surface M and a representation $\chi : SL(2, \mathbb{Z}) \to End(V)$ is defined as

(56) $$Z_S^\chi(\beta) = \prod_\gamma \prod_{k=0}^\infty \det(1 - \chi(\sigma_\gamma) e^{-(\beta+k)l(\gamma)})$$

where the product is again over the prime periodic orbits γ of $\Phi_t : SM \to SM$ and $\sigma_\gamma \in SL(2, \mathbb{Z})$ fixes the lift $\tilde{\gamma}$ of γ.

The Selberg zeta function Z_S^Γ on the other hand for the geodesic flow $\Phi_t^\Gamma : SM_\Gamma \to SM_\Gamma$ for any modular subgroup $\Gamma \in SL(2, \mathbb{Z})$ of finite index is defined as

(57) $$Z_S^\Gamma(\beta) = \prod_\gamma \prod_{k=0}^\infty (1 - e^{-(\beta+k)l(\gamma)})$$

where γ now denotes a prime periodic orbit of the geodesic flow on the unit tangent bundle SM_Γ of the modular surface M_Γ with prime period $l(\gamma)$.

The transfer operators $\mathcal{L}_\beta$ are quite generally sums of so called composition operators of the general form

(58) $$(\mathcal{L}f)(z) = \phi(z) f(\psi(z))$$

defined on a Banach space B of functions $f = f(z)$ holomorphic on some disc D with ϕ holomorphic on this disc and ψ some holomorphic map $\psi : D \to D$ with $\psi(\overline{D}) \in D$. Such operators are well known to be nuclear of order zero [**Gr**] and the trace of $\mathcal{L}$ is given by the trace formula [**Ma1**]

(59) $$trace\, \mathcal{L} = \frac{\phi(z^*)}{1 - \psi'(z^*)}$$

where $z*$ is the unique fixed point of the map ψ in D.

Applying this trace formula we get [**CM3**]

Proposition 5.2. *The partition functions $Z_n(P, \mathbb{A}_{\beta,\chi_\Gamma})$ respectively $Z_n(P_\Gamma, \mathbb{A}_{\beta,1})$ in (54) can be expressed as*

$$Z_n(P, \mathbb{A}_{\beta,\chi_\Gamma}) = trace(\mathcal{L}_\beta^{\chi_\Gamma})^n - trace(-\mathcal{L}_{\beta+1}^{\chi_\Gamma})^n$$

respectively

$$Z_n(P_\Gamma, \mathbb{A}_{\beta,1}) = trace(\mathcal{L}_\beta^\Gamma)^n - trace(-\mathcal{L}_{\beta+1}^\Gamma)^n$$

with $\mathcal{L}_\beta^{\chi_\Gamma}$ *and* $\mathcal{L}_\beta^\Gamma$ *defined in (42) and (44).*

The Ruelle zeta functions $\zeta_R(z, \mathbb{A}_{\beta,\chi_\Gamma})$ and $\zeta_R(z, \mathbb{A}_{\beta,1})$ therefore can be written as

$$\zeta_R(z, \mathbb{A}_{\beta,\chi_\Gamma}) = \frac{det(1 - z\mathcal{L}_{\beta+1}^{\chi_\Gamma})}{det(1 - z\mathcal{L}_\beta^{\chi_\Gamma})}$$

respectively

$$\zeta_R(z, \mathbb{A}_{\beta,1}) = \frac{det(1 - z\mathcal{L}_{\beta+1}^{\Gamma})}{det(1 - z\mathcal{L}_\beta^{\Gamma})}.$$

Since the Selberg functions Z_S^Γ respectively $Z_S^{\chi_\Gamma}$ in (56) and (**??**) are products of Ruelle-Smale functions with shifted arguments and these functions can be expressed by Lemma 5.1 in terms of the Ruelle zeta functions in (53) one gets finally

Theorem 5.3. *The Selberg zeta function* Z_S^Γ *for the geodesic flow* Φ_t^Γ *on the unit tangent bundle* SM_Γ *of the modular surface* M_Γ *coincides with the Selberg zeta function* $Z_S^{\chi_\Gamma}$ *of the geodesic flow on the unit tangent bundle* SM *of the modular surface* M *with representation* χ_Γ*. Both functions can be expressed as Fredholm determinants of the transfer operators* $\mathcal{L}_\beta^\Gamma$ *respectively* $\mathcal{L}_\beta^{\chi_\Gamma}$ *as* $Z_S^\Gamma(\beta) = \det(1 - \mathcal{L}_\beta^\Gamma) = \det(1 - \mathcal{L}_\beta^{\chi_\Gamma}) = Z_S^{\chi_\Gamma}(\beta)$

If $\chi_\Gamma = \oplus_i \chi_i$ is the decomposition of the representation χ_Γ into its irreducible components χ_i then one finds the factorization

$$Z_S^{\chi_\Gamma}(\beta) = \prod_i Z_S^{\chi_i}(\beta)$$

which was derived also earlier by A. Venkov and P. Zograf [**VZ**] using the Selberg trace formula.

When acting on the functions

$$\underline{\underline{f}} = \underline{\underline{f}}(z) = \begin{pmatrix} \underline{f}(z,1) \\ \underline{f}(z,-1) \end{pmatrix}$$

the transfer operator $\mathcal{L}_\beta^{\chi_\Gamma}$ can be written in matrix form

$$\mathcal{L}_\beta^{\chi_\Gamma} = \begin{pmatrix} 0 & \mathcal{L}_{\beta,-} \\ \mathcal{L}_{\beta,+} & 0 \end{pmatrix}$$

with

$$\mathcal{L}_{\beta,\pm}\underline{f}(z) = \sum_{n=1}^\infty \frac{1}{(z+n)^{2\beta}} \chi_\Gamma(QT^{\pm n})\, \underline{f}(\frac{1}{z+n}).$$

Hence

$$\det(1 - \mathcal{L}_\beta^{\chi_\Gamma}) = \det(1 - \mathcal{L}_{\beta,+}\mathcal{L}_{\beta,-}) = \det(1 - \mathcal{L}_{\beta,-}\mathcal{L}_{\beta,+})$$

.

Obviously $L_{\beta,+} = L_{\beta,-}$ iff $\chi_\Gamma\left(T^2\right) = 1$. In this case one finds

(60) $$\det\left(1 - L_\beta^{\chi_\Gamma}\right) = \det\left(1 - L_{\beta,+}\right)\det\left(1 + L_{\beta,+}\right)$$

and hence

$$Z_S^\Gamma(\beta) = \det(1 - L_{\beta,+})\det(1 + L_{\beta,+}) \tag{61}$$

Examples of modular groups for which the Selberg zeta function Z_S^Γ factorizes in this way are for instance $SL(2,\mathbb{Z}), \Gamma(2), \Gamma_0(2)$ and $\Gamma^0(2)$. It turns out that there is a slightly different transfer operator L_β^{MM} introduced by Y. Manin and M. Marcolli in [**MM**] for which the above factorization holds for a much wider class of modular surfaces.

5.1 The Manin-Marcolli operator

Consider modular subgroups $\Gamma \subset SL(2,\mathbb{Z})$ with the property

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma \quad . \tag{62}$$

For such a subgroup Γ define the extended group $\widetilde{\Gamma}$ with $\widetilde{\Gamma} = \Gamma \cup \Gamma \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \subset SL(2,\mathbb{Z})$
and the extended subgroup $\overline{\Gamma}$ of $GL(2,\mathbb{Z})$ with

$$\overline{\Gamma} = \widetilde{\Gamma} \cup \widetilde{\Gamma} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{63}$$

Then one shows that

$$\overline{\Gamma} \setminus GL(2,\mathbb{Z}) \cong \Gamma \setminus SL(2,\mathbb{Z})$$

and hence there is a $1-1$ correspondence between the rest classes of $GL(2,\mathbb{Z})$ modulo $\overline{\Gamma}$ and the rest classes of $SL(2,\mathbb{Z})$ modulo Γ .

Manin and Marcolli considered in [**MM**] the following transfer operator L_β^{MM} for the modular group $\Gamma \subset SL(2,\mathbb{Z})$:

$$\left(L_\beta^{MM} \underline{f}\right)(z) = \sum_{n-1}^{\infty} \left(\frac{1}{z+n}\right)^{z\beta} \chi_{\overline{\Gamma}} \begin{pmatrix} -n & 1 \\ 1 & 0 \end{pmatrix} \underline{f}\left(\frac{1}{z+n}\right) \tag{64}$$

where $\chi_{\overline{\Gamma}} : GL(2,\mathbb{Z}) \to \mathrm{End}\,\mathbb{C}^\mu$ is the induced representation of $GL(2,\mathbb{Z})$ induced from the trivial representation of $\overline{\Gamma}$. There is a simple relation between this operator L_β^{MM} and the transfer operators $L_{\beta,\pm}$:

$$\left(L_\beta^{MM}\right)^2 = \chi_{\overline{\Gamma}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L_{\beta,+} L_{\beta,-} \chi_{\overline{\Gamma}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{65}$$

that means $\left(L_\beta^{MM}\right)^2$ and $L_{\beta,+}L_{\beta,-}$ are conjugate and hence one gets for the Selberg zeta functions Z_s^Γ respectively $Z_s^{\chi_\Gamma}$:

$$Z_s^\Gamma(\beta) = Z_s^{\chi_\Gamma}(\beta) = \det\left(1 - \left(L_\beta^{MM}\right)^2\right) = \det\left(1 - L_\beta^{MM}\right)\det\left(1 + L_\beta^{MM}\right)$$

The Hecke congruence subgroups $\Gamma_0(n)$ fulfill property (**??**) and hence their Selberg zeta function have the factorization property as given in (66). From our discussion of the spectral properties of the operator $L_{\beta,+}$ and $L_{,-}$ it follows that also the operator L_β^{MM} is meromorphic in β and nuclear of order zero away from its poles. Hence relation (66) gives a new proof of the fact that for the modular groups $\Gamma \subset SL\,(2,\mathbb{Z})$ the Selberg zeta function is meromorphic in the entire β -plane. Its zeroes on the other hand correspond to those values of β where $\lambda = 1$ or $\lambda = -1$ belongs to the spectrum of the operator L_β^{MM} . Unfortunately we did not succeed up to now to derive the location of the zeroes of Selbergs zeta function in the β -plane alone from the spectral properties of the operator L_β^{MM} .

6 The period functions of Lewis and Zagier

The transfer operator approach to the Selberg zeta function relates the analytic properties of this function to spectral properties of the transfer operator. Both are purely classical objects: the Selberg function can be considered a generating function for the length spectrum of the geodesic flow whereas the transfer operator encodes further details of the dynamical properties of the flow like the Poincare map and the recurrence time function. It is therefore not surprizing that the transfer operator provides one with more informations also for the corresponding quantum system. Indeed, through the work of J. Lewis and D. Zagier and our own work for the group $SL(2,\mathbb{Z})$ it became clear that the eigenfunctions with eigenvalues $\lambda = \pm 1$ of the transfer operator $\mathcal{L}_\beta$ for β-values corresponding to nontrivial zeroes of the Selberg function are directly related to the Maass wave forms for this group. Furthermore it was found that such eigenfunctions for β-values corresponding to trivial zeroes of $Z_S(\beta)$ are related to the theory of both holomorphic and real analytic modular forms namely to the period polynomials of Eichler, Shimura and Manin for holomorphic cusp forms. To explain this connection in more detail consider any eigenfunction $\underline{f}$ of the Manin-Marcolli operator in (64) with eigenvalue λ:

$$\mathcal{L}_\beta^\Gamma \underline{f} = \lambda \underline{f},$$

where $\underline{f}$ is a holomorphic function in the domain $\mathbb{C} \setminus (-\infty,-1]$. If $\underline{\phi}$ denotes the function $\underline{\phi}(z) = \underline{f}(z-1)$ then this function fulfills the functional equation

$$\lambda(\underline{\phi}(z) - \chi_{\overline{\Gamma}}(T)\,\underline{\phi}(z+1)) = z^{-2\beta}\,\chi_{\overline{\Gamma}}(\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix})\,\underline{\phi}(\frac{z+1}{z}). \tag{66}$$

For the following discussion it is more convenient to work also with the space of functions $\psi = \psi(z, \overline{\Gamma}g)$ with

$$\psi(z, \overline{\Gamma}g) = \phi_{\overline{\Gamma}g}(z), \tag{67}$$

if $\underline{\phi} = (\phi_{\overline{\Gamma}g})_{g=g_i,\,1\le i\le\mu}$. For any subgroup $\overline{\Gamma} \subset GL(2,\mathbb{Z})$ the group $GL(2,\mathbb{Z})$ acts as follows on the functions $\psi : \mathbb{C}\setminus\mathbb{R} \times \overline{\Gamma} \setminus GL(2,\mathbb{Z}) \to \mathbb{C}$ for $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$:

$$(\psi_{|\beta} h)(z, \overline{\Gamma}g) = (cz+d)^{-2\beta}\,\psi(hz, \overline{\Gamma}gT^{-1}h^{-1}T) \tag{68}$$

. We call this action a T-twisted action of the group $GL(2,\mathbb{Z})$ contrary to the more common action without the conjugation by T on the cosets $\overline{\Gamma}g$. Using the twisted action in (68) one can write the Lewis equation (**??**) for the group $\Gamma \subset SL(2,\mathbb{Z})$ for $\lambda = \pm 1$ simply as

$$\psi = \psi_{|\beta} T \,+\, \lambda \psi_{|\beta} TM \tag{69}$$

with $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In the special case of the group $\Gamma = SL(2,\mathbb{Z})$ one finds $\overline{\Gamma} = GL(2,\mathbb{Z})$ and eq. (69) becomes in explicit form

$$\psi(z) \;=\; \psi(z+1) \,+\, \lambda\, z^{-2\beta}\, \psi(\frac{z+1}{z}). \tag{70}$$

A simple calculation then shows that this equation is equivalent to the following pair of equations

$$\psi(z) \;=\; \lambda\, z^{-2\beta}\, \psi(\frac{1}{z}), \tag{71}$$

$$\psi(z) \;=\; \psi(z+1) \,+\, (z+1)^{-2\beta}\, \psi(\frac{z}{z+1}). \tag{72}$$

Lewis and Zagier proved in [**LZ1**] the following theorem for the group $SL(2,\mathbb{Z})$

Theorem 6.1. *For $\beta \in \mathbb{C}$ with*
$Re\,\beta \;=\; \frac{1}{2}$ there exists a $1-1$ correspondence between the Maass cusp forms ϕ for $SL(2,\mathbb{Z})$ with $-\Delta_{LB}\,\phi \;=\; \beta(1-\beta)\,\phi$ and the real analytic solutions of Lewis functional equation (70) for the same β-value with asymptotic behaviour $\psi(x) = o(\frac{1}{x})$ for $x \to 0$ and $\psi(x) = o(1)$ for $x \to \infty$.

Remark 6.2. Obviously the space of solutions of the Lewis equation (70) with the asymptotic behaviour as in Theorem 6.1 coincides with the eigenspaces to the eigenvalues $\lambda = \pm 1$ of the transfer operator $\mathcal{L}_\beta$ for the group $SL(2,\mathbb{Z})$. To see this one only has to iterate the functional equation and use the asymptotic behaviour of the Hurwitz zeta function in the half plane
$Re\,\beta\, > 0$. □

In his Ph.D.-thesis [**Mar**] F. Martin extended the theory of period functions of Lewis and Zagier to holomorphic modular forms of weight 1 for arbitrary subgoups $\Gamma \subset SL(2,\mathbb{Z})$. Thereby he used also a Lewis-type functional equation of the following form

$$\psi \;=\; \psi_{||\beta} T \,+\, \psi_{||\beta}\, MTM \tag{73}$$

where the action $||\beta$ of the group $GL(2,\mathbb{Z})$ is similar to its action $|\,\beta$ in (68) but without the T-twist when acting on the rest classes. Obviously with ψ also the function $\psi_{||\beta} M$ solves equation (73). The functional equation (73) is closely related to the functional equation (69) as shown by

Lemma 6.3. *The function ψ solves equation (69) iff ψ solves the pair of equations*

$$\psi \;=\; \psi_{|\beta} T \,+\, \psi_{|\beta}\, MTM \tag{74}$$

$$\lambda\, \psi \;=\; \psi_{|\beta}\, M \tag{75}$$

Proof. Just act with the element M on equation (69) to get equation (75). Next replace $\lambda\,\psi$ in equation (69) by $\psi_{|\beta}M$ to get equation (74). Similar arguments show the inverse direction of the Lemma. $\square$

Lemma 6.4. *The function ψ solves equation (69) iff the function $\tilde{\psi} \;=\; \tilde{\psi}(z,\overline{\Gamma}g) \;=:\; \psi(z,\overline{\Gamma}gT)$ is a solution of the Lewis equation (73) with $\tilde{\phi}_{\overline{\Gamma}gM\,|\beta}M \;=\; \lambda\,\tilde{\phi}_{\overline{\Gamma}g}$ for $\tilde{\phi}_{\overline{\Gamma}g} \;=\; \tilde{\psi}(z,\overline{\Gamma}g)$*

Proof. From Lemma 6.3 it follows that ψ solves equation (69) iff it solves equations (74) and (75). Explicitly equation (74) reads as follows

$$\begin{aligned} 0 \;&=\; \phi_{\overline{\Gamma}gT} \;-\; \phi_{\overline{\Gamma}gTT^{-1}|\beta}T \;-\; \phi_{\overline{\Gamma}gTT^{-1}MT^{-1}MT|\beta}MTM \;= \\ &=\; \phi_{\overline{\Gamma}gT} \;-\; \phi_{\overline{\Gamma}gT^{-1}T|\beta}T \;-\; \phi_{\overline{\Gamma}gMT^{-1}MT|\beta}MTM \;= \\ &=\; \tilde{\phi}_{\overline{\Gamma}g} \;-\; \tilde{\phi}_{\overline{\Gamma}gT^{-1}|\beta}T \;-\; \tilde{\phi}_{\overline{\Gamma}gMT^{-1}M|\beta}MTM \end{aligned}$$

But this is just equation (73) for the components of the function $\tilde{\psi}$. Equation (75) for the components of ψ on the other hand reads as $\lambda\;\phi_{\overline{\Gamma}gT} \;=\; \phi_{\overline{\Gamma}gMT|\beta}\,M$ and hence $\tilde{\phi}_{\overline{\Gamma}gM\,|\beta}M \;=\; \lambda\,\tilde{\phi}_{\overline{\Gamma}g}$. The inverse direction of the Lemma is proved in a similar way. $\square$

The Lewis equation used by F. Martin hence also arizes from the transfer operator $\mathcal{L}_{\beta}^{MM}$ for the geodesic flow $\Phi_t^{\Gamma} : SM_{\Gamma} \to SM_{\Gamma}$, it is written down only in a different basis in $\mathbb{C}^{\mu}$ compared to ours.
Consider now any Maass wave form $\phi \;=\; \phi(z)$ for a subgroup $\Gamma \subset SL(2,\mathbb{Z})$ with $-\Delta_{LB}\phi(z) \;=\; \beta(1-\beta)\,\phi(z)$ and $\phi(gz) \;=\; \phi(z)$ for all $g \in \Gamma$. Following basically the arguments in [**Mar**] and of Lewis and Zagier [**LZ1**]for the case of $SL(2,\mathbb{Z})$ we can construct solutions of the Lewis equation for any modular group Γ.
For $\zeta \in \mathbb{C}$ and $z \in \mathbb{H}$ consider the function

$$R_{\zeta}(z) \;=\; \frac{y}{(x-\zeta)^2 + y^2}. \tag{76}$$

with $z = x + iy$. A simple calculation shows that R_{ζ}^{β} is an eigenfunction of the Laplace-Beltrami operator $-\Delta_{LB}$ with eigenvalue $\lambda = \beta(1-\beta)$

$$-\Delta_{LB}\,R_{\zeta}^{\beta}(z) \;=\; \beta(1-\beta)\,R_{\zeta}^{\beta}(z). \tag{77}$$

In [**LZ1**] it is shown that the following transformation formula holds for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$:

$$R_{g\zeta}(gz) = \frac{(c\zeta + d)^2}{\det\, g}\,R_{\zeta}(z). \tag{78}$$

Next one checks that for any two eigenfunctions $u\,,\,v\;:\mathbb{C} \to \mathbb{C}$ of $-\Delta$ with identical eigenvalues the 1-form $\{u\,,\,v\}(z)$ with

$$\{u\,,\,v\}(z) \;=\; (v\,u_y \;-\; u\,v_y)dx \;+\; (u\,v_x \;-\; v\,u_x)dy \tag{79}$$

is closed and hence $d\{u,v\}(z) = 0$. If $g : \mathbb{C} \to \mathbb{C}$ is any holomorphic change of coordinates then

$$\{u \circ g, v \circ g\}(z) \;=\; \{u,v\}(gz) \tag{80}$$

Given an eigenfunction ϕ of the Laplace Beltrami operator $-\Delta_{LB}$ with eigenvalue $\lambda = \beta(1-\beta)$ such that $\phi(gz) = \phi(z)$ for all $g \in \overline{\Gamma} \subset GL(2,\mathbb{Z})$, consider for
$Re\,\zeta >\ 0$ the following function

$$(81) \qquad \psi_\phi \ = \ \psi_\phi(\zeta, \overline{\Gamma}g) \ = \int_0^{i\infty} \{\phi \circ (gT^{-1}, R_\zeta^\beta)\}(z).$$

Obviously the function ψ_ϕ depends only on the cosets $\overline{\Gamma}g$ since $\phi(hz) = \phi(z)$ for all $h \in \overline{\Gamma}$. Then one shows

Theorem 6.5. *For ϕ a Maass wave form of the modular group $\Gamma \subset SL(2,\mathbb{Z})$ with eigenvalue $\lambda = \beta(1-\beta)$ the function ψ_ϕ in (81) solves the Lewis equation (74).*

Proof. By using the transformation properties of the function $R_\zeta(z)$ one finds for arbitrary $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

$$(82) \qquad (\psi_{\phi|\beta}h)(\zeta, \overline{\Gamma}g) = \int_{h(0)}^{h(i\infty} \{\phi \circ (gT^{-1}), R_\zeta^\beta(z)\}(z)$$

and hence

$$\begin{aligned} & \psi_\phi - \psi_{\phi|\beta}T - \psi_{\phi|\beta}MTM = \\ = & \int_0^{i\infty} \{\phi \circ (gT^{-1}), R_\zeta^\beta\}(z) - \int_1^{i\infty} \{\phi \circ (gT^{-1}), R_\zeta^\beta\}(z) - \\ - & \int_0^1 \{\phi \circ (gT^{-1}), R_\zeta^\beta\}(z) = 0. \end{aligned}$$

□

Remark 6.6. *Since the 1-form $\left\{\phi \circ gT^{-1}, R_\zeta^\beta\right\}(z)$ is closed one can move the path of integration from 0 to $i\infty$ in the complex $z-$plane and derives this way that the function $\psi_\phi\left(\zeta, \overline{\Gamma}g\right)$ is indeed holomorphic in the plane $\mathbb{C} \setminus [-\infty, 0]$.*

In the special case $\Gamma = SL\,(2,\mathbb{Z})$ formula (81) for the function ψ_ϕ in (**??**) simplifies to (see [**LZ1**])

$$(83) \qquad \psi_\phi\,(z) = \int_0^{i\infty} \left\{\phi, R_\zeta^\beta\right\}(z) \quad .$$

Obviously the function ψ_ϕ in (81) gives one direction of the Lewis-Zagier correspondence for arbitrary modular groups. One only has to check that the function has the correct asymptotic behaviour to define really an eigenfunction of the Manin-Marcolli transfer operator L_β^{MM} .

The converse correspondence, namely to attach a Maass wave form to an eigenfunction of the transfer operator is understood up to now only in the case of the full modular group $SL\,(2,\mathbb{Z})$. One possibility here is the use of L- functions and the converse theorem of Maass. We follow again the paper by Lewis and Zagier [**LZ1**] .

Proposition 6.7. *For $\beta \in \mathbb{C}$ denote by $\sigma = \Re\beta$ its real part. There exists a 1-1 correspondence between the following classes of functions:*

- *a solution u of $-\Delta_{LB}u = \beta(1-\beta)u$ in $\mathbb{H}$ with $u(z+1) = u(z)$ and $u(x+iy) = O\left(y^A\right)$ for some $A < \min(\sigma, 1-\sigma)$*
- *a pair of L-series $L_\epsilon(\rho) \in 0,1$ convergent in some half plane*
- *a periodic holomorphic function f in $\mathbb{C}\setminus\mathbb{R}$ with $f(z) = O\left(|Im\ z|^{-A}\right)$ for some A .*

Proof. We only sketch the main steps of the proof. Since u is a periodic eigenfunction of $-\Delta_{LB}$ it can be expanded in a Fourier series of the form

$$u(z) = \sqrt{y}\sum_{n\neq 0} A_n\ K_{\beta-\frac{1}{2}}(2\pi|n|y)\, e^{2\pi inx} \tag{84}$$

where the A_n have at most polynomial growth $in\,|n|$.

Consider next the following two L-series

$$L_\epsilon(\rho) := \sum_{n-1}^{\infty} \frac{A_{n,\epsilon}}{n^\rho}, \quad \epsilon = 0,1, \tag{85}$$

with $A_{n,\epsilon} = A_n + (-1)^\epsilon A_{-n}$ and the holomorphic function

$$f(z) = \begin{cases} \sum_{n>0} n^{\beta-\frac{1}{2}}A_n\ e^{2\pi inz} & \text{for}\ \Im z\ >\ 0 \\ -\sum_{n<0}|n|^{\beta-\frac{1}{2}}\ A_n\ e^{2\pi inz} & \text{for}\ \Im z\ <\ 0. \end{cases} \tag{86}$$

On the other hand let be given two Dirichlet series L_ϵ, $\epsilon = 0,1$, convergent in some half plane or a periodic function holomorphic in $\mathbb{C}\setminus\mathbb{R}$ with $f(z) = O\left(|Imz|^{-A}\right)$ for some $A > 0$ then the expansion coefficients $\{A_n\}_{n\neq 0}$ are at most of polynomial growth in n . Define then a function u on $\mathbb{H}$ as

$$u(z) = \sqrt{y}\sum_{n\neq 0} A_n K_{\beta-\frac{1}{2}}(2\pi|n|y)\, e^{2\pi inx} \tag{87}$$

Then the functions $u_0(y) := \frac{1}{\sqrt{y}}u(iy)$ respectively $u_1(y) := \frac{\sqrt{y}}{2\pi i}\frac{\partial u}{\partial x}(iy)$ can be written as

$$u_\epsilon(y) = \sum_{n=1}^{\infty}(ny)^\epsilon A_{n\epsilon}\ K_{\beta-\frac{1}{2}}(2\pi ny)\ , \ \epsilon = 0,1 \tag{88}$$

such that

$$L_\epsilon^*(\rho) := \int_0^\infty u_\epsilon(y)\, y^{\rho-1}dy \tag{89}$$

fulfills

$$L_\epsilon^*(\rho) = \frac{1}{4\pi^{\rho+\epsilon}}\Gamma\left(\frac{\rho+\epsilon-\beta+\frac{1}{2}}{2}\right)\Gamma\left(\frac{\rho+\epsilon+\beta-\frac{1}{2}}{2}\right)L_\epsilon(\rho)\,. \tag{90}$$

On the other hand one finds

$$(91)\qquad (2\pi)^{-\rho}\,\Gamma(\rho)\,L_\epsilon\left(\rho - s + \frac{1}{2}\right) = \int_0^\infty \left(f(iy) - (-1)^\epsilon f(-iy)\right) y^{\rho-1} d\rho.$$

□

Hence to any periodic eigenfunction u of $-\Delta_{LB}$ one can find a holomorphic function f on $\mathbb{C}\setminus\mathbb{R}$ which furthermore is periodic of period 1 .
The following Proposition establishes a direct relation between the solutions ψ of Lewis equation in the upper and lower half plane $\mathbb{H}$ and periodic holomorphic functions in the upper and lower half plane $\mathbb{H}$:

Proposition 6.8. *The function ψ is a holomorphic solution of the Lewis equation (72) in $\mathbb{H}$ respectively in $-\mathbb{H}$ iff the function $f = f(z) = \psi(z) + z^{-2\beta}\psi\left(-\frac{1}{z}\right)$ is a periodic holomorphic function in $\mathbb{H}$ respectively $-\mathbb{H}$*

.

Proof. If ψ solves equation (72) then one has

$$\begin{aligned}0 &= \psi(z+1) - \psi(z) + \left(\tfrac{1}{z+1}\right)^{2\beta}\psi\left(\tfrac{z}{z+1}\right)\\ &- (z+1)^{-2\beta}\left[\psi\left(\tfrac{z}{z+1}\right) - \psi\left(-\tfrac{1}{z+1}\right) + \left(\tfrac{z+1}{z}\right)^{2\beta}\psi\left(-\tfrac{1}{z}\right)\right]\\ &= \left[\psi(z+1) + (z+1)^{-2\beta}\psi\left(-\tfrac{1}{z+1}\right)\right] - \left[\psi(z) + z^{-2\beta}\psi\left(-\tfrac{1}{z}\right)\right].\end{aligned}$$

Therefore the function $f(z) = \psi(z) + z^{-2\beta}\psi(\frac{-1}{z})$ is holomorphic in $\mathbb{H}$ and periodic with period 1. On the other hand if there is given a holomorphic function f on $\mathbb{H}$ with $f(z+1) = f(z)$ then the function $\psi(z) := f(z) - z^{-2\beta} f(\frac{-1}{z})$ is a solution of the Lewis equation for $SL(2,\mathbb{Z})$. The same kind of arguments can be applied for the lower half plane $z \in -\mathbb{H}$. □

In the next step one uses an old result by Maass [**Maa**]

Theorem 6.9. *There is a $1-1$ correspondence between the Maass wave forms ϕ for $SL(2,\mathbb{Z})$ and pairs of L series $L^*_\epsilon = L^*_\epsilon(\rho)$, $\epsilon = 0,1$ with*

$$(92)\qquad L^*_\epsilon(1-\rho) = (-1)^\epsilon L^*_\epsilon(\rho).$$

Proof. We sketch the main steps of the proof. We know already from the periodicity of the Maass form ϕ that there is a $1-1$ correspondence with a pair of L-series L^*_ϵ convergent in some halfplane. Since u is real analytic the condition $u(z) = u(\frac{-1}{z})$ however is equivalent to $(u(z) - u(\frac{-1}{z}))\mid_{z=i\rho} = 0$ and $\frac{d}{dz}(u(z) - u(\frac{-1}{z}))\mid_{z=i\rho} = 0$. But these two identities are equivalent to the equations $u_\epsilon(\frac{1}{y}) = (-1)^\epsilon y u_\epsilon(y)$ for $\epsilon = 0,1$. The last two equations are again equivalent to the L-series $L^*_\epsilon = (-1)^\epsilon\int_0^\infty u_\epsilon(y)y^{\rho-1}dy$ to fullfill the functional equations $L^*_\epsilon(1-\rho) = L^*_\epsilon(\rho)$. Thereby one uses the fact that the function $u_\epsilon(y)$ can be expressed as the Mellin transform $u_\epsilon(y) = \frac{1}{2\pi i}\int_{Re\,\rho=C} L^*_\epsilon(\rho)\, y^{-\rho}\, d\rho$ of L^*_ϵ with $C >> 0$. □

To finish the proof of Theorem 6.1 one has to show that the existence of the L-series L^*_ϵ, $\epsilon = 0,1$ fulfilling the functional equations (**??**) is equivalent to the fact that the function $\psi(z) = f(z) - z^{-2\beta} f(\frac{-1}{z})$, holomorphic in the upper and lower Poincare plane, extends indeed across the positive real line to a holomorphic function on the cut plane $\mathbb{C}\setminus\mathbb{R}_-$ and fulfills the correct asymptotics for $x \to 0$ and $x \to \infty$. The details of this argument which are a little bit more cumbersome can be found in the Annals paper by Lewis and Zagier [**LZ1**].

7 The Eichler-Shimura-Manin period polynomials

In the preceding sections we have discussed the eigenfunctions of the transfer operator $\mathcal{L}_\beta^{MM}$ with eigenvalue $\lambda = \pm 1$ for β-values corresponding to the so called spectral or nontrivial zeroes of the Selberg zeta function $Z_S(\beta)$ for the group $SL(2,\mathbb{Z})$. These eigenfunctions are the period functions of Lewis and Zagier. One can obviously ask if there is also a similar interpretation for the eigenfunctions of this transfer operator at β-values corresponding to the so called trivial zeroes $\zeta_R(2\beta) = 0$ of the Selberg function with $Re\,\beta < 0$. Here ζ_R denotes Riemann's zeta function. For $\beta = -n$ it turns out that these eigenfunctions are either polynomials closely related to the holomorphic modular cusp forms or rational functions related to the holomorphic Eisenstein series for $SL(2,\mathbb{Z})$.
A function $f : \mathbb{H} \to \mathbb{C}$ holomorphic in $\mathbb{H}$ and at the cusp of $SL(2,\mathbb{Z})$ at $i\infty$ is called a modular form of weight $2k$ if

$$f_{|2k}g\,(z) = f(z) \tag{93}$$

for all $g \in SL(2,\mathbb{Z})$ where the slash action $_|$ is defined as

$$f_{|2k}g\,(z) = (cz+b)^{-2k} f(\frac{az+b}{cz+d}) \tag{94}$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let $f(z) = \sum_{n=1}^\infty a_n\,q^n$ with $q = e^{2\pi i\,z}$ be the Fourier expansion of f. Then the period polynomial $r_f(z)$ attached to f can be defined as

$$r_f(z) = \tilde{f}(z) - z^{k-2}\tilde{f}(\frac{-1}{z}) \tag{95}$$

with $\tilde{f}(z) = \sum_{n=1}^\infty \frac{a_n}{n^{2k-1}} q^n$ the Eichler integral [**Za**], respectively as

$$r_f(z) \sim \int_0^{i\infty} f(y)\,(y-z)^{2k-2} dy, \tag{96}$$

or respectively as

$$r_f(z) \sim \sum_{r=0}^{2k-2} \frac{(-2\pi\,i)^{-r}}{(2k-2-r)!}\, L_f(r+1)\, z^r \tag{97}$$

with $L_f(\rho) = \sum_{n=1}^\infty a_n\, n^{-\rho}$ the L-series of f.
It turns out that all three definitions are equivalent:
consider namely the differential operator

$$D := \frac{1}{2\pi\,i}\frac{d}{dz} = q\frac{d}{dq}. \tag{98}$$

Then one finds $D^{2k-1}(\tilde{f}) = f$. With Bol's identity

$$D^{2k-1}(F_{|2-2k}g) = (D^{2k-1}F)_{|2k}g \tag{99}$$

valid for any smooth function F in $\mathbb{C}$ one gets by a simple calculation

$$D^{2k-1}(r_f) = D^{2k-1}(\tilde{f} - \tilde{f}_{|2k}S) = f - f_{|2k}S = 0 \tag{100}$$

and hence r_f is a polynomial of degree $2k-2$. But one easily verifies

$$\tilde{f}(z) = \int_z^{i\infty} (z-\tau)^{2k-2} f(\tau) d\tau$$

and hence, since f is a modular form of weight $2k$ for $SL(2,\mathbb{Z})$

$$z^{-2k}\tilde{f}(\frac{-1}{z}) = \int_z^0 (z-\tau)^{2k-2} f(\tau) d\tau.$$

Subtracting the two equations leads finally to

$$r_f(z) = \tilde{f}(z) - z^{-2k}\tilde{f}(\frac{-1}{z}) = \int_0^{i\infty} (z-\tau)^{2k-2} f(\tau) d\tau.$$

The representation of r_f in terms of the values of the L-series at the integer values $z = r+1$ is obvious.
Consider then the space W_{2k-2} of polynomials P of degree less or equal to $2k-2$

$$W_{k-2} =: \{P \in \mathcal{P} \,:\, P_{|2-2k}(1+S) \;=\; P_{|2-2k}(1+U+U^2) = 0\} \tag{101}$$

with S and T the generators of $SL(2,\mathbb{Z})$ and $U = TS$ fulfilling the relations $S^2 = U^3 = 0$. Eichler showed in [**Ei**] that the space

$$Span\{r_f(z)\,,\, \overline{r_f(\overline{z})} : f \text{ a holomorphic modular cusp form of } SL(2,\mathbb{Z}) \text{ of weight } 2k\}$$

has codimension 1 in W_{2k-2}. The result of Lewis, Zagier [**Za**] and ourselves [**CM3**] is

Proposition 7.1. *Every $P \in W_{2k-2}$ is a solution of Lewis equation for $\beta = 1-k$. Every polynomial solution of Lewis equation for $\beta = 1-k$, $k > 1$ belongs to the space W_{2k-2}.*

This result can be extended also to the holomorphic Eisenstein series of weight $2k$ for the group $SL(2,\mathbb{Z})$ which lead to rational solutions of Lewis equation also at the β-values $1-k$ [**CM3**]. Obviously this is the reason why all the solutions of the Lewis equation are called "period functions". The solutions of Lewis equation just discussed belong to β-values with $\zeta_R(2\beta) = 0$ corresponding to the trivial zeroes of the Riemann zeta function $\zeta_R(\beta) = \sum_{n=1}^{\infty} n^{-\beta}$. What about the non-trivial zeroes which according to Riemann's hypothesis are located on the critical line
$Re\,\beta = \frac{1}{2}$? Indeed there exist explicitly known solutions (see [**CM3**]) of the Lewis equation at β- values with $\zeta_R(2\beta) = 0$ and
$Re\,\beta > 0$ which correspond to the real analytic Eisenstein series determining the continuos spectrum of the hyperbolic Laplacian on the surface M [**CM2**] .
What have the solutions of Lewis equation to do with eigenfunctions of the transfer operator $\mathcal{L}_\beta$ for the full modular group $SL(2,\mathbb{Z})$? We have seen already that any eigenfunction of the transfer operator with eigenvalue $\lambda = \pm 1$ determines a solution of the pair of equations (71) and (72) . On the other hand given a solution ψ of equation (72) then also the function $\psi \pm \psi^\tau$ with $\psi^\tau(z) =: z^{-2\beta}\,\psi(\frac{1}{z})$ solves this equation. But$(\psi \pm \psi^\tau)^\tau = \pm(\psi \pm \psi^\tau)$. Hence if ψ is a solution of equation (72) and $\psi^\tau = \lambda\psi$ for $\lambda = \pm 1$ then ψ solves equation (69). The solutions of this last equation holomorphic in the cut plane $\mathbb{C} \setminus R_-$ determine the eigenfunctions $f = f(z) = \psi(z+1)$ with eigenvalue $\lambda = \pm 1$ of the transfer operator $\mathcal{L}_\beta$ for $SL(2,\mathbb{Z})$ as long as they fulfill the appropriate asymptotic behaviour valid in the corresponding region of the complex β-plane.

8 Transfer operators and Hecke operators for $\Gamma_0(n)$

The transfer operator for the Hecke congruence subgroup $\Gamma_0(n)$ as given by Manin and Marcolli has the form

$$(\mathcal{L}_\beta^{MM}\underline{f})(z) = \sum_{n=1}^{\infty} \frac{1}{(z+n)^{2\beta}}\, \chi_{\overline{\Gamma}_0^n}(\begin{pmatrix} -n & 1 \\ 1 & 0 \end{pmatrix})\, \underline{f}(\frac{1}{z+n})$$

with $\overline{\Gamma}_0(n) = \Gamma_0(n) \cup \Gamma_0(n)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The Lewis equation derived from this operator on the other hand reads

$$\underline{\phi} - \underline{\phi}_{|\beta} T - \underline{\phi}_{|\beta} MTM = 0$$

with $\underline{\phi} = (\phi_{\overline{\Gamma}_0(n)\, g})$ and $\overline{\Gamma}_0(n)\, g$ describing the different rest classes of $\Gamma_0(n) \setminus GL(2,\mathbb{Z})$. Since $GL(2,\mathbb{Z}) \supset \Gamma_0(m) \supset \Gamma_0(nm)$ there is a canonical projection

$$\sigma_{n,m} : I_{nm} \to I_m. \tag{102}$$

In [**?**] we showed that to any index $i \in I_n$ is attached a unique matrix $A_i = \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix}$ with $c \geq 1$, $c \mid n$ and $\gcd(c, \frac{n}{c}, b) = 1$. Hence, if

$$S_n = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a > c \geq 0, d > b \geq 0, ad - bc = n\} \tag{103}$$

then $A_i \in S_n$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_n$ with $b \neq 0$ consider the map

$$K(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = T^{\lceil \frac{d}{b} \rceil} Q \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{104}$$

with $\lceil \frac{d}{b} \rceil = -[-\frac{d}{b}]$. Then $K(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in S_n$. For $i \in I_n$ denote by k_i the number such that $K^j\, A_i \in S_n$ for all $0 \leq j < k_i$ and $K^{k_i}\, A_i = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. Then one shows (see [**?**])

Lemma 8.1. *To $i \in I_{nm}$ and $0 \leq j \leq k_{\sigma_{n,m}(i)}$ there exists exactly one index $l = l(i,j) \in I_n$ such that $A_{l(i,j)}(K^j\, A_{\sigma_{n,m}(i)})\, A_i^{-1} \in SL(2,\mathbb{Z})$*

This then allowed us to prove in [**?**]

Theorem 8.2. *For any solution $\underline{\phi} = (\phi_r)_{r \in I_n}$ of the Lewis equation (69) for $\Gamma_0(n)$ with parameter β the function $\underline{\psi} = (\psi_i)_{i \in I_{nm}}$ with $\psi_i(z) = \sum_{j=0}^{k_{\sigma_{n,m}(i)}} \phi_{l(i,j)|\beta}\, K^j A_{\sigma_{n,m}(i)}(z)$ is a solution of the Lewis equation for the same β-value and the same λ.*

This shows that any solution of the Lewis equation for the group $\Gamma_0(n)$ determines a special nontrivial solution for the subgroup $\Gamma_0(nm)$. Besides this nontrivial solution $\underline{\phi}$ determines also a trivial solution $\underline{\psi}$ for the subgroup $\Gamma_0(nm)$ given by $\underline{\psi} = (\psi_j(z))_{j \in I_{nm}}$ with

$$\psi_j \;=\; \phi_{\sigma_{m,n}(j)}.$$

We call these solutions old solutions of the Lewis equation for the subgroup $\Gamma_0(nm)$. On the other hand one can show that any solution $\underline{\psi}$ of the Lewis equation for the subgroup $\Gamma_0(nm) \subset \Gamma_0(n)$ determines also a solution for $\overline{\Gamma}_0(n)$:

Lemma 8.3. *For any solution* $\underline{\psi} = (\psi_i)_{i \in I_{nm}}$ *of the Lewis equation (69) for* $\Gamma_0(nm)$ *with parameter* β *the function* $\underline{\phi} = (\phi_l)_{l \in I_n}$ *with*

$$\phi_l(z) = \sum_{i \in \sigma_{m,n}^{-1}(l)} \psi_i(z)$$

is a solution of the Lewis equation for $\Gamma_0(n)$ *with the same parameter* β.

In the special case $n = 1$ one hence gets

Corollary 8.4. *Any solution* ϕ *of the Lewis equation for* $SL(2,\mathbb{Z})$ *determines a nontrivial solution* $\underline{\psi} = (\psi_i)_{i \in I_m}$ *of this equation for the group* $\Gamma_0(m)$ *with*

$$\psi_i(z) = \sum_{j=0}^{k_i} \phi_{|2\beta}\, K^j A_i\,(z)$$

Remark 8.5. *Since the entries of the matrices* $K^j\, A_i$ *are nonnegative the action* $\mid \beta$ *is well defined also for complex* β.

Combining Theorem 8.2 and Lemma 8.3 one arrives at a family of linear operators $\tilde{T}_{n,m}$ mapping the space of solutions of the Lewis equation for $\Gamma_0(n)$ into itself:

$$(\tilde{T}_{n,m}\psi)_i = \sum_{r \in \sigma_{m,n}^{-1}(i)} \sum_{j=0}^{k_{\sigma_{n,m}(r)}} \psi_{l(i,j)}\, K^j A_{\sigma_{n,m}(r)},\ m = 1, 2, \ldots .$$

In the special case of the modular group $SL(2,\mathbb{Z})$ one finds

$$\tilde{T}_m =: \tilde{T}_{1,m} = \sum_{A \in S_m^*} A \tag{105}$$

with

$$S_m^* = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_m : \gcd(a,b,c,d) = 1\}. \tag{106}$$

By using the explicit form of the Lewis correspondence for $SL(2,\mathbb{Z})$ on the other hand T. Muehlenbruch recently transfered the Hecke operators T_m for the Maass wave forms to the space of period functions and derived this way the following explicit form for these operators when acting on the period functions for $SL(2,\mathbb{Z})$

$$T_m = \sum_{A \in S_m} A.$$

Obviously our operators $\tilde{T}_m$ coincide for m prime with these Hecke operators T_m. For general m one finds

$$T_m = \sum_{d^2|m} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \tilde{T}_{\frac{m}{d^2}}.$$

It is to be expected that our above operators $\tilde{T}_{n,m}$ for $m = 1, 2, ..$ are also closely related to the Hecke operators on the space of period functions for the modular groups $\Gamma_0(n)$.

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