Transfer operators and dynamical zeta functions for a class of lattice spin models

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Abstract: We investigate the location of zeros and poles of a dynamical zeta function for a family of subshifts of finite type with an interaction function depending on the parameters $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)$ with $0 \leqslant \lambda_i < 1$. The system corresponds to the well known Kac-Baker lattice spin model in statistical mechanics. Its dynamical zeta function can be expressed in terms of the Fredholm determinants of two transfer operators $\mathcal{L}_\beta$ and $\mathcal{G}_\beta$ with $\mathcal{L}_\beta$ the Ruelle operator acting in a Banach space of holomorphic functions, and $\mathcal{G}_\beta$ an integral operator introduced originally by Kac, which acts in the space $L^2(\mathbb{R}^m, d\underline{x})$ with a kernel which is symmetric and positive definite for positive β. By relating $\mathcal{L}_\beta$ via the Segal-Bargmann transform to an operator $\widetilde{\mathcal{G}}_\beta$ closely related to the Kac operator $\mathcal{G}_\beta$ we can prove equality of their spectra and hence reality respectively positivity for the eigenvalues of the operator $\mathcal{L}_\beta$ for real respectively positive β. For a restricted range of parameters $0 \leqslant \lambda_i < \frac{1}{2}$, $1 \leqslant i \leqslant m$ we can determine the asymptotic behavior of the eigenvalues of $\mathcal{L}_\beta$ for large positive and negative values of β and deduce from this the existence of infinitely many non trivial zeros and poles of the dynamical zeta functions on the real β line at least for generic $\underline{\lambda}$. For the special choice $\lambda_i = \frac{1}{2}$, $1 \leqslant i \leqslant m$, we find a family of eigenfunctions and eigenvalues of $\mathcal{L}_\beta$ leading to an infinite sequence of equally spaced "trivial" zeros and poles of the zeta function on a line parallel to the imaginary β-axis. Hence there seems to hold some generalized Riemann hypothesis also for this kind of dynamical zeta functions.

1 Introduction

The transfer matrix method has played an important role in statistical mechanics ever since E. Ising for the first time used this method to solve his 1-dimensional lattice spin model with nearest neighbour interaction. The method was extended later to treat higher dimensional models with arbitrary finite range interactions. The most satisfying theory for this method from the mathematical point of view goes back to D. Ruelle who introduced the so called transfer operator for 1-dimensional lattice spin systems with arbitrary long range interactions (see [Ru68]). Continued interest in such systems is related to the fact that these systems show up in a rather natural way within the so called thermodynamic formalism for dynamical systems (see [Ru78]).

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Thereby the transfer operator can be used for instance to construct invariant measures for such systems and to characterize their ergodic properties (see [Ba00]).

Another nice application of this method is to the theory of dynamical zeta functions (see [Ru92], [Ru94]). These functions can be interpreted as generating functions for the partition functions of the system constructed in complete analogy to the partition functions of lattice spin systems. It turned out that the transfer operator method had indeed been used already some time ago in the p-adic setup of zeta functions by B. Dwork (see [Dw60] or [Ro86]) who constructed such an operator to show rationality of the Artin-Weil zeta function for projective algebraic varieties over finite fields and proved in this way part of the Weil conjectures (see [We49]). More recently the method also yielded a completely new approach to Selberg's zeta function, which can also be viewed as a dynamical zeta function for the geodesic flow on surfaces of constant negative curvature (see [Ma91]). Indeed, the aforementioned Artin-Weil zeta function is nothing but the dynamical Artin-Mazur zeta function (see [ArMa65]) for the Frobenius map of the algebraic variety. Typically, such dynamical zeta functions can be expressed in terms of some kind of Fredholm determinant of the transfer operator, which therefore allows a spectral interpretation of the zeros and poles of these functions. The existence of such an interpretation is one of the challenging open problems for all arithmetic zeta functions of number theory and algebraic geometry (see [Be86], [Co96], [De99]). Obviously such a spectral interpretation is also closely related to another famous open problem for these zeta and more general L-functions, namely the general Riemann hypothesis: one expects that the zeros and poles of such functions are located on critical lines in the complex plane as one does in the special case of the well known Riemann zeta function. Presently it is not known whether such a conjecture makes sense also for general dynamical zeta functions which are, unlike the Selberg- or the Artin-Weil zeta functions, not related to arithmetics.

In the present paper we address this problem for the Ruelle zeta function of a certain subshift of finite type which in the physics literature has become known as the Kac-Baker model (see [Ka66]). M. Kac got interested in this model while trying to understand the mathematics behind the phenomenon of phase transitions in systems of statistical mechanics with weak long-range interactions like in the van der Waals gas (see [Ba61]). The model he considered is an Ising spin system on a 1-dimensional lattice with a 2-body interaction given by a finite superposition of terms decaying exponentially fast with the distance between the different spins. His real interest was certainly in a system with a continuous superposition of such exponentially decaying terms to model also interactions decaying only polynomially fast as is the case in the van der Waals gas. However his method did not allow him to treat this limiting case in a rigorous way.

We have chosen the Kac model since its dynamical zeta function can be understood rather well by the transfer operator method. On the other hand there seems to be no obvious connection of this zeta function to any arithmetic zeta functions for which a general Riemann hypothesis is known to hold.

There exist two rather different transfer operators for this model which allow to express its zeta function as Fredholm determinants of these operators (see [Ma80], [ViMa77], [Ka66]). Up to now, however, it was not known how these two operators are related to each other. Our investigations show that the Ruelle operator $\mathcal{L}_\beta$ is equivalent to a modified Kac operator $\widetilde{\mathcal{G}}_\beta$ basically through a Segal-Bargmann transformation. This transformation establishes a unitary map between the Hilbert space of square integrable functions on the real line, where the Kac operator acts, and the Fock space of entire functions on the complex plane square integrable with respect to a certain weight function, to which the Ruelle operator can be restricted. For positive values of the parameter β the operator $\widetilde{\mathcal{G}}_\beta$ has the same spectrum as Kac's original operator $\mathcal{G}_\beta$. Since this operator $\mathcal{G}_\beta$ is a symmetric, positive definite trace class operator for $\beta > 0$ its eigenvalues and hence also those of the modified operator $\widetilde{\mathcal{G}}_\beta$ and of the Ruelle operator $\mathcal{L}_\beta$ are positive for these β-values. Furthermore we can show that the operator $\widetilde{\mathcal{G}}_\beta$ has real spectrum also for negative β so that the Ruelle operator $\mathcal{L}_\beta$ indeed has real spectrum for real β. Since $\mathcal{L}_\beta$ defines a family of trace class operators holomorphic in the variable β we can show that the zeta functions of a whole class of Kac models extend meromorphically to the entire complex β-plane and have infinitely many nontrivial zeros on the real line. For a special case of the parameters we can also show the existence of infinitely many "trivial" zeros of this dynamical zeta function located on a line parallel to the imaginary axis in the complex β-plane. Thus for this function an analogue of the Riemann hypothesis seems plausible. The present paper generalizes analogous results in [HiMa01] for the case of an interaction consisting of a single exponentially decaying term.

In detail the present paper is organized as follows: in Section 2 we recall the definition of the Kac-Baker models and derive their Ruelle transfer operators. Further, we show how the dynamical zeta function of these models can be expressed through Fredholm determinants of the Ruelle operators. In Section 3 we derive, basically following Gutzwiller (see [Gu82]), the Kac operator appropriate to our problem and show that the zeta function can be expressed for positive β also in terms of Fredholm determinants of this Kac operator. In Section 4 we show how the kernel of the Kac operator is related to a certain form of Mehler's formula for the Hermite functions which allows us to diagonalize an integral operator closely related to the Kac operator. In Section 5 we introduce the Fock spaces and the Segal-Bargmann transform and show how the Ruelle and a modified Kac operator can be directly related to each other. There we show that for real β the two operators have the same spectrum and give explicit expressions relating the eigenfunctions of the two operators for nonvanishing eigenvalues. In Section 6 we derive the asymptotic behavior of the eigenvalues of the Ruelle operator for large positive and negative values of β and apply it to get the results on the location of poles and zeros of the zeta function on the real line. In Section 7 we give explicit expressions for the matrix elements of a modified Kac-Gutzwiller operator in the Hilbert space basis given by the Hermite functions which seem best suited for future numerical calculations.

2 The Ruelle operator for the Kac-Baker model

The generalized Kac model describes a 1-dimensional lattice spin system with a 2-body interaction which is a superposition of finitely many exponentially decaying terms. More precisely, for $F := \{\pm 1\}$, $\underline{\xi} = (\xi_n)_{n\in\mathbb{Z}_+} \in F^{\mathbb{Z}_+}$, and $i, j \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ we set

$$\phi_{ij}(\underline{\xi}) := -\xi_i\xi_j \sum_{l=1}^{m} J_l \lambda_l^{|i-j|},$$

where $m \in \mathbb{N}$ and the parameters $J_l > 0$ and $0 < \lambda_l < 1$ are fixed and describe the interaction strengths and the different decay rates. The interaction energy of a configuration $\underline{\xi} \in F^{\mathbb{Z}_+}$ when restricted to the finite sublattice $\mathbb{Z}_{[n-1]} = \{0, 1, \ldots, n-1\}$ is then given as

$$U_n(\underline{\xi}) := U_{\mathbb{Z}_{[n-1]}}(\underline{\xi}) = \sum_{i=0}^{n-1} \sum_{j=1}^{\infty} \phi_{i,i+j}(\underline{\xi}).$$

When inserting the explicit form of ϕ one gets

$$U_n(\underline{\xi}) = -\sum_{i=0}^{n-1} \sum_{j=1}^{\infty} \xi_i \xi_{i+j} \sum_{l=1}^{m} J_l \lambda_l^j.$$

For $\beta \in \mathbb{C}$ the partition functions $Z_n(\beta)$ for the finite sublattices $\mathbb{Z}_{[n-1]}$ with periodic boundary conditions are defined as

$$Z_n(\beta) := \sum_{\underline{\xi} \in \mathrm{Per}_n} \exp\left(-\beta U_n(\underline{\xi})\right), \tag{1}$$

where Per_n denotes the set of configurations $\underline{\xi} \in F^{\mathbb{Z}_+}$ which are periodic with period n. That means $\underline{\xi} \in \mathrm{Per}_n$ if and only if $\xi_{i+n} = \xi_i$ for all $i \in \mathbb{Z}_+$. Defining the shift $\tau : F^{\mathbb{Z}_+} \to F^{\mathbb{Z}_+}$ by

$$\left(\tau\underline{\xi}\right)_i := \xi_{i+1} \quad \text{if } \underline{\xi} = (\xi_i)_{i \in \mathbb{Z}_+},$$

one has $\mathrm{Per}_n = \mathrm{Fix}\,\tau^n = \left\{\underline{\xi} \in F^{\mathbb{Z}_+} : \tau^n \underline{\xi} = \underline{\xi}\right\}$.

To the dynamical system $\left(F^{\mathbb{Z}_+}, \tau\right)$ one can associate the Ruelle zeta function

$$\zeta_R(z, \beta) := \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\beta)\right). \tag{2}$$

Note that $|U_n(\underline{\xi})| \leq n \sum_{l=1}^{m} \frac{\lambda_l J_l}{1-\lambda_l} =: nc$ so that $|Z_n(\beta)| \leq (2e^{|\beta| c})^n$. Therefore the series defining ζ_R converges in a neighborhood of $(0,0)$ in $\mathbb{C}^2$. We will show in Proposition 2.4 that ζ_R can in fact be extended to a meromorphic function on $\mathbb{C}^2$.

To determine the analytic properties of this function one makes use of the transfer operator technique. Note first that the configuration space $F^{\mathbb{Z}_+}$ is compact and metrizable with respect to the product topology. For each $\beta \in \mathbb{C}$ one can define the Ruelle transfer operator $\mathcal{L}_\beta$ which will act on the space of observables of the lattice spin system, i.e. on $\mathcal{C}(F^{\mathbb{Z}_+})$, the space of continuous functions on $F^{\mathbb{Z}_+}$. In this framework one sets

$$\left(\mathcal{L}_\beta f\right)\left(\underline{\xi}\right) := \sum_{\underline{\eta} \in \tau^{-1}(\underline{\xi})} \exp\left(-\beta U_1(\underline{\eta})\right) f(\underline{\eta}).$$

Inserting the explicit expression for $U_1(\underline{\eta})$ one finds

$$\left(\mathcal{L}_\beta f\right)(\underline{\xi}) = \sum_{\sigma = \pm 1} \exp\left(\beta\sigma \sum_{j=0}^{\infty} \xi_j \sum_{l=1}^{m} J_l \lambda_l^{j+1}\right) f\left((\sigma, \underline{\xi})\right), \tag{3}$$

where $(\sigma, \underline{\xi}) := \underline{\eta}$ with $\eta_0 = \sigma$ and $\eta_j = \xi_{j-1}$ for all $j \in \mathbb{N}$. Generalizing the arguments for the case $m = 1$ in [Ma80] one introduces the map $\underline{z} = (z_l)_{l=1,..,m} : F^{\mathbb{Z}_+} \to \mathbb{R}^m$ defined by

$$z_l(\underline{\xi}) := \sum_{i=0}^{\infty} \xi_i \lambda_l^{i+1}.$$

Since $z_l(\sigma, \underline{\xi}) = \sigma\lambda_l + \lambda_l z_l(\underline{\xi})$ for all l, the operator $\mathcal{L}_\beta$ leaves the space of functions $\underline{z}^*(\varphi) := \varphi \circ \underline{z}$ with $\varphi \in \mathcal{C}(\mathbb{R}^m)$ invariant. Thus we obtain a factorization

$$\begin{tikzcd} \mathcal{C}(F^{\mathbb{Z}_+}) \arrow[r, "\mathcal{L}_\beta"] & \mathcal{C}(F^{\mathbb{Z}_+}) \\ \mathcal{C}(\mathbb{R}^m) \arrow[u, "\underline{z}^*"] \arrow[r, dotted] & \mathcal{C}(\mathbb{R}^m) \arrow[u, "\underline{z}^*"'] \end{tikzcd}$$

of $\mathcal{L}_\beta$ through $\mathcal{C}(\mathbb{R}^m)$, which we will denote again by $\mathcal{L}_\beta$ and which is of the form

$$(\mathcal{L}_\beta g)(\underline{z}) = e^{\beta \underline{J} \cdot \underline{z}} g(\mathbf{\Lambda}\underline{z} + \underline{\lambda}) + e^{-\beta \underline{J} \cdot \underline{z}} g(\mathbf{\Lambda}\underline{z} - \underline{\lambda}), \tag{4}$$

where $\mathbf{\Lambda}$ is the diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_m$ and $\underline{J} = (J_1, \ldots, J_m)$.

Indeed, by the Ruelle-Perron-Frobenius Theorem (see [Zi00, Chap.4]) the iterates under $\mathcal{L}_\beta$ of the constant function $f(\underline{\xi}) = 1$ in $\mathcal{C}(F^{\mathbb{Z}_+})$ converge uniformly to the eigenfunction belonging to the leading eigenvalue of $\mathcal{L}_\beta$ and hence this eigenfunction belongs to the space $\mathcal{C}(\mathbb{R}^m)$. Therefore, when restricting the operator to the space $\mathcal{C}(\mathbb{R}^m)$ one does not lose the leading eigenvalue which is the most important one from the physical point of view. The physically most satisfying operator is obtained in fact by restricting the domain of the operator still further.

From the form of the operator $\mathcal{L}_\beta$ in the space $\mathcal{C}(\mathbb{R}^m)$ we see that, if g is a holomorphic function on the polycylinder

$$D = \{\underline{z} \in \mathbb{C}^m : |z_l| < R_l, l = 1, \ldots, m\}$$

with $R_l > \frac{\lambda_l}{1-\lambda_l}$, then also $(\mathcal{L}_\beta g)(\underline{z})$ is such a function. In fact, much more is true: Denote by $\mathcal{B}(D)$ the Banach space of holomorphic functions on D which extend continuously to the closure $\overline{D}$ of D. Then, according to [Ma80, Appendix B], we have

Proposition 2.1 *$\mathcal{L}_\beta \colon \mathcal{B}(D) \to \mathcal{B}(D)$ is with respect to the parameter $\beta \in \mathbb{C}$ a holomorphic family of nuclear operators of degree 0 in the sense of Grothendieck (see [Gr55]). In particular all the $\mathcal{L}_\beta$ are of trace class.* □

The trace can be computed using the holomorphic version of the Atiyah-Bott fixed point formula (see [AtBo67] and [Ma80], Appendix B or [Ru94],§1.12):

Lemma 2.2 *Fix $\varphi \in \mathcal{B}(D)$ and a continuous map $\psi : \overline{D} \to D$ which is holomorphic on D. Then ψ has a unique fixed point $\underline{z}^{\mathrm{fix}} \in D$ and the composition operator $A : \mathcal{B}(D) \to \mathcal{B}(D)$ defined by $Ag := \varphi \cdot (g \circ \psi)$ is trace class with trace*

$$\operatorname{trace}(A) = \frac{\varphi(\underline{z}^{\mathrm{fix}})}{\det\left(1 - \psi'(\underline{z}^{\mathrm{fix}})\right)}.$$

□

Proposition 2.3 *The partition function $Z_n(\beta)$ of the Kac-Baker model can be expressed through the traces of the powers of the Ruelle transfer operator $\mathcal{L}_\beta$ via*

$$Z_n(\beta) = \left(\prod_{l=1}^{m}(1-\lambda_l^n)\right)\operatorname{trace}\mathcal{L}_\beta^n.$$

Proof. The defining equation (1) for the partition function $Z_n(\beta)$ can be rewritten as

$$Z_n(\beta) = \sum_{\underline{\xi}\in \mathrm{Per}_n} \exp\left(\beta \sum_{l=1}^{m} J_l \sum_{k=0}^{n-1}\sum_{i=1}^{\infty} \xi_k \xi_{k+i}\lambda_l^i\right).$$

Using the fact that $\underline{\xi} \in \mathrm{Per}_n$ implies $\xi_{i+n} = \xi_i$ for all $i \in \mathbb{Z}_+$ one gets

$$Z_n(\beta) = \sum_{\underline{\sigma}\in F^n} \exp\left(\beta \sum_{l=1}^{m} \frac{J_l}{1-\lambda_l^n} \sum_{k=0}^{n-1}\sum_{i=1}^{n} \sigma_k\sigma_{k+i}\lambda_l^i\right),$$

where $\sigma_{n+i} = \sigma_i$ for all i. On the other hand the n-th iterate of the transfer operator $\mathcal{L}_\beta$ from (4) acting on the Banach space $\mathcal{B}(D)$ is given by

$$\begin{aligned}&\left(\mathcal{L}_\beta^n g\right)(\underline{z}) = \\ &= \sum_{\underline{\sigma}\in F^n} \exp\left(\beta\sum_{l=1}^{m} J_l \left(\sum_{k=1}^{n}\sigma_k\lambda_l^{n-k}z_l + \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sigma_k\sigma_{k+i}\lambda_l^i\right)\right) g\left(\sum_{i=1}^{n}\sigma_i\underline{\lambda}^i + \mathbf{\Lambda}^n\underline{z}\right),\end{aligned}$$

where $\underline{\lambda}^i := (\lambda_1^i,\ldots,\lambda_m^i)$. We apply Lemma 2.2 to the maps $\psi_{\underline{\sigma}}$ defined by

$$\psi_{\underline{\sigma}}(\underline{z}) := \left(\sum_{i=1}^{n}\sigma_i\underline{\lambda}^i + \mathbf{\Lambda}^n\underline{z}\right)$$

for which the fixed points are given by

$$\underline{z}_{\underline{\sigma}}^{\mathrm{fix}} = (1-\mathbf{\Lambda}^n)^{-1}\sum_{i=1}^{n}\sigma_i\underline{\lambda}^i.$$

The result is

$$\begin{aligned}\operatorname{trace}\mathcal{L}_\beta^n \quad &= \quad \frac{1}{\prod_{l=1}^{m}(1-\lambda_l^n)}\sum_{\underline{\sigma}\in F^n}\exp\left(\beta\sum_{l=1}^{m}\frac{J_l}{1-\lambda_l^n}\left(\sum_{k=1}^{n}\sum_{i=1}^{n}\sigma_k\sigma_i\lambda_l^{n-k+i} + \right.\right.\\ &\left.\left. + \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sigma_k\sigma_{k+i}\lambda_l^i - \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sigma_k\sigma_{k+i}\lambda_l^{n+i}\right)\right).\end{aligned}$$

But

$$\sum_{k=1}^{n-1}\sum_{i=k+1}^{n}\sigma_k\sigma_i\lambda_l^{n-k+i} = \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sigma_k\sigma_{k+i}\lambda_l^{n+i}$$

and hence

$$\begin{aligned}\operatorname{trace}\mathcal{L}_\beta^n \quad &= \quad \frac{1}{\prod_{l=1}^{m}(1-\lambda_l^n)}\sum_{\underline{\sigma}\in F^n}\exp\left(\beta\sum_{l=1}^{m}\frac{J_l}{1-\lambda_l^n}\left(\sigma_n\sum_{i=1}^{n}\sigma_i\lambda_l^i + \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sigma_k\sigma_{k+i}\lambda_l^i + \right.\right.\\ &\left.\left. + \sum_{k=1}^{n-1}\sum_{i=1}^{k}\sigma_k\sigma_i\lambda_l^{n-k+i}\right)\right).\end{aligned}$$

Changing the order of summation in the last double sum we finally get

$$\operatorname{trace} \mathcal{L}_\beta^n = \frac{1}{\prod_{l=1}^m (1-\lambda_l^n)} \sum_{\underline{\sigma} \in F^n} \exp \left(\beta \sum_{l=1}^m \frac{J_l}{1-\lambda_l^n} \sum_{k=1}^n \sum_{i=1}^n \sigma_k \sigma_{k+i} \lambda_l^i \right)$$

which up to the factor $\prod_{l=1}^m \frac{1}{1-\lambda_l^n}$ is just the partition function $Z_n(\beta)$. □

In view of the identity $\prod_{l=1}^m (1-\lambda_l^n) = \sum_{\underline{\alpha} \in \{0,1\}^m} (-1)^{|\underline{\alpha}|} \lambda_1^{n\alpha_l} \cdots \lambda_m^{n\alpha_m}$ Proposition 2.3 yields

$$Z_n(\beta) = \sum_{\underline{\alpha} \in \{0,1\}^m} (-1)^{|\underline{\alpha}|} \operatorname{trace} \left(\left(\prod_{l=1}^m \lambda_l^{\alpha_l} \right)^n \mathcal{L}_\beta^n \right),$$

so that

$$\begin{aligned} \zeta_R(z,\beta) &= \exp \left(\sum_{n=1}^\infty \frac{z^n}{n} \sum_{\underline{\alpha} \in \{0,1\}^m} (-1)^{|\underline{\alpha}|} \operatorname{trace} \left(\left(\prod_{l=1}^m \lambda_l^{\alpha_l} \right) \mathcal{L}_\beta \right)^n \right) \\ &= \exp \left(\operatorname{trace} \sum_{\underline{\alpha} \in \{0,1\}^m} (-1)^{|\underline{\alpha}|} (-1) \log \left(1 - z \left(\prod_{l=1}^m \lambda_l^{\alpha_l} \right) \mathcal{L}_\beta \right) \right) \\ &= \prod_{\underline{\alpha} \in \{0,1\}^m} \det \left(1 - z \prod_{l=1}^m (\lambda_l^{\alpha_l}) \mathcal{L}_\beta \right)^{(-1)^{|\underline{\alpha}|+1}}. \end{aligned}$$

Together with Proposition 2.1 this proves the following proposition.

Proposition 2.4 *The Ruelle zeta function $\zeta_R(z,\beta)$ defined in (2) for the Kac-Baker model with decay rates $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m) \in]0,1[^m$ can be extended to a meromorphic family $(\zeta_R(\cdot,\beta))_{\beta \in \mathbb{C}}$ of meromorphic functions on $\mathbb{C}$ via the formula*

$$\zeta_R(z,\beta) = \prod_{\underline{\alpha} \in \{0,1\}^m} \det \left(1 - z \underline{\lambda}^{\underline{\alpha}} \mathcal{L}_\beta \right)^{(-1)^{|\underline{\alpha}|+1}}.$$

□

Remark 2.5 The analytic properties in the variable z are well known for general lattice spin systems with exponentially fast decaying interactions. For $m = 1$ Proposition 2.4 was proved in [Ma80]. □

3 The Kac-Gutzwiller operator

In [Ka59] M. Kac found another operator whose traces are directly related to the partition functions $Z_n(\beta)$ of the Kac-Baker model. Kac did not work with periodic but open boundary conditions and hence his operator has to be modified a bit to give the partition functions we use here. In the case $m = 1$ M. Gutzwiller already derived this operator (see [Gu82]) and the case of general $m \geqslant 1$ can be handled similarly. Just like Gutzwiller and Kac we start with an identity for Gaussian integrals known already to C. Cramér: For a positive definite $(n\times n)$-matrix $\mathbb{A}$ and $\underline{x}\in\mathbb{R}^n$ we have (see [Cr46] or the appendix of [Fo89])

$$e^{\frac{1}{2}(\underline{x}\cdot\mathbb{A}\underline{x})} = (2\pi)^{-\frac{n}{2}}|\det\mathbb{B}|^{\frac{1}{2}}\int_{\mathbb{R}^n} e^{\underline{x}\cdot\underline{z}}e^{-\frac{1}{2}(\underline{z}\cdot\mathbb{B}\underline{z})}d\underline{z}, \tag{5}$$

where $\mathbb{B}=\mathbb{A}^{-1}$.

Any periodic configuration $\underline{\xi}^p\in\mathrm{Per}_n$ can be extended to a periodic configuration on the entire lattice $\mathbb{Z}$ with the same period which we again denote by $\underline{\xi}^p$.

Now consider the $(n\times n)$-matrix $\mathbb{A}^{(l)}$ given by

$$\mathbb{A}^{(l)}_{i,j} = \beta J_l \sum_{k=-\infty}^{+\infty}\exp\left(-\gamma_l\,|i-j+nk|\right), \quad 0\le i,j\le n-1,$$

where we choose the constants γ_l so that $e^{-\gamma_l}=\lambda_l$ with the coupling constants λ_l of the Kac-Baker model. Using $\underline{\xi}^p_n=\left(\xi^p_0,\dots,\xi^p_{n-1}\right)$ one finds (see [Gu82, §6] for this and the following results on matrix calculations)

$$\frac{\beta}{2}J_l\sum_{i=0}^{n-1}\sum_{j=-\infty}^{+\infty}\xi^p_i\xi^p_j\exp\left(-\gamma_l\,|i-j|\right) = \frac{1}{2}\left(\underline{\xi}^p_n\cdot\mathbb{A}^{(l)}\underline{\xi}^p_n\right).$$

But

$$\frac{\beta}{2}\sum_{l=1}^{m}J_l\sum_{i=0}^{n-1}\sum_{j=-\infty}^{+\infty}\xi^p_i\xi^p_j\lambda_l^{|i-j|} = -\beta U_n\left(\underline{\xi}^p\right)+\frac{\beta}{2}n\sum_{l=1}^{m}J_l, \tag{6}$$

i.e. the left hand side is, up to the constant term given by the sum of the J_l, just the interaction energy of the periodic configuration $\underline{\xi}^p\in\mathrm{Per}_n$ for the Kac model.
The matrix $\mathbb{B}^{(l)}=\left(\mathbb{A}^{(l)}\right)^{-1}$ has for $n>2$ the following form

$$\mathbb{B}^{(l)} = \frac{1}{\beta J_l\sinh\gamma_l}\begin{pmatrix}
\cosh\gamma_l & -\frac{1}{2} & 0 & \dots & \dots & 0 & -\frac{1}{2}\\
-\frac{1}{2} & \cosh\gamma_l & -\frac{1}{2} & 0 & \dots & 0 & 0\\
0 & -\frac{1}{2} & \cosh\gamma_l & -\frac{1}{2} & 0 & \dots & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
0 & \dots & 0 & -\frac{1}{2} & \cosh\gamma_l & -\frac{1}{2} & 0\\
0 & 0 & \dots & 0 & -\frac{1}{2} & \cosh\gamma_l & -\frac{1}{2}\\
-\frac{1}{2} & 0 & \dots & \dots & 0 & -\frac{1}{2} & \cosh\gamma_l
\end{pmatrix},$$

which for positive β is a positive definite matrix with determinant

$$|\det\mathbb{B}^{(l)}| = \frac{4}{(2\beta J_l\sinh\gamma_l)^n}\left(\sinh\frac{n\gamma}{2}\right)^2.$$

This formula holds also for $n=1$ and $n=2$. That $\mathbb{B}^{(l)}$ is indeed positive definite one can see as follows: for arbitrary $\underline{x}\in\mathbb{R}^n$ one finds

$$\left(\underline{x}\cdot\mathbb{B}^{(l)}\underline{x}\right) = \frac{1}{\beta J_l}\left(\sum_{i=1}^{n}\coth(\gamma_l)x_i^2 - \frac{\sum_{i=1}^{n}x_ix_{i-1}}{\sinh\gamma_l}\right)$$

where $x_0 = x_n$. A simple calculation (see also Proposition 4.1) shows that the following identity holds

$$\coth(\gamma_l)\sum_{i=1}^{n} x_i^{(l)2} - \frac{\sum\limits_{i=1}^{n} x_i^{(l)} x_{i-1}^{(l)}}{\sinh\gamma_l} = \frac{1}{2}\left(\tanh\left(\frac{\gamma_l}{2}\right)\sum_{i=1}^{n}\left(x_i^{(l)2} + x_{i-1}^{(l)2}\right) + \frac{\sum_{i=1}^{n}\left(x_i^{(l)} - x_{i-1}^{(l)}\right)^2}{\sinh\gamma_l}\right) \tag{7}$$

and hence for $\beta J_l > 0$ and $\gamma_l > 0$ one finds $\left(\underline{x}, \mathbb{B}^{(l)}\underline{x}\right) \geqslant 0$. Inserting (6) into formula (5) one calculates

$$\begin{aligned}
e^{-\beta U_n(\underline{\xi}^p)+\frac{n\beta}{2}\sum\limits_{l=1}^{m} J_l} &= \prod_{l=1}^{m} e^{\frac{1}{2}\left(\underline{\xi}_n^p\cdot\mathbb{A}^{(l)}\underline{\xi}_n^p\right)} \\
&= \prod_{l=1}^{m}\left((2\pi)^{-\frac{n}{2}}\frac{2}{(2\beta J_l\sinh\gamma_l)^{\frac{n}{2}}}\sinh\left(\frac{n\gamma_l}{2}\right)\int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\left(\underline{z}^{(l)}\cdot\mathbb{B}^{(l)}\underline{z}^{(l)}\right)} e^{\underline{\xi}_n^p\cdot\underline{z}^{(l)}} d\underline{z}^{(l)}\right) \\
&= \frac{2^m}{2^{nm}}\left(\prod_{l=1}^{m}\frac{\sinh\frac{n\gamma_l}{2}}{(\beta\pi J_l\sinh\gamma_l)^{\frac{n}{2}}}\right)\int\limits_{\mathbb{R}^n}\dots\int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\sum\limits_{l=1}^{m}\left(\underline{z}^{(l)}\cdot\mathbb{B}^{(l)}\underline{z}^{(l)}\right)} e^{\sum\limits_{l=1}^{m}\left(\underline{\xi}_n^p\cdot\underline{z}^{(l)}\right)} d\underline{z}^{(1)}\dots d\underline{z}^{(m)}.
\end{aligned}$$

The change $\underline{z}^{(l)} =: \sqrt{\beta J_l}\,\underline{x}^{(l)}$ of integration variables yields

$$\begin{aligned}
&e^{-\beta U_n(\underline{\xi}^p)+\frac{n\beta}{2}\sum_{l=1}^{m} J_l} = \\
&= \frac{2^m}{2^{nm}}\prod_{l=1}^{m}\frac{\left(\sinh\frac{n\gamma_l}{2}\right)(\beta J_l)^{\frac{n}{2}}}{(\beta J_l\pi\sinh\gamma_l)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\dots\int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\sum_{l=1}^{m}\beta J_l\left(\underline{x}^{(l)}\cdot\mathbb{B}^{(l)}\underline{x}^{(l)}\right)} e^{\sum_{l=1}^{m}\sqrt{\beta J_l}\left(\underline{\xi}_n^p\cdot\underline{x}^{(l)}\right)} d\underline{x}^{(1)}\dots d\underline{x}^{(m)}.
\end{aligned}$$

Performing the summation over all the periodic configurations $\underline{\xi} \in \mathrm{Per}_n$ then gives

$$\begin{aligned}
e^{\frac{n\beta}{2}\sum\limits_{l=1}^{m} J_l}\sum_{\underline{\xi}\in\mathrm{Per}_n} e^{-\beta U_n(\underline{\xi})} &= \frac{2^m}{2^{n(m-1)}}\prod_{l=1}^{m}\left(\frac{\sinh\frac{n\gamma_l}{2}}{(\pi\sinh\gamma_l)^{\frac{n}{2}}}\right)\cdot \\
&\cdot\int\limits_{\mathbb{R}^n}\dots\int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\sum_{l=1}^{m}\beta J_l\left(\underline{x}^{(l)}\cdot\mathbb{B}^{(l)}\underline{x}^{(l)}\right)}\prod_{i=1}^{n}\cosh\left(\sum_{l=1}^{m}\sqrt{\beta J_l}x_i^{(l)}\right) d\underline{x}^{(1)}\dots d\underline{x}^{(m)}.
\end{aligned}$$

Inserting the explicit form of the matrix $\mathbb{B}^{(l)} = \left(\mathbb{A}^{(l)}\right)^{-1}$ we find

$$\begin{aligned}
e^{\frac{n\beta}{2}\sum_{l=1}^{m} J_l} Z_n(\beta) &= \frac{2^m}{2^{n(m-1)}}\prod_{l=1}^{m}\frac{\sinh\frac{n\gamma_l}{2}}{(\pi\sinh\gamma_l)^{\frac{n}{2}}}\cdot \\
&\cdot\int\limits_{\mathbb{R}^n}\dots\int\limits_{\mathbb{R}^n}\exp\left(-\frac{1}{2}\sum_{l=1}^{m}\left((\coth\gamma_l)\sum_{i=1}^{n}(x_i^{(l)})^2 - \frac{\sum\limits_{i=1}^{n} x_i^{(l)}x_{i-1}^{(l)}}{\sinh\gamma_l}\right)\right)\cdot \\
&\cdot\prod_{i=1}^{n}\cosh\left(\sum_{l=1}^{m}\sqrt{\beta J_l}x_i^{(l)}\right) d\underline{x}^{(1)}\dots d\underline{x}^{(m)},
\end{aligned}$$

where $x_0^{(l)} = x_n^{(l)}$.

For $\beta \geqslant 0$ we introduce the kernel function

$$\mathcal{K}_\beta\left(\underline{\xi},\underline{\eta}\right) := \frac{\left(\cosh\left(\sum\limits_{l=1}^m \sqrt{\beta J_l}\xi_l\right)\cosh\left(\sum\limits_{l=1}^m \sqrt{\beta J_l}\eta_l\right)\right)^{\frac{1}{2}}}{2^{(m-1)}\prod\limits_{l=1}^m(\pi\sinh\gamma_l)^{\frac{1}{2}}}\exp\left(-\frac{1}{4}\left(\sum_{l=1}^m\left((\tanh\frac{\gamma_l}{2})\left(\xi_l^2+\eta_l^2\right)+\frac{(\xi_l-\eta_l)^2}{\sinh\gamma_l}\right)\right)\right), \tag{8}$$

where $\underline{\xi}=(\xi_1,\dots,\xi_m)\in\mathbb{R}^m$ and $\underline{\eta}=(\eta_1,\dots,\eta_m)\in\mathbb{R}^m$. We call the associated operator $\mathcal{K}_\beta$ on $L^2(\mathbb{R}^m,d\underline{\xi})$ defined by

$$(\mathcal{K}_\beta f)\left(\underline{\xi}\right) = \int\limits_{\mathbb{R}^m}\mathcal{K}_\beta\left(\underline{\xi},\underline{\eta}\right)f\left(\underline{\eta}\right)d\underline{\eta}$$

the Kac-Gutzwiller transfer operator. Note that the kernel of $\mathcal{K}_\beta$ decreases fast enough to ensure that $\mathcal{K}_\beta$ is of trace class with trace $\mathcal{K}_\beta = \int_{\mathbb{R}^m}\mathcal{K}_\beta(\underline{\xi},\underline{\xi})d\underline{\xi}$. Calculating the trace of the iterates of $\mathcal{K}_\beta$ and comparing the result to the above formula for the partition function $Z_n(\beta)$ after inserting (7) we find

$$Z_n(\beta) = 2^m\prod_{l=1}^m\left(\sinh\frac{n\gamma_l}{2}\right)e^{-\frac{n\beta}{2}\sum_{l=1}^m J_l}\mathrm{trace}\,\mathcal{K}_\beta^n. \tag{9}$$

To simplify the expression for $Z_n(\beta)$ we introduce the rescaled Kac-Gutzwiller operator $\mathcal{G}_\beta$: $L^2\left(\mathbb{R}^m,d\underline{\xi}\right)\to L^2\left(\mathbb{R}^m,d\underline{\xi}\right)$ defined by

$$\mathcal{G}_\beta := \prod_{l=1}^m\left(\lambda_l e^{\beta J_l}\right)^{-\frac{1}{2}}\mathcal{K}_\beta. \tag{10}$$

In view of

$$Z_n(\beta) = (\prod_{l=1}^m(1-\lambda_l^n))\mathrm{trace}\,(\prod_{l=1}^m e^{\frac{n\gamma_l}{2}}e^{-\frac{\beta J_l n}{2}})\mathcal{K}_\beta^n,$$

which is a simple reformulation of (9) we finally have shown the following proposition.

Proposition 3.1 *For $\beta \geqslant 0$ the partition function $Z_n(\beta)$ of the Kac-Baker model can be expressed through the traces of the powers of the rescaled Kac-Gutzwiller operator $\mathcal{G}_\beta$ via*

$$Z_n(\beta) = \left(\prod_{l=1}^m(1-\lambda_l^n)\right)\mathrm{trace}\,\mathcal{G}_\beta^n.$$

□

An argument similar to the one we used for the Ruelle operator $\mathcal{L}_\beta$ in the proof of Proposition 2.4 now shows

Proposition 3.2 *For $\beta \geqslant 0$ the Ruelle zeta function $\zeta_R(z,\beta)$ for the Kac-Baker model can be written in terms of the modified Kac-Gutzwiller operator via*

$$\zeta_R(z,\beta) = \prod_{\underline{\alpha}\in\{0,1\}^m}\det\left(1-z\underline{\lambda}^{\underline{\alpha}}\mathcal{G}_\beta\right)^{(-1)^{|\underline{\alpha}|+1}}.$$

□

Since the zeta function $\zeta_R(z,\beta)$ is meromorphic and its divisor for fixed β uniquely determined, it is not too difficult to see that at least for generic values of the parameters $\beta > 0$ and $\underline{\lambda} \in]0,1[^m$ the spectra of the two operators $\mathcal{L}_\beta$ and $\mathcal{G}_\beta$ have to be identical. This indeed has been shown in the case $m = 1$ already by B. Moritz in (see [Mo89]). We will show however (see Theorem 5.12) that the spectra of the two operators coincide for any real β and all parameters $\underline{\lambda}$. Hence the Fredholm determinants $\det(1 - z\mathcal{G}_\beta)$ and $\det(1 - z\mathcal{L}_\beta)$ of the Kac-Gutzwiller operator and the Ruelle operator coincide on the real axis and extend to a holomorphic function in the entire β plane even if the operator $\mathcal{G}_\beta$ contrary to the operator $\mathcal{L}_\beta$ has itself no such analytic continuation to the entire β-plane.

4 Hermite Functions and Mehler's Formula

Consider the operators

$$Z_j = m_{x_j} + \frac{1}{2\pi}\frac{\partial}{\partial x_j} \quad , \quad Z_j^* = m_{x_j} - \frac{1}{2\pi}\frac{\partial}{\partial x_j} \quad , \quad j = 1, \dots, n,$$

where m_{x_j} denotes the multiplication operator

$$\left(m_g f\right)(\underline{x}) = g(\underline{x}) f(\underline{x})$$

in the space $L^2\left(\mathbb{R}^m, d\underline{x}\right)$ for $g(\underline{x}) = x_j$ and $\underline{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$.

The Hermite functions $h_{\underline{\alpha}} \in L^2\left(\mathbb{R}^m, d\underline{x}\right)$ with $\underline{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}_0^m$ are given by (see [Fo89, p.51])

$$\begin{aligned} h_{\underline{0}}(\underline{x}) &= 2^{\frac{m}{4}} e^{-\pi \underline{x}\cdot\underline{x}} \\ h_{\underline{\alpha}}(\underline{x}) &= \sqrt{\frac{\pi^{|\underline{\alpha}|}}{\underline{\alpha}!}} \left(\underline{Z}^{*\underline{\alpha}} h_{\underline{0}}\right)(\underline{x}) = \frac{2^{\frac{m}{4}}}{\sqrt{\underline{\alpha}!}} \left(\frac{-1}{2\sqrt{\pi}}\right)^{|\underline{\alpha}|} e^{\pi \underline{x}\cdot\underline{x}} \left(\frac{\partial}{\partial \underline{x}}\right)^{\underline{\alpha}} e^{-2\pi \underline{x}\cdot\underline{x}}, \end{aligned}$$

where $\underline{\alpha} = (\alpha_1, \dots, \alpha_m)$ with $\alpha_i \in \mathbb{N}_0$ for all $1 \le i \le m$, $|\underline{\alpha}| = \sum_{i=1}^m \alpha_i$ and $\underline{\alpha}! = \prod_{i=1}^m \alpha_i!$. Moreover, $\underline{Z}^{*\underline{\alpha}}$ denotes the operator $Z_1^{*\alpha_1} \cdots Z_m^{*\alpha_m}$.

The Hermite functions are known to be an orthonormal basis of the Hilbert space $L^2\left(\mathbb{R}^m, d\underline{x}\right)$ and the following formula due to Mehler holds (see[Fo89]):

$$\sum_{\underline{\alpha}\in\mathbb{N}_0^m} w^{|\underline{\alpha}|} h_{\underline{\alpha}}(\underline{x})\, h_{\underline{\alpha}}(\underline{y}) = \left(\frac{2}{1-w^2}\right)^{\frac{m}{2}} \exp\left(\frac{-\pi\left(1+w^2\right)\left(\underline{x}^2 + \underline{y}^2\right) + 4\pi w\, \underline{x}\cdot\underline{y}}{1-w^2}\right),$$

where $|w| < 1$ and $\Re\frac{2}{1-w^2} > 0$. From the case $m = 1$ we then get for $\underline{\lambda} = (\lambda_1, \dots, \lambda_m)$ with $0 < \lambda_i < 1$ for $1 \le i \le m$ the identity

$$\prod_{l=1}^m \sum_{\alpha_l=0}^\infty \lambda_l^{\alpha_l} h_{\alpha_l}(x_l)\, h_{\alpha_l}(y_l) = \prod_{l=1}^m \left(\frac{2}{1-\lambda_l^2}\right)^{\frac{1}{2}} \cdot \exp\left(\sum_{l=1}^m \frac{-\pi\left(1+\lambda_l^2\right)\left(x_l^2+y_l^2\right) + 4\pi\lambda_l x_l y_l}{1-\lambda_l^2}\right).$$

A simple calculation presented for $m = 1$ in [HiMa01] shows that the following proposition is true.

Proposition 4.1 (i) *For $\lambda_l = e^{-\gamma_l}$ with $\Re\gamma_l > 0$ one has*

$$\frac{-\pi\left(1+\lambda_l^2\right)\left(x^2+y^2\right)+4\pi\lambda_l xy}{1-\lambda_l^2} = \frac{2\pi}{\sinh\gamma_l}\left(-\frac{1}{2}\left(x^2+y^2\right)\cosh\gamma_l + xy\right)$$

(ii) $\frac{1}{2\sinh\gamma_l}\left(-\frac{1}{2}\left(x^2+y^2\right)\cosh\gamma_l+xy\right) = -\frac{1}{4}\left(\left(x^2+y^2\right)\tanh\left(\frac{\gamma_l}{2}\right)+\frac{(x-y)^2}{\sinh\gamma_l}\right)$

(iii) *For $\lambda_l = e^{-\gamma_l}$ and $x_l = \xi_l\frac{1}{2\sqrt{\pi}}$, $y_l = \eta_l\frac{1}{2\sqrt{\pi}}$ one has the following version of Mehler's formula*

$$\sum_{\underline{\alpha}\in\mathbb{N}_0^m} \underline{\lambda}^{\underline{\alpha}} h_{\underline{\alpha}}\left(\underline{x}\right) h_{\underline{\alpha}}\left(\underline{y}\right) = \prod_{l=1}^m\left(\frac{1}{\lambda_l\sinh\gamma_l}\right)^{\frac{1}{2}}\exp\left(-\frac{1}{4}\sum_{l=1}^m\left(\left(\xi_l^2+\eta_l^2\right)\tanh\left(\frac{\gamma_l}{2}\right)+\frac{\left(\xi_l-\eta_l\right)^2}{\sinh\gamma_l}\right)\right)$$

□

In Proposition 4.1 (iii) we used only the fact that

$$h_{\underline{\alpha}}\left(\underline{x}\right) = h_{\alpha_1}\left(x_1\right)\cdots h_{\alpha_m}\left(x_m\right)$$

for $\underline{\alpha} = (\alpha_1,\ldots,\alpha_m)$, $\underline{x} = (x_1,\ldots,x_m)$ (see [Fo89, p.52]) and $\frac{2}{1-\lambda_l^2} = \frac{2e^{\gamma_l}}{e^{\gamma_l}-e^{-\gamma_l}} = (\lambda_l\sinh\gamma_l)^{-1}$.

Define next the kernel function

$$\widetilde{\mathcal{K}}\left(\underline{\xi},\underline{\eta}\right) = 2\prod_{l=1}^m\left(\frac{1}{4\pi\sinh\gamma_l}\right)^{\frac{1}{2}}\exp\left(-\frac{1}{4}\left(\sum_{l=1}^m\left(\xi_l^2+\eta_l^2\right)\tanh\frac{\gamma_l}{2}+\frac{\left(\xi_l-\eta_l\right)^2}{\sinh\gamma_l}\right)\right). \tag{11}$$

Then the kernel $\mathcal{K}_\beta(\underline{\xi},\underline{\eta})$ of the Kac-Gutzwiller transfer operator defined in (8) satisfies

$$\mathcal{K}_\beta(\underline{\xi},\underline{\eta}) = \left(\cosh\left(\sum_{l=1}^m\sqrt{\beta J_l}\xi_l\right)\cosh\left(\sum_{l=1}^m\sqrt{\beta J_l}\eta_l\right)\right)^{\frac{1}{2}}\widetilde{\mathcal{K}}(\underline{\xi},\underline{\eta}) \tag{12}$$

Lemma 4.2 *For an invertible real $(m\times m)$-matrix C and a smooth function $a:\mathbb{R}^m\to]0,\infty[$ consider the map*

$$\left(R_C f\right)\left(\underline{x}\right) = |\det(C)|^{\frac{1}{2}} f(C\underline{x}).$$

Then

$$R_C : L^2\left(\mathbb{R}^m, a(C^{-1}\underline{\xi})d\underline{\xi}\right)\to L^2\left(\mathbb{R}^m, a(\underline{x})d\underline{x}\right)$$

is an isomorphism of Hilbert spaces.

Proof.

$$\begin{aligned}\int_{\mathbb{R}^m}\left|\left(R_C f\right)\left(\underline{x}\right)\right|^2 a\left(\underline{x}\right)d\underline{x} &= \int_{\mathbb{R}^m}|\det(C)|\left|f\left(C\underline{x}\right)\right|^2 a(\underline{x})d\underline{x}\\ &= \int_{\mathbb{R}^m}\left|f\left(\underline{\xi}\right)\right|^2 a\left(C^{-1}\underline{\xi}\right)d\underline{\xi}\\ &= \|f\|^2_{L^2\left(\mathbb{R}^m,a\left(C^{-1}\underline{\xi}\right)d\underline{\xi}\right)}\end{aligned}$$

$\square$

If $K : L^2\left(\mathbb{R}^m, d\underline{\xi}\right) \to L^2\left(\mathbb{R}^m, d\underline{\xi}\right)$ is given by an integral kernel $K\left(\underline{\xi}, \underline{\eta}\right)$ then the induced operator $K_C := R_C \circ K \circ R_C^{-1} : L^2\left(\mathbb{R}^m, d\underline{x}\right) \to L^2\left(\mathbb{R}^m, d\underline{x}\right)$ has kernel

$$K_C\left(\underline{x}, \underline{y}\right) = |\det(C)| K\left(C\underline{x}, C\underline{y}\right)$$

as one easily verifies by a straightforward calculation using the transformation formula.

If $\underline{c} = (c_1, \ldots, c_m) \in \mathbb{R}^m$ with $c_i \neq 0$ for $1 \leq i \leq m$ and $\boldsymbol{C}$ is the diagonal matrix with the c_l on the diagonal, we simply write $R_{\underline{c}}$ for $R_{\boldsymbol{C}}$ and $K_{\underline{c}}$ for $K_{\boldsymbol{C}}$. Note that $\boldsymbol{C}\underline{x} = (c_1 x_1, \ldots, c_m x_m)$.

Lemma 4.3 *Let $a : \mathbb{R}^m \to [1, \infty[$ be a smooth function and K be a bounded operator on $L^2\left(\mathbb{R}^m, d\underline{x}\right)$ given by the kernel $K(\underline{x}, \underline{y})$ as*

$$\left(Kf\right)\left(\underline{x}\right) = \int\limits_{\mathbb{R}^m} K\left(\underline{x}, \underline{y}\right) f(\underline{y}) d\underline{y}.$$

Then

(a) *the operator $K \circ m_{\sqrt{a}}$ is an unbounded operator on $L^2\left(\mathbb{R}^m, d\underline{x}\right)$ with kernel $K\left(\underline{x}, \underline{y}\right) \sqrt{a(\underline{y})}$.*

(b) *the operator $m_{\frac{1}{\sqrt{a}}} \circ K$ is a bounded operator on $L^2\left(\mathbb{R}^m, d\underline{x}\right)$ with kernel $\frac{1}{\sqrt{a(\underline{x})}} K\left(\underline{x}, \underline{y}\right)$.*

Proof.

(a) This follows from

$$\left(K \circ m_{\sqrt{a}} f\right)\left(\underline{x}\right) = \int\limits_{\mathbb{R}^m} K\left(\underline{x}, \underline{y}\right) \sqrt{a(\underline{y})} f(\underline{y}) d\underline{y}$$

and

$$\int\limits_{\mathbb{R}^m} \left|\sqrt{a(\underline{x})} f(\underline{x})\right|^2 d\underline{x} = \int\limits_{\mathbb{R}^m} \left|f(\underline{x})\right|^2 a(\underline{x}) d\underline{x}.$$

(b) Calculate

$$\left(m_{\frac{1}{\sqrt{a}}} \circ Kf\right)(x) = \frac{1}{\sqrt{a(\underline{x})}} \int\limits_{\mathbb{R}^m} K(\underline{x}, \underline{y}) f(\underline{y}) d\underline{y} = \int\limits_{\mathbb{R}^m} \frac{1}{\sqrt{a(\underline{x})}} K(\underline{x}, \underline{y}) f(\underline{y}) d\underline{y}.$$

$\square$

Fix $a \colon \mathbb{R}^m \to [1, \infty[$. Then Lemma 4.3 yields the following commutative diagram

$$\begin{tikzcd}
L^2\left(\mathbb{R}^m, d\underline{x}\right) & L^2\left(\mathbb{R}^m, a(\underline{x}) d\underline{x}\right) \arrow[l, hook'] \arrow[r, "K"] \arrow[d, "m_{\sqrt{a}}"] & L^2\left(\mathbb{R}^m, d\underline{x}\right) \arrow[d, "\mathrm{id}"] \\
 & L^2\left(\mathbb{R}^m, d\underline{x}\right) \arrow[r, "K'"] \arrow[d, "\mathrm{id}"] & L^2\left(\mathbb{R}^m, d\underline{x}\right) \arrow[d, "m_{\frac{1}{\sqrt{a}}}"] \\
 & L^2\left(\mathbb{R}^m, d\underline{x}\right) \arrow[r, "K''"] & L^2\left(\mathbb{R}^m, d\underline{x}\right)
\end{tikzcd}$$

with integral operators K' and K'' given by the kernels

$$K'\left(\underline{x},\underline{y}\right) = K\left(\underline{x},\underline{y}\right)\frac{1}{\sqrt{a(\underline{y})}} \quad \text{and} \quad K''\left(\underline{x},\underline{y}\right) = \frac{1}{\sqrt{a(\underline{x})}} K\left(\underline{x},\underline{y}\right)\frac{1}{\sqrt{a(\underline{y})}}.$$

If the kernel $|K(x,y)|$ defines a bounded operator on $L^2(\mathbb{R}^m, d\underline{x})$, then also the operator K'' is bounded.

Now consider the Kac-Gutzwiller operator $\mathcal{K}_\beta$ with kernel $\mathcal{K}_\beta(\underline{\xi},\underline{\eta})$. For $\underline{s},\underline{x} \in \mathbb{R}^m$ we set $\cosh_{\underline{s}}(\underline{x}) := \cosh(\underline{s}\cdot\underline{x})$. With $\underline{J} := (J_1,\ldots,J_l)$ and $\underline{s} = \underline{s}_0 := 2\sqrt{\beta\pi}\underline{J}$ we choose the function $a := \cosh_{\underline{s}_0}$ and for $\underline{c} = \underline{c}_0 := 2\sqrt{\pi}(1,\ldots,1)$ we obtain

$$\begin{aligned}
\mathcal{K}_{\beta,\underline{c}_0}\left(\underline{x},\underline{y}\right) &= \left(2\sqrt{\pi}\right)^m \mathcal{K}_\beta\left(2\sqrt{\pi}\underline{x}, 2\sqrt{\pi}\underline{y}\right)\\
&= \left(2\sqrt{\pi}\right)^m \mathcal{K}_\beta\left(\underline{\xi},\underline{\eta}\right)\\
&= 2\left(\cosh_{\underline{s}_0}(\underline{x})\cosh_{\underline{s}_0}(\underline{x})\right)^{\frac{1}{2}} \prod_{l=1}^m \left(e^{-\frac{\gamma_l}{2}}\right) \sum_{\underline{\alpha}\in\mathbb{N}_0^m} \underline{\lambda}^{\underline{\alpha}} h_{\underline{\alpha}}(\underline{x})\, h_{\underline{\alpha}}(\underline{y})
\end{aligned}$$

Hence for the kernel $\mathcal{K}''_{\underline{c}_0}(\underline{x},\underline{y})$ of the operator $\mathcal{K}''_{\beta,\underline{c}_0} =: \mathcal{K}''_{\underline{c}_0}$, which does not depend on the variable β, one finds

$$\mathcal{K}''_{\underline{c}_0}\left(\underline{x},\underline{y}\right) := 2\sum_{\underline{\alpha}\in\mathbb{N}_0^m}\prod_{l=1}^m \left(e^{-\frac{\gamma_l}{2}}\right)\underline{\lambda}^{\underline{\alpha}} h_{\underline{\alpha}}(\underline{x})\, h_{\underline{\alpha}}(\underline{y})\,. \tag{13}$$

From the fact that the Hermite functions $h_{\underline{\alpha}}$ determine an orthonormal basis of $L^2(\mathbb{R}^m, d\underline{x})$ one concludes

$$\left(\mathcal{K}''_{\underline{c}_0} h_{\underline{\alpha}}\right)(\underline{x}) = 2\underline{\lambda}^{\underline{\alpha}}\underline{\lambda}^{\frac{1}{2}} h_{\underline{\alpha}}(\underline{x}), \tag{14}$$

i.e., the $h_{\underline{\alpha}}$ are the complete set of eigenfunctions of the operator $\mathcal{K}''_{\underline{c}_0}$ with eigenvalue

$$\rho_{\underline{\alpha}} := 2\underline{\lambda}^{\underline{\alpha}+\frac{1}{2}} = 2\prod_{l=1}^m \lambda_l^{\alpha_l+\frac{1}{2}}. \tag{15}$$

In particular, $\mathcal{K}''_{\underline{c}_0}$ is bounded. We will need this result later on. Note that Lemma 4.3 yields the following commutative diagram for the Kac-Gutzwiller operator $\mathcal{K}_\beta$:

$$\begin{tikzcd}
L^2\left(\mathbb{R}^m, a\left(\underline{c}_0^{-1}\circ\underline{\xi}\right)d\underline{\xi}\right) \arrow[r, "\mathcal{K}_\beta"] \arrow[d, "R_{\underline{c}_0}"'] & L^2\left(\mathbb{R}^m d\underline{\xi}\right) \arrow[d, "R_{\underline{c}_0}"] \\
L^2\left(\mathbb{R}^m, a\left(\underline{x}\right)d\underline{x}\right) \arrow[r, "\mathcal{K}_{\beta,\underline{c}_0}"] \arrow[d, "m_{\sqrt{a}}"'] & L^2\left(\mathbb{R}^m, d\underline{x}\right) \arrow[d, "\mathrm{id}"] \\
L^2\left(\mathbb{R}^m, d\underline{x}\right) \arrow[r, "\mathcal{K}'_{\beta,\underline{c}_0}"] \arrow[d, "\mathrm{id}"'] & L^2\left(\mathbb{R}^m, d\underline{x}\right) \arrow[d, "m_{\frac{1}{\sqrt{a}}}"] \\
L^2\left(\mathbb{R}^m, d\underline{x}\right) \arrow[r, "\mathcal{K}''_{\underline{c}_0}"] & L^2\left(\mathbb{R}^m, d\underline{x}\right)
\end{tikzcd}$$

The functions $e^{\pi\underline{x}^2}h_{\underline{\alpha}}(\underline{x})$ are polynomials of degree $|\underline{\alpha}|$ in $\underline{x}$ (see [Fo89, p.52]). In view of the estimate

$$\sqrt{\cosh_{\underline{R}}(\underline{\xi})} = \sqrt{\cosh\left(\sum_{i=1}^m R_i\xi_i\right)} \leqslant e^{\frac{1}{2}\sum_{i=1}^m |R_i||\xi_i|}$$

one concludes that the functions $\underline{\xi} \mapsto h_{\underline{\alpha}}(\underline{\xi})\sqrt{\cosh_{\underline{R}}(\underline{\xi})}$ are in $L^2\left(\mathbb{R}^m, d\underline{\xi}\right)$. Hence all the $h_{\underline{\alpha}}$ are contained in $L^2\left(\mathbb{R}^m, \cosh_{\underline{R}}(\underline{\xi})d\underline{\xi}\right)$. This together with Mehler's formula in Proposition 4.1 shows immediately that the Kac-Gutzwiller operator $\mathcal{K}_\beta$ with kernel $\mathcal{K}_\beta(\underline{\xi}, \underline{\eta})$ in (12) is symmetric and positive definite for $\beta \geqslant 0$. In fact,

$$(f, \mathcal{K}_\beta f) = 2 \sum_{\underline{\alpha}\in\mathbb{N}_0^m} \prod_{l=1}^{m} \left(\frac{\lambda_l}{4\pi}\right)^{\frac{1}{2}} \underline{\lambda}^{\underline{\alpha}} \left| \int_{\mathbb{R}^m} f(\underline{\xi})\sqrt{\cosh_{\underline{R}_0}(\underline{\xi})} h_{\underline{\alpha}}\left(\frac{\underline{\xi}}{2\sqrt{\pi}}\right) d\underline{\xi} \right|^2 \geqslant 0$$

where $R_{0,i} = \sqrt{\beta J_i}, 0 \leqslant i \leqslant m$ so that the eigenvalues of the operator $\mathcal{K}_\beta$ are nonnegative for $\beta \geqslant 0$.

5 Fock Space and Segal-Bargmann Transformation

For $t > 0$ consider the Hilbert space $\mathcal{H}L^2\left(\mathbb{C}^m, \mu_t\right)$ of entire functions $F : \mathbb{C}^m \to \mathbb{C}$ with

$$\|F\|_t^2 := \int\limits_{\mathbb{C}^m} |F(\underline{z})|^2 \mu_t(\underline{z})\, d\underline{z} < \infty,$$

where μ_t denotes the weight function

$$\mu_t\left(\underline{z}\right) = t^m \exp\left(-\pi t \left|\underline{z}\right|^2\right).$$

The Bargmann transform $B_t : L^2\left(\mathbb{R}^m, d\underline{x}\right) \to \mathcal{H}L^2\left(\mathbb{C}^m, \mu_t\right)$ defined via

$$\left(B_t f\right)\left(\underline{z}\right) = \left(\frac{2}{t}\right)^{\frac{m}{4}} \int\limits_{\mathbb{R}^m} f\left(\underline{x}\right) \exp\left(2\pi \underline{x}\cdot\underline{z} - \frac{\pi}{t}\underline{x}^2 - \frac{\pi t}{2}\underline{z}^2\right) d\underline{x}$$

determines a unitary operator in the two Hilbert-spaces (see [Fo89, p.47]).

In the following we are primarily interested in the case $t = 1$. In this case we denote the space $\mathcal{H}L^2\left(\mathbb{C}^m, \mu_t\right)$ simply by $\mathcal{F}_m$ and call it the Fock space over $\mathbb{C}^m$. The transform B_1 is then denoted by B and hence

$$\left(Bf\right)\left(\underline{z}\right) = 2^{\frac{m}{4}} \int\limits_{\mathbb{R}^m} f(\underline{x}) \exp\left(2\pi \underline{x}\cdot\underline{z} - \pi\underline{x}^2 - \frac{\pi}{2}\underline{z}^2\right) d\underline{x}. \tag{16}$$

Its inverse $B^{-1} : \mathcal{F}_m \to L^2\left(\mathbb{R}^m, d\underline{x}\right)$ is given by (see [Fo89, p.45])

$$\left(B^{-1}F\right)\left(\underline{x}\right) = 2^{\frac{m}{4}} \int\limits_{\mathbb{C}^m} F\left(\underline{z}\right) \exp\left(2\pi \underline{x}\cdot\underline{z}^* - \pi\underline{x}^2 - \frac{\pi}{2}\underline{z}^{*2}\right) \exp\left(-\pi\left|\underline{z}\right|^2\right) d\underline{z},$$

where $z_l^* = \overline{z}_l$ simply is the complex conjugate of z_l. An orthonormal basis of Fock space $\mathcal{F}_m$ is given by the functions

$$\zeta_{\underline{\alpha}}\left(\underline{z}\right) = \sqrt{\frac{\pi^{|\underline{\alpha}|}}{\underline{\alpha}!}}\,\underline{z}^{\underline{\alpha}}, \tag{17}$$

where $\underline{z}^{\underline{\alpha}} = \prod_{l=1}^{m} z_i^{\alpha_i}$ and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$. Indeed one has (see [Fo89, p.51])

$$\zeta_{\underline{\alpha}} = B h_{\underline{\alpha}},$$

where as before the $h_{\underline{\alpha}}$ are the Hermite functions in $\mathbb{R}^m$. This we use to prove the following proposition.

Proposition 5.1 *For $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m) \in (0,1)^m$ the bounded operator $M_{\underline{\lambda}} : \mathcal{F}_m \to \mathcal{F}_m$ defined as*

$$M_{\underline{\lambda}} := B \circ \mathcal{K}''_{\underline{c}_0} \circ B^{-1}$$

is given by the expression

$$\left(M_{\underline{\lambda}} F\right)(\underline{z}) = 2\sqrt{\prod_{l=1}^{m} \lambda_l}\, F\left(\lambda_1 z_1, \ldots, \lambda_m z_m\right) = 2\underline{\lambda}^{\frac{1}{2}} F\left(\mathbf{\Lambda}\underline{z}\right),$$

where $\underline{\frac{1}{2}} = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ and $\mathbf{\Lambda}\underline{z} := (\lambda_1 z_1, \ldots, \lambda_m z_m)$.

Proof. Consider first $F(\underline{z}) = \zeta_{\underline{\alpha}}(\underline{z})$. In view of (14) and (15) we have

$$M_{\underline{\lambda}} \zeta_{\underline{\alpha}} = B \circ \mathcal{K}''_{\underline{c}_0} \circ B^{-1} \zeta_{\underline{\alpha}} = B \circ \mathcal{K}''_{\underline{c}_0} h_{\underline{\alpha}} = B\left(2\underline{\lambda}^{\underline{\alpha}+\underline{\frac{1}{2}}} h_{\underline{\alpha}}\right) = 2\underline{\lambda}^{\underline{\alpha}+\underline{\frac{1}{2}}} \zeta_{\underline{\alpha}}.$$

But $\zeta_{\underline{\alpha}}(\underline{z})$ is homogeneous of degree α_i in z_i and hence

$$\underline{\lambda}^{\underline{\alpha}} \zeta_{\underline{\alpha}}(\underline{z}) = \zeta_{\underline{\alpha}}(\mathbf{\Lambda}\underline{z}).$$

Therefore the claim is true for the basis elements $\zeta_{\underline{\alpha}}$ of $\mathcal{F}_m$ and hence also for any $F \in \mathcal{F}_m$. $\square$

For $\underline{r} \in \mathbb{R}^m$ define the translation operator $\tau_{\underline{r}} : L^2(\mathbb{R}^m, d\underline{x}) \to L^2(\mathbb{R}^m, dx)$ by

$$\left(\tau_{\underline{r}} f\right)(\underline{x}) := f(\underline{x} - \underline{r})$$

and for $s \in \mathbb{R} \setminus \{0\}$ define the multiplication operator $\mu_s : L^2(\mathbb{R}^m, d\underline{x}) \to L^2(\mathbb{R}^m, d\underline{x})$ by

$$(\mu_s f)(\underline{x}) := s f(\underline{x})$$

For $\underline{s} \in (\mathbb{R} \setminus \{0\})^m$ we define $\mu_{\underline{s}}^{\underline{\alpha}} := \prod_{l=1}^m \mu_{s_l}^{\alpha_l}$. Since Z_l^* and μ_s commute, it makes sense to write $\left(\underline{Z}^* + \mu_{\underline{s}}\right)^{\underline{\alpha}} := \prod_{j=1}^{m} \left(Z_j^* + \mu_{s_j}\right)^{\alpha_j}$.

Proposition 5.2 *For any $\underline{\alpha} \in \mathbb{N}_0^m$ we have*

$$\underline{Z}^{*\underline{\alpha}} \circ \tau_{\underline{r}} = \tau_{\underline{r}} \circ \left(\underline{Z}^* + \mu_{\underline{r}}\right)^{\underline{\alpha}},$$

Proof. For $f \in \mathcal{C}^1(\mathbb{R}^m)$ and $1 \leq j \leq m$ we calculate

$$\begin{aligned}
\left(Z_j^* \circ \tau_{\underline{r}} f\right)(\underline{x}) &= \left(x_j - \frac{1}{2\pi}\frac{\partial}{\partial x_j}\right) f(\underline{x} - \underline{r}) \\
&= (x_j - r_j) f(\underline{x} - \underline{r}) - \frac{1}{2\pi}\frac{\partial}{\partial x_j} f(\underline{x} - \underline{r}) + r_j f(\underline{x} - \underline{r}) \\
&= \left(\tau_{\underline{r}} \circ Z_j^* f\right)(\underline{x}) + \left(\tau_{\underline{r}} \circ \mu_{r_j} f\right)(\underline{x}) \\
&= \left(\tau_{\underline{r}} \circ \left(Z_j^* + \mu_{r_j}\right) f\right)(\underline{x})
\end{aligned}$$

From this it follows immediately that $\underline{Z}^{*\underline{\alpha}} \circ \tau_{\underline{r}} = \tau_{\underline{r}} \circ \left(\underline{Z}^* + \mu_{\underline{r}}\right)^{\underline{\alpha}}$. $\square$

For the Hermite function $h_{\underline{0}}(\underline{x})$ one now gets

$$\left(\underline{Z}^{*\underline{\alpha}} \circ \tau_{\underline{r}} h_{\underline{0}}\right)(\underline{x}) = \tau_{\underline{r}} \circ \left(\underline{Z}^* + \mu_{\underline{r}}\right)^{\underline{\alpha}} h_{\underline{0}}(\underline{x}) = \tau_{\underline{r}} \sum_{\underline{l}=\underline{0}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{l}} \mu_{\underline{r}}^{\underline{l}} \underline{Z}^{*(\underline{\alpha}-\underline{l})} \underline{h}_0(\underline{x}),$$

where we used the notation

$$\sum_{\underline{l}=\underline{0}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{l}} \mu_{\underline{r}}^{\underline{l}} \underline{Z}^{*(\underline{\alpha}-\underline{l})} = \sum_{l_1=0}^{\alpha_1} \cdots \sum_{l_m=0}^{\alpha_m} \binom{\alpha_1}{l_1} \cdots \binom{\alpha_m}{l_m} \mu_{r_1}^{l_1} \cdots \mu_{r_m}^{l_m} Z_1^{*(\alpha_1-l_1)} \cdots Z_m^{*(\alpha_m-l_m)}.$$

For $\underline{Z}^{*(\underline{\alpha}-\underline{l})} h_{\underline{0}} =: \widetilde{h}_{\underline{\alpha}-\underline{l}}$ we find

$$\underline{Z}^{*\underline{\alpha}} \circ \tau_{\underline{r}} h_{\underline{0}} = \sum_{\underline{l}=\underline{0}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{l}} \mu_{\underline{r}}^{\underline{l}} \tau_{\underline{r}} \widetilde{h}_{\underline{\alpha}-\underline{l}}.$$

For $\underline{s} \in \mathbb{R}^m$ denote by $\exp_{\underline{s}} : \mathbb{R}^m \to \mathbb{R}$ the function defined by $\exp_{\underline{s}}(\underline{x}) := e^{\underline{s}\cdot\underline{x}}$. Then one has

Proposition 5.3 *For $\underline{\alpha} \in \mathbb{N}_0^m$ and $\underline{s} \in \mathbb{R}^m$ the following identities hold*

$$\begin{aligned}
\underline{Z}^{*\underline{\alpha}} \circ m_{\exp_{\underline{s}}} &= m_{\exp_{\underline{s}}} \circ \left(\underline{Z}^* - \frac{\underline{s}}{2\pi}\right)^{\underline{\alpha}} \\
m_{\exp_{\underline{s}}} \circ \underline{Z}^{*\underline{\alpha}} &= \left(\underline{Z}^* + \frac{\underline{s}}{2\pi}\right)^{\underline{\alpha}} \circ m_{\exp_{\underline{s}}}
\end{aligned}$$

Proof. By definition of Z_j^* we obtain for smooth f

$$\begin{aligned}
\left(Z_j^* \circ m_{\exp_{\underline{s}}} f\right)(\underline{x}) &= x_j e^{\underline{s}\cdot\underline{x}} f(\underline{x}) - \frac{1}{2\pi} \frac{\partial}{\partial x_j} \left(e^{\underline{s}\cdot\underline{x}} f(\underline{x})\right) \\
&= x_j e^{\underline{s}\cdot\underline{x}} f(\underline{x}) - \frac{1}{2\pi} \left(s_j e^{\underline{s}\cdot\underline{x}} f(\underline{x}) + e^{\underline{s}\cdot\underline{x}} \frac{\partial}{\partial x_j} f(\underline{x})\right) \\
&= e^{\underline{s}\cdot\underline{x}} \left(Z_j^* f\right)(\underline{x}) - \frac{s_j}{2\pi} e^{\underline{s}\cdot\underline{x}} f(\underline{x}) \\
&= \left(m_{\exp_{\underline{s}}} \circ \left(Z_j^* - \frac{s_j}{2\pi}\right) f\right)(\underline{x}).
\end{aligned}$$

Iterating this calculation proves the first identity of the proposition. The second identity is proved in the same way. $\square$

From this one derives

Proposition 5.4 *For $\underline{s} \in \mathbb{R}^m$ and $\widetilde{h}_{\underline{\alpha}} = \underline{Z}^{*\underline{\alpha}} h_{\underline{0}}$ one has*

$$m_{\exp_{\underline{s}}} \widetilde{h}_{\underline{\alpha}} = e^{\frac{\underline{s}^2}{4\pi}} \sum_{\underline{k}=\underline{0}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{k}} \left(\frac{\underline{s}}{\pi}\right)^{\underline{\alpha}-\underline{k}} \tau_{\frac{\underline{s}}{2\pi}} \widetilde{h}_{\underline{k}}.$$

Proof. For $\underline{\alpha} = \underline{0}$ we have $\widetilde{h}_{\underline{0}} = h_{\underline{0}}$ and hence we get

$$\begin{aligned}
\left(m_{\exp_{\underline{s}}} h_{\underline{0}}\right)(\underline{x}) &= e^{\underline{s}\cdot\underline{x}} 2^{\frac{m}{4}} e^{-\pi \underline{x}^2} \\
&= 2^{\frac{m}{4}} e^{\underline{s}\cdot\underline{x} - \pi \underline{x}^2} \\
&= 2^{\frac{m}{4}} e^{-\pi\left(\underline{x} - \frac{\underline{s}}{2\pi}\right)^2 + \frac{\underline{s}^2}{4\pi}} \\
&= e^{\frac{\underline{s}^2}{4\pi}} \tau_{\frac{\underline{s}}{2\pi}} h_{\underline{0}}(\underline{x}).
\end{aligned}$$

For general $\underline{\alpha} \in \mathbb{N}_0^m$ we then get using Proposition 5.3 and Proposition 5.2

$$\begin{aligned}
m_{\exp_{\underline{s}}} \widetilde{h}_{\underline{\alpha}} &= m_{\exp_{\underline{s}}} \circ \underline{Z}^{*\underline{\alpha}} h_{\underline{0}} \\
&= \left(\underline{Z}^* + \frac{\underline{s}}{2\pi}\right)^{\underline{\alpha}} m_{\exp_{\underline{s}}} h_{\underline{0}} \\
&= e^{\frac{\underline{s}^2}{4\pi}} \left(\underline{Z}^* + \frac{\underline{s}}{2\pi}\right)^{\underline{\alpha}} \circ \tau_{\frac{\underline{s}}{2\pi}} h_{\underline{0}} \\
&= e^{\frac{\underline{s}^2}{4\pi}} \sum_{\underline{l}=\underline{0}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{l}} \left(\frac{\underline{s}}{2\pi}\right)^{\underline{\alpha}-\underline{l}} \underline{Z}^{*\underline{l}} \circ \tau_{\frac{\underline{s}}{2\pi}} h_{\underline{0}} \\
&= e^{\frac{\underline{s}^2}{4\pi}} \sum_{\underline{l}=\underline{0}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{l}} \left(\frac{\underline{s}}{2\pi}\right)^{\underline{\alpha}-\underline{l}} \tau_{\frac{\underline{s}}{2\pi}} \circ \left(\underline{Z}^* + \mu_{\frac{\underline{s}}{2\pi}}\right)^{\underline{l}} h_{\underline{0}} \\
&= e^{\frac{\underline{s}^2}{4\pi}} \sum_{\underline{l}=\underline{0}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{l}} \left(\frac{\underline{s}}{2\pi}\right)^{\underline{\alpha}-\underline{l}} \sum_{\underline{k}=\underline{0}}^{\underline{l}} \binom{\underline{l}}{\underline{k}} \left(\frac{\underline{s}}{2\pi}\right)^{\underline{l}-\underline{k}} \tau_{\frac{\underline{s}}{2\pi}} \circ \underline{Z}^{*\underline{k}} h_{\underline{0}} \\
&= e^{\frac{\underline{s}^2}{4\pi}} \sum_{\underline{l}=\underline{0}}^{\underline{\alpha}} \sum_{\underline{k}=\underline{0}}^{\underline{l}} \binom{\underline{\alpha}}{\underline{l}} \binom{\underline{l}}{\underline{k}} \left(\frac{\underline{s}}{2\pi}\right)^{\underline{\alpha}-\underline{k}} \tau_{\frac{\underline{s}}{2\pi}} \widetilde{h}_{\underline{k}} \\
&= e^{\frac{\underline{s}^2}{4\pi}} \sum_{\underline{k}=\underline{0}}^{\underline{\alpha}} \sum_{\underline{l}=\underline{k}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{l}} \binom{\underline{l}}{\underline{k}} \left(\frac{\underline{s}}{2\pi}\right)^{\underline{\alpha}-\underline{k}} \tau_{\frac{\underline{s}}{2\pi}} \widetilde{h}_{\underline{k}}.
\end{aligned}$$

But $\sum_{\underline{l}=\underline{k}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{l}} \binom{\underline{l}}{\underline{k}} = \underline{2}^{\underline{\alpha}-\underline{k}} \binom{\underline{\alpha}}{\underline{k}}$ since $\sum_{\underline{l}_i=\underline{k}_i}^{\underline{\alpha}_i} \binom{\underline{\alpha}_i}{\underline{l}_i} \binom{\underline{l}_i}{\underline{k}_i} = \underline{2}^{\underline{\alpha}_i-\underline{k}_i} \binom{\underline{\alpha}_i}{\underline{k}_i}$, and therefore

$$m_{\exp_{\underline{s}}} \widetilde{h}_\alpha = e^{\frac{\underline{s}^2}{4\pi}} \sum_{\underline{k}=\underline{0}}^{\underline{\alpha}} \underline{2}^{\underline{\alpha}-\underline{k}} \binom{\underline{\alpha}}{\underline{k}} \left(\frac{\underline{s}}{2\pi}\right)^{\underline{\alpha}-\underline{k}} \tau_{\frac{\underline{s}}{2\pi}} \widetilde{h}_{\underline{k}} = e^{\frac{\underline{s}^2}{4\pi}} \sum_{\underline{k}=\underline{0}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{k}} \left(\frac{\underline{s}}{\pi}\right)^{\underline{\alpha}-\underline{k}} \tau_{\frac{\underline{s}}{2\pi}} \widetilde{h}_{\underline{k}}.$$

□

Proposition 5.5 *For $\underline{r} \in \mathbb{R}^m$ and $B : L^2(\mathbb{R}^m, d\underline{x}) \to \mathcal{F}_m$ the Bargmann transform one has*

$$B \circ \tau_{\underline{r}} = e^{-\frac{\pi}{2}\underline{r}^2} m_{\exp_{\pi \underline{r}}} \circ \tau_{\underline{r}} \circ B,$$

where we have denoted the function $e^{\underline{r}\cdot\underline{z}}$ also by $\exp_{\underline{r}}$.

Proof. Using (16) we calculate

$$
\begin{aligned}
\left(B \circ \tau_{\underline{r}} f\right)(\underline{z}) &= 2^{\frac{m}{4}} \int_{\mathbb{R}^m} f\left(\underline{x}-\underline{r}\right) e^{2\pi \underline{x}\cdot\underline{z}-\pi \underline{x}^2-\frac{\pi}{2}\underline{z}^2} d\underline{x} \\
&= 2^{\frac{m}{4}} \int_{\mathbb{R}^m} f(\underline{y}) e^{2\pi\left(\underline{y}+\underline{r}\right)\cdot\underline{z}-\pi\left(\underline{y}+\underline{r}\right)^2-\frac{\pi}{2}\underline{z}^2} d\underline{y} \\
&= 2^{\frac{m}{4}} \int_{\mathbb{R}^m} f\left(\underline{y}\right) e^{2\pi \underline{y}\cdot(\underline{z}-\underline{r})-\pi\underline{y}^2-\frac{\pi}{2}(\underline{z}-\underline{r})^2} e^{\pi \underline{r}\cdot\underline{z}-\frac{\pi}{2}\underline{r}^2} d\underline{y} \\
&= e^{\pi \underline{r}\cdot\underline{z}-\frac{\pi}{2}\underline{r}^2} \left(\tau_{\underline{r}} \circ Bf\right)(\underline{z}) \\
&= e^{-\frac{\pi \underline{r}^2}{2}} \left(m_{\exp_{\pi\underline{r}}} \circ \tau_{\underline{r}} \circ B\, f\right)(\underline{z}).
\end{aligned}
$$

□

Consider next the operator in $\mathcal{F}_m$ induced from the multiplication operator $m_{\exp_{\underline{s}}}$ in $L^2\left(\mathbb{R}^m, d\underline{x}\right)$. One finds

Proposition 5.6 *For $\underline{s} \in \mathbb{R}^m$ and $F: \mathbb{C}^m \to \mathbb{C}$ polynomial we have*

$$
\left(B \circ m_{\exp_{\underline{s}}} \circ B^{-1} F\right)(\underline{z}) = e^{\frac{\underline{s}^2}{8\pi}} e^{\frac{\underline{s}\cdot\underline{z}}{2}} F\left(\underline{z} + \frac{\underline{s}}{2\pi}\right)
$$

Proof. For the functions $\widetilde{\zeta}_{\underline{\alpha}}(\underline{z}) := \underline{z}^{\underline{\alpha}} = B\tilde{h}_{\underline{\alpha}}(\underline{z})$ one finds

$$
\begin{aligned}
B \circ m_{\exp_{\underline{s}}} \circ B^{-1}\widetilde{\zeta}_{\underline{\alpha}} &= B \circ m_{\exp_{\underline{s}}} \widetilde{h}_{\underline{\alpha}} \\
&= B e^{\frac{\underline{s}^2}{4\pi}} \sum_{\underline{k}=\underline{0}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{k}} \left(\frac{\underline{s}}{\pi}\right)^{\underline{\alpha}-\underline{k}} \tau_{\frac{\underline{s}}{2\pi}} \widetilde{h}_{\underline{k}} \\
&= e^{\frac{\underline{s}^2}{4\pi}} \sum_{\underline{k}=\underline{0}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{k}} \left(\frac{\underline{s}}{\pi}\right)^{\underline{\alpha}-\underline{k}} e^{-\frac{\underline{s}^2}{8\pi}} m_{\exp_{\frac{\underline{s}}{2}}} \circ \tau_{\frac{\underline{s}}{2\pi}} \circ B\widetilde{h}_{\underline{k}} \\
&= e^{\frac{\underline{s}^2}{8\pi}} e^{\frac{\underline{s}\cdot\underline{z}}{2}} \sum_{\underline{k}=\underline{0}}^{\underline{\alpha}} \binom{\underline{\alpha}}{\underline{k}} \left(\frac{\underline{s}}{\pi}\right)^{\underline{\alpha}-\underline{k}} \left(\underline{z} - \frac{\underline{s}}{2\pi}\right)^{\underline{k}} \\
&= e^{\frac{\underline{s}^2}{8\pi}+\frac{\underline{s}\cdot\underline{z}}{2}} \left(\underline{z} + \frac{\underline{s}}{2\pi}\right)^{\underline{\alpha}}.
\end{aligned}
$$

Since the $\left\{\widetilde{\zeta}_{\underline{\alpha}}\right\}$ form a basis in the space $\mathcal{F}_m$ the claim of the proposition is true. □

Remark 5.7 According to Proposition 5.6 we can view $B \circ m_{\exp_{\underline{s}}} \circ B^{-1}$ as an unbounded operator on $\mathcal{F}_m$ which is defined on a dense linear subspace, namely the space of polynomial functions. □

For the following we need the densely defined unbounded operator $C_{\underline{s}}: \mathcal{F}_m \to \mathcal{F}_m$ defined as

$$
C_{\underline{s}} := B \circ m_{\cosh_{\underline{s}}} \circ B^{-1} = \frac{1}{2}\left(B \circ m_{\exp_{\underline{s}}} \circ B^{-1} + B \circ m_{\exp_{-\underline{s}}} \circ B^{-1}\right). \tag{18}
$$

From Proposition 5.6 we deduce

$$\left(C_{\underline{s}}F\right)(\underline{z}) = \frac{1}{2}e^{\frac{\underline{s}^2}{8\pi}}\left(e^{\frac{\underline{s}\cdot\underline{z}}{2}}F\left(\underline{z}+\frac{\underline{s}}{2\pi}\right)+e^{-\frac{\underline{s}\cdot\underline{z}}{2}}F\left(\underline{z}-\frac{\underline{s}}{2\pi}\right)\right)$$

for polynomial F, so indeed $C_{\underline{s}}$ is densely defined. Composing this unbounded operator with the bounded operator $M_{\underline{\lambda}}$ of Proposition 5.1 we actually get a bounded operator on $\mathcal{F}_m$ as the following proposition shows.

Proposition 5.8 *For $\underline{s}\in\mathbb{R}^m$, $\underline{\lambda}=(\lambda_1,\ldots,\lambda_m)\in(0,1)^m$, and polynomial $F\in\mathcal{F}_m$ one has*

$$\left(C_{\underline{s}}\circ M_{\underline{\lambda}}F\right)(\underline{z}) = \underline{\lambda}^{\frac{1}{2}}e^{\frac{\underline{s}^2}{8\pi}}\left(e^{\frac{\underline{s}\cdot\underline{z}}{2}}F\left(\mathbf{\Lambda}\underline{z}+\frac{\mathbf{\Lambda}\underline{s}}{2\pi}\right)+e^{-\frac{\underline{s}\cdot\underline{z}}{2}}F\left(\mathbf{\Lambda}\underline{z}-\frac{\mathbf{\Lambda}\underline{s}}{2\pi}\right)\right)$$

and hence $C_{\underline{s}}\circ M_{\underline{\lambda}}$ extends to a bounded operator $\mathcal{F}_m\to\mathcal{F}_m$.

Proof.

$$\left(\underline{C}_{\underline{s}}\circ M_{\underline{\lambda}}F\right)(\underline{z}) = \frac{1}{2}e^{\frac{\underline{s}^2}{8\pi}}\left(e^{\frac{\underline{s}\cdot\underline{z}}{2}}\left(M_{\underline{\lambda}}F\right)\left(\underline{z}+\frac{\underline{s}}{2\pi}\right)+e^{-\frac{\underline{s}\cdot\underline{z}}{2}}\left(M_{\lambda}F\right)\left(\underline{z}-\frac{\underline{s}}{2\pi}\right)\right).$$

But we have $\left(M_{\underline{\lambda}}F\right)(\underline{z}) = 2\underline{\lambda}^{\frac{1}{2}}F\left(\mathbf{\Lambda}\underline{z}\right)$ and hence the claim follows. $\square$

For $\underline{\alpha}\in\mathbb{R}_*^m$ with $\mathbb{R}_* = \{r\in\mathbb{R}: r\neq 0\}$ define the map $\nu_{\underline{\alpha}}:\mathcal{F}_m^{\underline{\alpha}^{-1}}\to\mathcal{F}_m$ by

$$\left(\nu_{\underline{\alpha}}F\right)(\underline{z}) := F\left(\boldsymbol{A}\underline{z}\right) = F\left(\alpha_1 z_1,\ldots,\alpha_m z_m\right), \tag{19}$$

where

$$\mathcal{F}_m^{\underline{\alpha}} = \left\{F:\mathbb{C}^m\to\mathbb{C}\quad\text{entire},\quad \|F\|_{\underline{\alpha}}^2 := \int_{\mathbb{C}^m}|F(z)|^2\, e^{-(\pi\sum_{i=1}^m|\alpha_i z_i|^2)}dz<\infty\right\}$$

and $\boldsymbol{A}$ is the diagonal matrix with diagonal entries $\alpha_1,\ldots,\alpha_m$.

Then consider the induced operator

$$\nu_{\underline{\alpha}^{-1}}\circ C_{\underline{s}}\circ M_{\underline{\lambda}}\circ\nu_{\underline{\alpha}}:\mathcal{F}_m^{\underline{\alpha}^{-1}}\to\mathcal{F}_m^{\underline{\alpha}^{-1}}\quad .$$

Inserting the expressions for $\nu_{\underline{\alpha}}$ and $C_{\underline{s}}\circ M_{\underline{\lambda}}$ one gets

$$\left(\nu_{\underline{\alpha}^{-1}}\circ C_{\underline{s}}\circ M_{\underline{\lambda}}\circ\nu_{\underline{\alpha}}F\right)(\underline{z}) \quad = \quad \underline{\lambda}^{\frac{1}{2}}e^{\frac{\underline{s}^2}{8\pi}}\left(e^{\frac{\underline{s}\cdot\left(\boldsymbol{A}^{-1}\underline{z}\right)}{2}}F\left(\mathbf{\Lambda}\underline{z}+\frac{\mathbf{\Lambda}\boldsymbol{A}\underline{s}}{2\pi}\right)+e^{-\frac{\underline{s}\cdot\left(\boldsymbol{A}^{-1}\underline{z}\right)}{2}}F\left(\mathbf{\Lambda}\underline{z}-\frac{\mathbf{\Lambda}\boldsymbol{A}\underline{s}}{2\pi}\right)\right).$$

For $0\neq\beta\in\mathbb{C}$ choose next the parameters $\underline{s}=\underline{s}_0$ and $\underline{\alpha}=\underline{\alpha}_0$ with

$$s_{0,i} = 2\sqrt{\pi J_i}\sqrt{\beta}\quad\text{and}\quad \alpha_{0,i}=2\pi s_{0,i}^{-1},\quad 1\leq i\leq m.$$

We then get

$$\left(\nu_{\underline{\alpha}_0^{-1}}\circ C_{\underline{s}_0}\circ M_{\underline{\lambda}}\circ\nu_{\underline{\alpha}_0}F\right)(\underline{z}) = \prod_{i=1}^m\left(\lambda_i\exp\beta J_i\right)^{\frac{1}{2}}\left(e^{\beta\sum_{i=1}^m J_i z_i}F\left(\mathbf{\Lambda}\underline{z}+\underline{\lambda}\right)+e^{-\beta\sum_{i=1}^m J_i z_i}F\left(\mathbf{\Lambda}\underline{z}-\underline{\lambda}\right)\right).$$

But this operator has up to the multiplicative factor $\prod_{i=1}^{m}(\lambda_i \exp \beta J_i)^{\frac{1}{2}}$ exactly the form of the Ruelle transfer operator of the Kac model for the parameters $\underline{\lambda}$.

The operator $\nu_{\underline{\alpha}_0^{-1}} \circ C_{\underline{s}_0} \circ M_{\underline{\lambda}} \circ \nu_{\underline{\alpha}_0}$ is defined in the Hilbert space $\mathcal{F}_m^{\underline{\alpha}_0^{-1}}$ with $\alpha_{0,i}^{-1} = \sqrt{\frac{\beta J_i}{\pi}}$. All eigenfunctions of the operator $\mathcal{L}_\beta : \mathcal{B}(D) \to \mathcal{B}(D)$ besides the ones belonging to the eigenvalue zero belong to this space.

This can be seen as follows. From the functional equation

$$\rho f(\underline{z}) = e^{\beta \underline{J}\cdot\underline{z}} f(\mathbf{\Lambda}\underline{z} + \underline{\lambda}) + e^{-\beta \underline{J}\cdot\underline{z}} f(\mathbf{\Lambda}\underline{z} - \underline{\lambda})$$

one concludes that for $\rho \neq 0$ any eigenfunction of $\mathcal{L}_\beta$ is an entire function in $\underline{z}$ which can grow for $|\underline{z}| \to \infty$ at most like $e^{C\sum_{l=1}^{m}|z_i|}$ for some positive constant C. Such functions, however, belong to any of the Fock spaces $\mathcal{F}_m^{\underline{\alpha}}$. Hence the operators $\mathcal{L}_\beta$ and $\frac{1}{\prod_{i=1}^{m}(\lambda_i \exp \beta J_i)^{\frac{1}{2}}}\nu_{\underline{\alpha}_0^{-1}} \circ C_{\underline{s}_0} \circ M_{\underline{\lambda}} \circ \nu_{\underline{\alpha}_0}$ have the same spectra in this space. The eigenfunctions with eigenvalue zero of the operator $\mathcal{L}_\beta : \mathcal{B}(D) \to \mathcal{B}(D)$ can be determined explicitly. They are given by the functions

$$f_{\underline{n},\underline{\alpha}}(\underline{z}) = e^{-\left(\beta \sum_{l=1}^{m} \frac{J_l}{2\lambda_l^2} z_l^2\right)} \prod_{l=1}^{m} \left(\exp \frac{(2n_l+1)\pi i z_l}{2\lambda_l}\right)^{\alpha_l}$$

with $\underline{n} \in \mathbb{N}_0^m$ and $\underline{\alpha} \in \mathbb{Z}^m$ such that $|\underline{\alpha}| = 1 \mod 2$ and hence do not belong to the space $\mathcal{F}_m^{\underline{\alpha}_0^{-1}}$. Summarizing we have shown

Proposition 5.9 *For $\beta \in \mathbb{C}, \beta \neq 0$ the operators $\prod_{l=1}^{m}(\lambda_l \exp \beta J_l)^{\frac{1}{2}} \mathcal{L}_\beta : \mathcal{F}_m^{\underline{\alpha}_0^{-1}} \to \mathcal{F}_m^{\underline{\alpha}_0^{-1}}$ and $C_{\underline{s}_0} \circ M_{\underline{\lambda}} : \mathcal{F}_m \to \mathcal{F}_m$ are conjugate.* □

This leads to

Proposition 5.10 *For $\beta \in \mathbb{C}, \beta \neq 0$ the operators $\mathcal{L}_\beta : \mathcal{F}_m^{\underline{\alpha}_0^{-1}} \to \mathcal{F}_m^{\underline{\alpha}_0^{-1}}$ and*

$$\frac{1}{\prod_{l=1}^{m}(\lambda_l \exp \beta J_l)^{\frac{1}{2}}} m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}_0} : L^2(\mathbb{R}^m, d\underline{x}) \to L^2(\mathbb{R}^m, d\underline{x})$$

are conjugate.

Proof. Inserting the definitions of the operator $C_{\underline{s}}$ and $M_{\underline{\lambda}}$ we find

$$C_{\underline{s}} \circ M_{\underline{\lambda}} = B \circ m_{\cosh_{\underline{s}}} \circ B^{-1} \circ B \circ \mathcal{K}''_{\underline{c}_0} \circ B^{-1} = B \circ m_{\cosh_{\underline{s}}} \circ \mathcal{K}''_{\underline{c}_0} \circ B^{-1}$$

and hence $m_{\cosh_{\underline{s}}} \circ \mathcal{K}''_{\underline{c}_0}$ is conjugate to $C_{\underline{s}} \circ M_{\underline{\lambda}}$. Therefore $f \in L^2(\mathbb{R}^m, d\underline{x})$ is an eigenfunction of $m_{\cosh_{\underline{s}}} \circ \mathcal{K}''_{\underline{c}_0}$ with eigenvalue ϱ iff Bf is an eigenfunction of the operator $C_{\underline{s}} \circ M_{\underline{\lambda}}$ for the same eigenvalue ϱ. For $\underline{s} = \underline{s}_0 = 2(\pi\beta\underline{J})^{\frac{1}{2}}$, however, the operator $C_{\underline{s}_0} \circ M_{\underline{\lambda}} : \mathcal{F}_m \to \mathcal{F}_m$ is conjugate to $\prod_{i=1}^{m}(\lambda_i \exp \beta J_i)^{\frac{1}{2}} \mathcal{L}_\beta : \mathcal{F}_m^{\underline{\alpha}_0^{-1}} \to \mathcal{F}_m^{\underline{\alpha}_0^{-1}}$ by Proposition 5.9. □

Hence one concludes that $f \in L^2(\mathbb{R}^m, dx)$ is an eigenfunction of the operator

$$\frac{1}{\prod\limits_{l=1}^{m} (\lambda_l \exp \beta J_l)^{\frac{1}{2}}} m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}_0}$$

iff the function $\left(\nu_{\underline{\alpha}_0^{-1}} \circ B\right) f \in \mathcal{F}_m^{\underline{\alpha}_0^{-1}}$ is an eigenfunction of the operator $\mathcal{L}_\beta$ for the same eigenvalue. If therefore $f(\underline{x})$ is an eigenfunction of the operator

$$\widetilde{\mathcal{G}}_{\beta,\underline{c}_0} := \frac{1}{\prod\limits_{l=1}^{m} (\lambda_l \exp \beta J_l)^{\frac{1}{2}}} m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}_0}$$

in $L^2(\mathbb{R}^m, d\underline{x})$, where the operator $\mathcal{K}''_{\underline{c}_0}$ has been defined in (13), then the corresponding eigenfunction $F(\underline{z})$ of the operator $\mathcal{L}_\beta : \mathcal{F}_m^{\underline{\alpha}_0^{-1}} \to \mathcal{F}_m^{\underline{\alpha}_0^{-1}}$ has the following explicit form

$$\begin{aligned}
F(z) &= \left(\nu_{\underline{\alpha}_0^{-1}} \circ Bf\right)(\underline{z}) \\
&= 2^{\frac{m}{4}} \int\limits_{\mathbb{R}^m} f(\underline{x}) \exp\left(2\pi \underline{x} \cdot \left(\frac{\beta}{\pi}\underline{J}\right)^{\frac{1}{2}} \circ \underline{z} - \pi \underline{x}^2 - \frac{\pi}{2}\left(\left(\frac{\beta}{\pi}\underline{J}\right)^{\frac{1}{2}} \circ \underline{z}\right)^2\right) d\underline{x} \\
&= 2^{\frac{m}{4}} \int\limits_{\mathbb{R}^m} f(\underline{x}) \exp\left(2\pi \sum_{i=1}^{m}\left(x_i \sqrt{\frac{\beta J_i}{\pi}} z_i\right) - \pi \underline{x}^2 - \frac{\beta}{2}\sum_{i=1}^{m} J_i z_i^2\right) d\underline{x}.
\end{aligned}$$

On the other hand given an eigenfunction $F = F(z)$ of the operator $\mathcal{L}_\beta : \mathcal{F}_m^{\underline{\alpha}_0^{-1}} \to \mathcal{F}_m^{\underline{\alpha}_0^{-1}}$, the corresponding eigenfunction $f = f(x)$ of the operator $\widetilde{\mathcal{G}}_{\beta,\underline{c}_0}$ has the form

$$f(x) = \left(B^{-1} \circ \nu_{\underline{\alpha}_0} F\right)(x) = 2^{\frac{m}{4}} \int\limits_{\mathbb{C}^m} F(\underline{\alpha}_0 \circ \underline{z}) \exp\left(2\pi \underline{x} \cdot \underline{z}^* - \pi \underline{x}^2 - \frac{\pi}{2}\underline{z}^{*2}\right) \exp\left(-\pi |\underline{z}|^2\right) d\underline{z}.$$

Inserting the explicit expression for $\underline{\alpha}_0 = \left(\frac{1}{\sqrt{J_1}}, \ldots, \frac{1}{\sqrt{J_m}}\right)\sqrt{\frac{\pi}{\beta}}$ we therefore get

$$f(\underline{x}) = 2^{\frac{m}{4}} \int\limits_{\mathbb{C}^m} F\left(\sqrt{\frac{\pi}{\beta J_1}} z_1, \ldots, \sqrt{\frac{\pi}{\beta J_m}} z_m\right) \exp\left(2\pi \underline{x} \cdot \underline{z}^* - \pi \underline{x}^2 - \frac{\pi}{2}(\underline{z}^*)^2 - \pi |\underline{z}|^2\right) d\underline{z}$$

Obviously the integral operator $\widetilde{\mathcal{G}}_{\beta,\underline{c}_0}$ depends on β holomorphically and hence defines a holomorphic family of trace class operators in the Hilbert space $L^2(\mathbb{R}^m, d\underline{x})$. Its Fredholm determinant $\det(1 - z\widetilde{\mathcal{G}}_{\beta,\underline{c}_0})$ hence is an entire function in the entire β plane coinciding with the Fredholm determinant of the Ruelle operator $\mathcal{L}_\beta$. To relate finally the operator $m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}^0}$ and its eigenfunctions to those of the Kac-Gutzwiller operator $\mathcal{K}_\beta$ with kernel $\mathcal{K}_\beta(\xi, \eta)$ we use

Proposition 5.11 *For real $\beta \geq 0$ consider the two operators $m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}_0}$ and $m_{\sqrt{\cosh_{\underline{s}_0}}} \circ \mathcal{K}''_{\underline{c}_0} \circ m_{\sqrt{\cosh_{\underline{s}_0}}}$ acting on the space $L^2(\mathbb{R}^m, d\underline{x})$. Then the following two statements hold:*

(i) *If $f \in L^2(\mathbb{R}^m, d\underline{x})$ is an eigenfunction of the operator $m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}_0}$ with eigenvalue $\rho \neq 0$, then the function $g := \frac{f}{\sqrt{\cosh_{\underline{s}_0}}}$ is an eigenfunction of the operator $m_{\sqrt{\cosh_{\underline{s}_0}}} \circ \mathcal{K}''_{\underline{c}_0} \circ m_{\sqrt{\cosh_{\underline{s}_0}}}$ in $L^2(\mathbb{R}^m, d\underline{x})$ for the same eigenvalue.*

(ii) *Conversely, if $g \in L^2(\mathbb{R}^m, d\underline{x})$ is an eigenfunction of the operator $m_{\sqrt{\cosh_{\underline{s}_0}}} \circ \mathcal{K}''_{\underline{c}_0} \circ m_{\sqrt{\cosh_{\underline{s}_0}}}$ with eigenvalue $\rho \neq 0$, then the function $f = \sqrt{\cosh_{\underline{s}_0}} \cdot g$ is an eigenfunction of the operator $m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}_0}$ in $L^2(\mathbb{R}^m, d\underline{x})$ for the same eigenvalue.*

Proof.

(i) If $f \in L^2(\mathbb{R}^m, d\underline{x})$ is an eigenfunction of $m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}_0}$, then $g := \frac{f}{\sqrt{\cosh_{\underline{s}_0}}}$ is in $L^2(\mathbb{R}^m, d\underline{x})$ since $\cosh_{\underline{s}_0} \geq 1$ and a simple calculation shows that this function is an eigenfunction of the operator $m_{\sqrt{\cosh_{\underline{s}_0}}} \circ \mathcal{K}''_{\underline{c}_0} \circ m_{\sqrt{\cosh_{\underline{s}_0}}}$ for the same eigenvalue.

(ii) For $\phi \in L^2(\mathbb{R}^m, \cosh_{\underline{s}_0}(\underline{x})^{-1} d\underline{x})$ set

$$h(\underline{x}) := \int_{\mathbb{R}^m} \tilde{\mathcal{K}}(2\sqrt{\pi}\underline{x}, 2\sqrt{\pi}\underline{y})\phi(\underline{y})\, d\underline{y}.$$

Then (11) shows that there exist constants $c, d > 0$ such that $|h(\underline{x})| \leq c e^{-d|\underline{x}|^2}$. In particular, it follows that $\cosh_{\underline{s}_0} h \in L^2(\mathbb{R}^m, d\underline{x})$.

If now $g \in L^2(\mathbb{R}^m, d\underline{x})$ is an eigenfunction of the operator $m_{\sqrt{\cosh_{\underline{s}_0}}} \circ \mathcal{K}''_{\underline{c}_0} \circ m_{\sqrt{\cosh_{\underline{s}_0}}}$ for the nonzero eigenvalue ρ, then a simple calculation shows that the function $f := g\sqrt{\cosh_{\underline{s}_0}} \in L^2(\mathbb{R}^m, \frac{1}{\cosh_{\underline{s}_0}} d\underline{x})$ is an eigenfunction of the operator $m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}_0}$ for the eigenvalue $\rho \neq 0$ and the above argument applied to $\phi = \rho(4\pi)^{-\frac{m}{2}} f$ shows that $f \in L^2(\mathbb{R}^m, d\underline{x})$.

□

This leads us to the main result of this section:

Theorem 5.12 *The Ruelle operator $\mathcal{L}_\beta : \mathcal{F}_m^{\underline{\alpha}_0^{-1}} \to \mathcal{F}_m^{\underline{\alpha}_0^{-1}}$ and the modified Kac-Gutzwiller operator $\widetilde{\mathcal{G}}_\beta : L^2\left(\mathbb{R}^m, d\underline{\xi}\right) \to L^2\left(\mathbb{R}^m, d\underline{\xi}\right)$ with kernel*

$$\widetilde{\mathcal{G}}_\beta\left(\underline{\xi}, \underline{\eta}\right) := \prod_{l=1}^m \left(\lambda_l \exp \beta J_l\right)^{-\frac{1}{2}} \cosh\left(\sum_{l=1}^m \sqrt{\beta J_l}\xi_l\right) \widetilde{\mathcal{K}}\left(\underline{\xi}, \underline{\eta}\right)$$

with $\widetilde{\mathcal{K}}(\underline{\xi}, \underline{\eta})$ defined in (11) have the same spectrum. For positive β this spectrum coincides also with the spectrum of the Kac-Gutzwiller operator $\mathcal{G}_\beta$ defined in (10) on the Hilbert space $L^2(\mathbb{R}, d\underline{\xi})$. For nonvanishing eigenvalues the eigenfunctions $F(\underline{z})$ and $f(\underline{\xi})$ of $\mathcal{L}_\beta$ and $\mathcal{G}_\beta$ can be related to each other as follows:

$$\begin{aligned}
f\left(\underline{\xi}\right) &= \left(\frac{1}{2\pi}\right)^{\frac{m}{4}} \frac{1}{\sqrt{\cosh \sum_{i=1}^m \sqrt{\beta J_i}\xi_i}} \int_{\mathbb{C}^m} F\left(\left(\frac{\pi}{\beta J_1}\right)^{\frac{1}{2}} z_1, \ldots, \left(\frac{\pi}{\beta J_m}\right)^{\frac{1}{2}} z_m\right) \cdot \\
&\quad \cdot \exp\left(\sqrt{\pi}\underline{\xi} \cdot \underline{z}^* - \frac{1}{4}\underline{\xi}^2 - \frac{\pi}{2}\underline{z}^{*2} - \pi\left|\underline{z}\right|^2\right) d\underline{z} \\
F\left(\underline{z}\right) &= (8\pi)^{\frac{m}{4}} \int_{\mathbb{R}^m} \sqrt{\cosh\left(2\sqrt{\pi\beta}\sum_{l=1}^m \sqrt{J_l}x_l\right)}\, f\left(2\sqrt{\pi}\underline{x}\right) \cdot \\
&\quad \cdot \exp\left(2\sqrt{\pi\beta}\sum_{l=1}^m \sqrt{J_l x_l} z_l - \pi\underline{x}^2 - \frac{\beta}{2} \cdot \sum_{l=1}^m J_l z_l^2\right) d\underline{x}.
\end{aligned}$$

Proof. We have to consider the two cases $\beta = 0$ and $\beta \neq 0$. For $\beta = 0$ the spectra of the two operators are given by $\{2\underline{\lambda}^{\underline{\alpha}}, \underline{\alpha} \in \mathbb{N}_0^m\}$ (see Remark 6.1 and formula (14)) and hence are identical. On the other hand we have seen already in Proposition 5.10 that for $\beta \neq 0$ the operators $\prod_{l=1}^m (\lambda_l \exp \beta J_l)^{-\frac{1}{2}} m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}_0}$ and $\mathcal{L}_\beta$ are conjugate via the map $\nu_{\underline{\alpha}_0^{-1}} \circ B$. In fact, we have the commutative diagram

$$
\begin{tikzcd}
L^2\left(\mathbb{R}^m, d\underline{\xi}\right) \arrow[r, "\mathcal{K}_\beta"] \arrow[d, "R_{\underline{c}_0}"'] & L^2\left(\mathbb{R}^m d\underline{\xi}\right) \arrow[d, "R_{\underline{c}_0}"] & \\
L^2\left(\mathbb{R}^m, a\left(\underline{x}\right) d\underline{x}\right) \arrow[r] \arrow[d, "m_{\sqrt{\cosh_{\underline{s}_0}}}"'] & L^2\left(\mathbb{R}^m, d\underline{x}\right) \arrow[d, "m_{\frac{1}{\sqrt{\cosh_{\underline{s}_0}}}}"] & \\
L^2\left(\mathbb{R}^m, d\underline{x}\right) \arrow[r, "\mathcal{K}''_{\underline{c}_0}"'] \arrow[d, "B"'] & L^2\left(\mathbb{R}^m, d\underline{x}\right) \arrow[r, "m_{\cosh_{\underline{s}_0}}"'] \arrow[d, "B"] & L^2\left(\mathbb{R}^m, d\underline{x}\right) \arrow[d, "B"] \\
\mathcal{F}_m \arrow[r, "M_{\underline{c}_0}"'] \arrow[d, "\nu_{\underline{\alpha}_0}"'] & \mathcal{F}_m \arrow[r, "C_{\underline{s}_0}"'] & \mathcal{F}_m \arrow[d, "\nu_{\underline{\alpha}_0}"] \\
\mathcal{F}_m^{\underline{\alpha}_0^{-1}} \arrow[rr, "\mathcal{L}_\beta \sqrt{\prod_{l=1}^m \lambda_l e^{\beta J_l}}"'] & & \mathcal{F}_m^{\underline{\alpha}_0^{-1}}
\end{tikzcd}
$$

with $\underline{c}_0 = 2\sqrt{\pi}(1, \ldots, 1)$ and $\underline{s}_0 = 2\sqrt{\beta \pi \underline{J}}$. This proves the first claim. Furthermore Proposition 5.11 shows that for $\beta \geq 0$ the operators $m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}_0}$ and $m_{\sqrt{\cosh_{\underline{s}_0}}} \circ \mathcal{K}''_{\underline{c}_0} \circ m_{\sqrt{\cosh_{\underline{s}_0}}}$ have the same nonvanishing eigenvalues. But the last operator is conjugate to the operator $\mathcal{K}_\beta$ with kernel $\mathcal{K}_\beta(\underline{\xi}, \underline{\eta})$ through the map $R_{\underline{c}_0}$. Hence if $f \in L^2\left(\mathbb{R}^m, d\underline{\xi}\right)$ is an eigenfunction of the integral operator $\mathcal{G}_\beta$, then $\left(R_{\underline{c}_0} f\right)(\underline{x}) = \underline{c}_0^{\frac{1}{2}} f\left(\boldsymbol{C}_0 \underline{x}\right)$ is an eigenfunction of the integral operator $\prod_{l=1}^m \left(\lambda_l e^{\beta J_l}\right)^{-\frac{1}{2}} m_{\sqrt{\cosh_{\underline{s}_0}}} \circ \mathcal{K}''_{\underline{c}_0} \circ m_{\sqrt{\cosh_{\underline{s}_0}}}$ for the same eigenvalue. But then

$$
\sqrt{\cosh_{\underline{s}_0}(\underline{x})} \left(R_{\underline{c}_0} f\right)(\underline{x}) = \sqrt{\cosh_{\underline{s}_0}(\underline{x})} \quad \underline{c}_0^{\frac{1}{2}} f(\boldsymbol{C}_0 \underline{x})
$$

is an eigenfunction of the operator $\frac{1}{\prod\limits_{l=1}^m \left(\lambda_l e^{\beta J_l}\right)^{\frac{1}{2}}} \cdot m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}_0}$ again for the same eigenvalue. Therefore

$$
F\left(\underline{z}\right) = 2^{\frac{m}{4}} \int\limits_{\mathbb{R}^m} \sqrt{\cosh \underline{s}_0 \cdot \underline{x}} \quad \underline{c}_0^{\frac{1}{2}} f\left(\boldsymbol{C}_0 \underline{x}\right) \cdot \exp\left(2\pi \sum_{i=1}^m \sqrt{\frac{\beta J_i}{\pi}} x_i z_i - \pi \underline{x}^2 - \frac{\beta}{2} \sum_{i=1}^m J_i z_i^2\right) d\underline{x}
$$

is an eigenfunction of the operator $\mathcal{L}_\beta : F_m^{\underline{\alpha}_0^{-1}} \to F_m^{\underline{\alpha}_0^{-1}}$ for yet again the same eigenvalue.

Inserting $\underline{c}_0 = (2\sqrt{\pi}, \ldots, 2\sqrt{\pi})$ one therefore finds for $F(z)$

$$
\begin{aligned}
F(\underline{z}) \quad &= \quad (8\pi)^{\frac{m}{4}} \int_{\mathbb{R}^m} \sqrt{\cosh\left(2\sqrt{\beta\pi}\sum_{l=1}^m \sqrt{J_l}\cdot x_l\right)} f\left(2\sqrt{\pi}\underline{x}\right) \cdot \\
&\cdot \exp\left(2\sqrt{\pi\beta}\sum_{l=1}^m \sqrt{J_l} x_l z_l - \pi \underline{x}^2 - \frac{\beta}{2}\sum_{l=1}^m J_l z_l^2\right) d\underline{x}
\end{aligned}
$$

On the other hand, given an eigenfunction $F \in \mathcal{F}_m^{\underline{\alpha}_0^{-1}}$ of the operator $\mathcal{L}_\beta$ we know that $h(\underline{x}) = \left(B^{-1}\circ \nu_{\underline{\alpha}_0} \circ F\right)(\underline{x})$ is an eigenfunction of the operator $\frac{1}{\prod_{l=1}^m \sqrt{\lambda_l \exp\beta J_l}} m_{\cosh_{\underline{s}_0}} \circ \mathcal{K}''_{\underline{c}_0}$.

Then by Proposition 5.11 for $\beta \geq 0$ the function $\frac{1}{\sqrt{\cosh_{\underline{s}_0}(\underline{x})}} h(x)$ is an eigenfunction of the operator

$$
\frac{1}{\prod_{l=1}^m (\lambda_l \exp\beta J_l)^{\frac{1}{2}}} m_{\sqrt{\cosh_{\underline{s}_0}}} \circ \mathcal{K}''_{\underline{c}_0} \circ m_{\sqrt{\cosh_{\underline{s}_0}}},
$$

which is again conjugate to $\mathcal{G}_\beta$ via $R^{-1}_{\underline{c}_0}$ and hence $\left(R_{\underline{c}_0^{-1}}\left(\frac{1}{\sqrt{\cosh_{\underline{s}_0}}}\cdot h\right)\right)(\underline{\xi})$ is an eigenfunction of $\mathcal{G}_\beta$. Inserting all the transforms involved we finally get for the corresponding eigenfunction $f = f\left(\underline{\xi}\right) \in L^2\left(\mathbb{R}^m, d\underline{\xi}\right)$

$$
\begin{aligned}
f\left(\underline{\xi}\right) \quad &= \quad \left(\frac{1}{2\pi}\right)^{\frac{m}{4}} \frac{1}{\sqrt{\cosh\left(\sum_{l=1}^m \sqrt{\beta J_l}\xi_l\right)}} \cdot \int_{\mathbb{C}^m} F\left(\sqrt{\frac{\pi}{\beta J_1}} z_1, \ldots, \sqrt{\frac{\pi}{\beta J_m}} z_m\right) \cdot \\
&\cdot \exp\left(\sqrt{\pi}\underline{\xi}\cdot\underline{z}^* - \frac{1}{4}\underline{\xi}^2 - \frac{\pi}{2}\underline{z}^{*2} - \pi\left|\underline{z}\right|^2\right) d\underline{z}
\end{aligned}
$$

$\square$

To discuss the zeros and poles of the Ruelle zeta function for the Kac-Baker models we need the following proposition.

Proposition 5.13 *For β real the operators $\widetilde{\mathcal{G}}_\beta$ and $\mathcal{L}_\beta$ have real spectrum.*

Proof. For $\beta = 0$ this follows from the explicit form of the eigenvalues. For $\beta > 0$ and $f \in L^2\left(\mathbb{R}^m, d\underline{\xi}\right)$ an eigenfunction of the operator $\widetilde{\mathcal{G}}_\beta$ with eigenvalue ϱ consider the scalar product

$$
\left(\frac{f}{\cosh_{\underline{R}_0}}, \widetilde{\mathcal{G}}_\beta f\right) = \overline{\varrho}\left(\frac{f}{\cosh_{\underline{R}_0}}, f\right),
$$

where $\underline{R}_0 = \sqrt{\beta \underline{J}}$. But if

$$
\int_{\mathbb{R}^m} \prod_{l=1}^m (\lambda_l \exp\beta J_l)^{-\frac{1}{2}} \cosh_{\underline{R}_0}(\underline{\xi})\, \widetilde{\mathcal{K}}\left(\underline{\xi},\underline{\eta}\right)\, f(\underline{\eta})\, d\underline{\eta} = \varrho f(\underline{\xi}),
$$

then

$$\int_{\mathbb{R}^m} \prod_{l=1}^{m} (\lambda_l \exp \beta J_l)^{-\frac{1}{2}} \cosh_{\underline{R}_0}(\underline{\xi}) \widetilde{\mathcal{K}}(\underline{\xi},\underline{\eta}) \frac{f(\underline{\xi})}{\cosh_{\underline{R}_0}(\underline{\xi})} d\underline{\xi} = \varrho \frac{f(\underline{\eta})}{\cosh_{\underline{R}_0}(\underline{\eta})}$$

since $\widetilde{\mathcal{K}}(\underline{\xi},\underline{\eta}) = \widetilde{\mathcal{K}}(\underline{\eta},\underline{\xi})$ and the function $\frac{f(\underline{\xi})}{\cosh_{\underline{R}_0}(\underline{\xi})}$ belongs to the space $L^2(\mathbb{R}^m, d\underline{\xi})$ if f is an eigenfunction of $\widetilde{\mathcal{G}}_\beta$. Since $\widetilde{\mathcal{G}}$ is real valued we can now calculate

$$\begin{aligned}
\left(\frac{f}{\cosh_{\underline{R}_0}}, \widetilde{\mathcal{G}}_\beta f\right) &= \int_{\mathbb{R}^m} \frac{f(\underline{\xi})}{\cosh_{\underline{R}_0}(\underline{\xi})} \overline{\int_{\mathbb{R}^m} \widetilde{\mathcal{G}}_\beta(\underline{\xi},\underline{\eta}) f(\underline{\eta})\, d\underline{\eta}}\, d\underline{\xi} \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{f(\underline{\xi})}{\cosh_{\underline{R}_0}(\underline{\xi})} \widetilde{\mathcal{G}}_\beta(\underline{\xi},\underline{\eta})\, d\underline{\xi}\, \overline{f(\underline{\eta})}\, d\underline{\eta} \\
&= \left(\varrho \frac{f}{\cosh_{\underline{R}_0}}, f\right) \\
&= \varrho \left(\frac{f}{\cosh_{\underline{R}_0}}, f\right)
\end{aligned}$$

On the other hand $\left(\frac{f}{\cosh_{\underline{R}_0}}, f\right) \neq 0$, since according to Mehlers formula in Proposition 4.1

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{f(\underline{\xi})}{\cosh_{\underline{R}_0}(\underline{\xi})} \widetilde{\mathcal{G}}_\beta(\underline{\xi},\underline{\eta})\, \overline{f(\underline{\eta})}\, d\underline{\xi}\, d\underline{\eta} = 2 \prod_{l=1}^{m} (4\pi \exp \beta J_l)^{-\frac{1}{2}} \sum_{\underline{\alpha} \in \mathbb{N}_0^m} \underline{\lambda}^{\underline{\alpha}} \left| \int_{\mathbb{R}^m} h_{\underline{\alpha}}\left(\frac{\underline{\xi}}{2\sqrt{\pi}}\right) \overline{f(\underline{\xi})}\, d\underline{\xi} \right|^2 > 0$$

for f an eigenfunction of $\widetilde{\mathcal{G}}_\beta$. But then it is clear that ϱ must be identical to $\overline{\varrho}$ and hence real. For $\beta < 0$ and $f \in L^2(\mathbb{R}^m, d\underline{\xi})$ an eigenfunction of $\widetilde{\mathcal{G}}_\beta$ the function $\frac{f}{cos_{\underline{R}_0}}$ with $\underline{R}_0 = \sqrt{-\beta \underline{J}}$ is obviously in $L^2(\mathbb{R}^m, d\underline{\xi})$ and the same reasoning as in the case $\beta > 0$ applies. □

6 The Ruelle zeta function for the Kac-Baker model

Recall from Proposition 2.4 that the Ruelle zeta function $\zeta_R(z,\beta) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\beta)$ with $Z_n(\beta)$ the partition function for the Kac-Baker model can be written as

$$\zeta_R(z,\beta) = \prod_{\underline{\alpha} \in \{0,1\}^m} \det\left(1 - z\underline{\lambda}^{\underline{\alpha}} \mathcal{L}_\beta\right)^{(-1)^{|\underline{\alpha}|+1}}$$

with the operator $\mathcal{L}_\beta : \mathcal{B}(D) \to \mathcal{B}(D)$ given by

$$(\mathcal{L}_\beta f)(\underline{z}) = e^{\beta \underline{J}\cdot\underline{z}} f(\underline{\lambda} + \mathbf{\Lambda}\underline{z}) + e^{-\beta \underline{J}\cdot\underline{z}} f(-\underline{\lambda} + \mathbf{\Lambda}\underline{z}),$$

where $\mathbf{\Lambda}$ is the $m \times m$-diagonal matrix with the λ_j, $j = 1, \ldots, m$, as diagonal elements.

Remark 6.1 For $\beta = 0$ one then finds for the spectrum $\sigma(\mathcal{L}_0)$ of $\mathcal{L}_0$

$$\sigma(\mathcal{L}_0) = \{2\underline{\lambda}^{\underline{\alpha}} \mid \underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m\}.$$

In fact, by induction over $|\underline{\alpha}|$ one finds a polynomial eigenfunction for each $2\underline{\lambda}^{\underline{\alpha}}$. In the case $m = 1$ one obtains $1, z, z^2 + \frac{\lambda^2}{\lambda^2-1}, z^3 + \frac{3\lambda^2}{\lambda^2-1} z$ and so on. On the other hand, one can show that all eigenfunctions are polynomial, since the derivative of an eigenfunction for the eigenvalue ρ is again an eigenfunction for the eigenvalue $\rho\lambda^{-1}$ (in the case $m = 1$) so that infinitely non-vanishing derivatives of an eigenfunction would contradict the compactness of the operator. For $m > 1$ the argument is similar. $\square$

Hence the Ruelle function $\zeta_R(z, \beta)$ for $\beta = 0$ has the form

$$\zeta_R(z,0) = \prod_{\underline{\alpha}\in\{0,1\}^m} \prod_{\underline{\beta}\in\mathbb{N}_0^m} \left(1 - 2z\underline{\lambda}^{\underline{\alpha}+\underline{\beta}}\right)^{(-1)^{|\underline{\alpha}|+1}}.$$

By induction on m one shows

$$\zeta_R(z,0) = \frac{1}{1-2z}.$$

Indeed for $m = 1$ one has

$$\zeta_R(z,0) = \frac{\det(1 - z\lambda\mathcal{L}_0)}{\det(1 - z\mathcal{L}_0)} = \frac{\prod_{n=0}^{\infty}(1 - 2z\lambda^{n+1})}{\prod_{n=0}^{\infty}(1 - 2z\lambda^{n})} = \frac{1}{1-2z}.$$

For $m = n + 1$, on the other hand, one calculates

$$\begin{aligned}
\zeta_R(z,0) &= \prod_{\alpha_{n+1}\in\{0,1\}} \prod_{\underline{\alpha}\in\{0,1\}^n} \prod_{\beta_{n+1}\in\mathbb{N}_0} \prod_{\underline{\beta}\in\mathbb{N}_0^n} \left(1 - 2z\lambda_{n+1}^{\alpha_{n+1}+\beta_{n+1}}\underline{\lambda}^{\underline{\alpha}+\underline{\beta}}\right)^{(-1)^{1+|\underline{\alpha}|+\alpha_{n+1}}} \\
&= \prod_{\underline{\alpha}\in\{0,1\}^n} \prod_{\beta_{n+1}\in\mathbb{N}_0} \prod_{\underline{\beta}\in\mathbb{N}_0^n} \left(1 - 2z\lambda_{n+1}^{\beta_{n+1}}\underline{\lambda}^{\underline{\alpha}+\underline{\beta}}\right)^{(-1)^{1+|\underline{\alpha}|}} \cdot \\
&\quad \cdot \prod_{\underline{\alpha}\in\{0,1\}^n} \prod_{\beta_{n+1}\in\mathbb{N}_0} \prod_{\underline{\beta}\in\mathbb{N}_0^n} \left(\left(1 - 2z\lambda_{n+1}^{\beta_{n+1}+1}\underline{\lambda}^{\underline{\alpha}+\underline{\beta}}\right)^{(-1)^{1+|\underline{\alpha}|-1}}\right) \\
&= \prod_{\beta_{n+1}\in\mathbb{N}_0} \left(\frac{1}{1 - 2z\lambda_{n+1}^{\beta_{n+1}}}\right) \prod_{\beta_{n+1}\in\mathbb{N}_0} \left(1 - 2z\lambda_{n+1}^{\beta_{n+1}+1}\right) \\
&= \frac{1}{1-2z}.
\end{aligned}$$

This result does not come as a surprise since for $\beta = 0$ the Ruelle zeta function for the Kac-Baker model is just the Artin-Mazur zeta function for the full subshift over two symbols determined by Bowen and Lanford in [BoLa75].

To determine the zeros and poles of the Ruelle function $\zeta_R(\beta) := \zeta_R(1, \beta)$ on the real β-axis one has to investigate the zeros of the Fredholm determinants $\det(1 - \underline{\lambda}^{\underline{\alpha}}\mathcal{L}_\beta)$. Obviously this function takes the special value -1 at the point $\beta = 0$. We know already from our discussion of the Kac-Gutzwiller operator $\mathcal{G}_\beta$ that all its eigenvalues and hence also those of the Ruelle operator $\mathcal{L}_\beta$ are real for real β and nonnegative for $\beta \geqslant 0$. The poles and zeros of the Ruelle zeta function $\zeta_R(1, \beta)$ can be located only at those values of β where one of the numbers $(\underline{\lambda}^{\underline{\alpha}})^{-1}, \underline{\alpha} \in \{0,1\}^m$ belongs to the spectrum $\sigma(\mathcal{L}_\beta)$ of $\mathcal{L}_\beta$. For $\beta = 0$ we have seen that $\sigma(\mathcal{L}_0) = 2\underline{\lambda}^{\underline{\beta}}$ with $\underline{\beta} \in \mathbb{N}_0^m$. But

$$(\underline{\lambda}^{\underline{\alpha}})^{-1} = 2\underline{\lambda}^{\underline{\beta}} \quad \Leftrightarrow \quad \frac{1}{2} = \underline{\lambda}^{\underline{\alpha}+\underline{\beta}}$$

can be true only for finitely many $\underline{\beta} \in \mathbb{N}_0^m$ since $\underline{\alpha} \in \{0,1\}^m$ can take only finitely many values.

Furthermore, for infinitely many $\underline{\beta} \in \mathbb{N}_0^m$ the eigenvalue $2\underline{\lambda}^{\underline{\beta}}$ is strictly smaller than $(\underline{\lambda}^{\underline{\alpha}})^{-1}$ for all $\underline{\alpha} \in \{0,1\}^m$. Hence, if we can show that infinitely many eigenvalues $\varrho(\beta)$ of $\mathcal{L}_\beta$ tend to $+\infty$ for $|\beta| \to \infty$, then the Ruelle zeta function will have infinitely many "nontrivial" zeros and poles on the real β-axis at least for generic values of $\underline{\lambda}$ for which possible cancellations of zeros in the quotients of the different Fredholm determinants do not occur.

To derive the asymptotic behavior of the eigenvalues of the transfer operator for $|\beta| \to \infty$ we remark first that the eigenspace of any eigenvalue ϱ of $\mathcal{L}_\beta$ has a basis consisting of eigenfunctions which are also eigenfunctions of the operator $P : \mathcal{B}(D) \to \mathcal{B}(D)$ defined as

$$Pf(\underline{z}) = f(-\underline{z}).$$

We call the eigenfunctions with $Pf = f$ even and those with $Pf = -f$ odd. Consider then the two operators $\mathcal{L}_\beta^\pm : \mathcal{B}(D) \to \mathcal{B}(D)$ defined via

$$\mathcal{L}_\beta^+ f(\underline{z}) = e^{\beta \underline{J}\cdot \underline{z}} f(\underline{\lambda} + \mathbf{\Lambda}\underline{z}) + e^{-\beta \underline{J}\cdot \underline{z}} f(\underline{\lambda} - \mathbf{\Lambda}\underline{z}),$$

respectively,

$$\mathcal{L}_\beta^- f(\underline{z}) = e^{\beta \underline{J}\cdot \underline{z}} f(\underline{\lambda} + \mathbf{\Lambda}\underline{z}) - e^{-\beta \underline{J}\cdot \underline{z}} f(\underline{\lambda} - \mathbf{\Lambda}\underline{z}).$$

The eigenfunctions of $\mathcal{L}_\beta^+$ and $\mathcal{L}_\beta^-$ are just the even, respectively odd, eigenfunctions of $\mathcal{L}_\beta$. We call the corresponding eigenvalues even, respectively odd. Since $\mathcal{L}_\beta$ and P commute one therefore finds for the spectrum $\sigma(\mathcal{L}_\beta)$ of $\mathcal{L}_\beta$:

$$\sigma(\mathcal{L}_\beta) = \sigma(\mathcal{L}_\beta^+) \cup \sigma(\mathcal{L}_\beta^-)$$

Extending a result of B. Moritz (see [Mo89]) for $m = 1$ to the general case $m \in \mathbb{N}$ we find for the restricted range $0 \leqslant \lambda_i < \frac{1}{2}$ for $1 \leqslant i \leqslant m$ of the parameters $\underline{\lambda}$ and arbitrary $N \geqslant 1$:

Theorem 6.2 (i) *For $\beta \to \pm\infty$ the N leading even eigenvalues $\rho_{\underline{\alpha}}$ of $\mathcal{L}_\beta$ behave like*

$$\underline{\lambda}^{\underline{\alpha}} (\pm 1)^{|\underline{\alpha}|} \exp\left(\beta \underline{J} \cdot (1 - \mathbf{\Lambda})^{-1} \underline{\lambda}\right).$$

(ii) *For $\beta \to \pm\infty$ the N leading odd eigenvalues $\rho_{\underline{\alpha}}$ of $\mathcal{L}_\beta$ behave like*

$$\underline{\lambda}^{\underline{\alpha}} (\pm 1)^{|\underline{\alpha}|+1} \exp\left(\beta \underline{J} \cdot (1 - \mathbf{\Lambda})^{-1} \underline{\lambda}\right).$$

Proof. Consider first the even eigenvalues and their asymptotic behavior for $\beta \to +\infty$. Then

$$\mathcal{L}_\beta^+ g(\underline{z}) = \exp(\beta \underline{J} \cdot \underline{z})\, g(\underline{\lambda} + \mathbf{\Lambda}\underline{z}) + \exp(-\beta \underline{J} \cdot \underline{z})\, g(\underline{\lambda} - \mathbf{\Lambda}\underline{z}) = \varrho\, g(z).$$

Writing $g(z) = \exp\left(\beta \underline{J} \cdot (1 - \mathbf{\Lambda})^{-1} \underline{z}\right) u(\underline{z})$ we get for u the equation

$$\begin{aligned} u(\underline{\lambda} + \mathbf{\Lambda}\underline{z}) + \exp\left(-2\beta \underline{J} \cdot (1 - \mathbf{\Lambda})^{-1} \mathbf{\Lambda}\underline{z}\right) \exp(-2\beta \underline{J} \cdot \underline{z})\, u(\underline{\lambda} - \mathbf{\Lambda}\underline{z}) &= \\ = \quad \varrho \exp\left(-\beta \underline{J} \cdot (1 - \mathbf{\Lambda})^{-1} \underline{\lambda}\right) u(\underline{z}) & \end{aligned}$$

and hence

$$u\left(\underline{\lambda}+\mathbf{\Lambda}\underline{z}\right)+\exp\left(-2\beta\underline{J}\cdot(1-\mathbf{\Lambda})^{-1}\underline{z}\right)u(\underline{\lambda}-\mathbf{\Lambda}\underline{z})=\tilde{\varrho}u(\underline{z}),$$

where

$$\tilde{\varrho}=\varrho\exp\left(-\beta\underline{J}\cdot(1-\mathbf{\Lambda})^{-1}\underline{\lambda}\right).$$

Replacing $\underline{z}$ by $\underline{\lambda}+\underline{z}$ and introducing the function $h(\underline{z}):=u(\underline{\lambda}+\underline{z})$ one arrives at the equation

$$\tilde{\varrho}\,h\left(\underline{z}\right) \;=\; h(\mathbf{\Lambda}\underline{\lambda}+\mathbf{\Lambda}\underline{z})+\exp\left(-2\beta\underline{J}\cdot(1-\mathbf{\Lambda})^{-1}\underline{\lambda}\right)\exp\left(-2\beta\underline{J}\cdot(1-\mathbf{\Lambda})^{-1}\underline{z}\right)h(-\mathbf{\Lambda}\underline{\lambda}-\mathbf{\Lambda}\underline{z}).$$

Define an operator $\mathcal{T}^{+}_{\beta;\underline{\lambda}}:\mathcal{B}\left(D_{\underline{\widetilde{R}}}\right)\to\mathcal{B}\left(D_{\underline{\widetilde{R}}}\right)$ for $D_{\underline{\widetilde{R}}}$ the polydisc with $\widetilde{R}_i>\frac{\lambda_i^2}{1-\lambda_i}$ for $1\leqslant i\leqslant m$ via

$$\mathcal{T}^{+}_{\beta;\underline{\lambda}}h(\underline{z}):=h(\mathbf{\Lambda}\underline{\lambda}+\mathbf{\Lambda}\underline{z})+\exp\left(-2\beta\underline{J}\cdot(1-\mathbf{\Lambda})^{-1}\underline{\lambda}\right)\exp\left(-2\beta\underline{J}\cdot(1-\mathbf{\Lambda})^{-1}\underline{z}\right)h(-\mathbf{\Lambda}\underline{\lambda}-\mathbf{\Lambda}\underline{z}).$$

Then $\mathcal{T}^{+}_{\beta,\underline{\lambda}}$ is nuclear and its eigenvalues are just the numbers $\tilde{\varrho}=\varrho\exp\left(-\beta\underline{J}\cdot(1-\mathbf{\Lambda})^{-1}\underline{\lambda}\right)$, where ϱ is an eigenvalue of the operator $\mathcal{L}^{+}_{\beta}$. If $\mathcal{T}_{\underline{\lambda}}:\mathcal{B}\left(D_{\underline{\widetilde{R}}}\right)\to\mathcal{B}\left(D_{\underline{\widetilde{R}}}\right)$ denotes then the composition operator

$$\mathcal{T}_{\underline{\lambda}}h\left(\underline{z}\right)=h(\mathbf{\Lambda}\underline{\lambda}+\mathbf{\Lambda}\underline{z}),$$

one finds for $\underline{\lambda}=(\lambda_1,\ldots,\lambda_m)$ with $0<\lambda_i<\frac{1}{2}$ for all $1\le i\le m$:

$$\lim_{\beta\to+\infty}\left\|\mathcal{T}^{+}_{\beta,\underline{\lambda}}-\mathcal{T}_{\underline{\lambda}}\right\|=0$$

since for these values of λ_i one can find $\widetilde{R}_i$ such that for all $1\leqslant i\leqslant m$

$$\lambda_i>\widetilde{R}_i>\frac{\lambda_i^2}{1-\lambda_i}.$$

But the eigenvalues of the operator $\mathcal{T}_{\underline{\lambda}}$ can be determined explicitly: they are given by the numbers $\underline{\lambda}^{\underline{\alpha}}$ with $\underline{\alpha}\in\mathbb{N}_0^m$. From this the asymptotic behavior of the leading eigenvalues $\varrho_{\underline{\alpha}}$ of $\mathcal{L}^{+}_{\beta}$ and hence the asymptotic behavior of the leading even eigenvalues of $\mathcal{L}_{\beta}$ follows immediately.

The proof of the behavior of the odd eigenvalues for $\beta\to+\infty$ follows the same line of arguments applied to the operator $\mathcal{L}^{-}_{\beta}$ instead of $\mathcal{L}^{+}_{\beta}$.

For the asymptotic behavior of the even eigenvalues for $\beta\to-\infty$ consider the operator

$$\widetilde{\mathcal{L}}^{+}_{\beta}g(\underline{z})=\exp\left(-\beta\underline{J}\cdot\underline{z}\right)g(\underline{\lambda}+\mathbf{\Lambda}\underline{z})+\exp\left(\beta\underline{J}\cdot\underline{z}\right)g(\underline{\lambda}-\mathbf{\Lambda}\underline{z})$$

and the behavior of its eigenvalues for $\beta\to+\infty$. In this case we write an eigenfunction g as

$$g(\underline{z})=\exp\left(\beta\underline{J}\cdot(1+\mathbf{\Lambda})^{-1}\underline{z}\right)u(\underline{z})$$

and get for u the equation

$$\exp\left(-2\beta\underline{J}\cdot\underline{z}\right)\exp\left(2\beta\underline{J}\cdot(1+\mathbf{\Lambda})^{-1}\mathbf{\Lambda}\underline{z}\right)u(\underline{\lambda}+\mathbf{\Lambda}z)+u(\underline{\lambda}-\mathbf{\Lambda}\underline{z})=\tilde{\varrho}u(\underline{z}),$$

where $\tilde{\varrho}=\varrho\exp\left(-\beta\underline{J}\cdot(1+\mathbf{\Lambda})^{-1}\underline{\lambda}\right)$. Introducing the function $h(\underline{z}):=u(\underline{\lambda}+\mathbf{\Lambda}\underline{z})$ we finally arrive at

$$\tilde{\varrho}\,h(\underline{z})=h(-\mathbf{\Lambda}\underline{\lambda}-\mathbf{\Lambda}\underline{z})+\exp\left(-2\beta\underline{J}\cdot(1+\mathbf{\Lambda})^{-1}\underline{\lambda}\right)\exp\left(-2\beta\underline{J}\cdot(1+\mathbf{\Lambda})^{-1}\underline{z}\right)h(\mathbf{\Lambda}\underline{\lambda}+\mathbf{\Lambda}\underline{z}).$$

Hence $\tilde{\varrho}$ is an eigenvalue of the operator $\mathcal{T}^{-}_{\beta;\underline{\lambda}} : \mathcal{B}\left(D_{\underline{\widetilde{R}}}\right) \to \mathcal{B}\left(D_{\underline{\widetilde{R}}}\right)$ with

$$\mathcal{T}^{-}_{\beta;\underline{\lambda}} h(\underline{z}) \quad = \quad h(-\mathbf{\Lambda}\underline{\lambda} - \mathbf{\Lambda}\underline{z}) + \exp\left(-2\beta\underline{J}\cdot(1+\mathbf{\Lambda})^{-1}\underline{\lambda}\right)\exp\left(-2\beta\underline{J}\cdot(1+\mathbf{\Lambda})^{-1}\underline{z}\right) h(\mathbf{\Lambda}\underline{\lambda} + \mathbf{\Lambda}\underline{z}).$$

For $0 < \lambda_i < \frac{1}{2}$, $1 \leqslant i \leqslant m$ we can find again $\widetilde{R}_i$ with $\lambda_i > \widetilde{R}_i > \frac{\lambda_i^2}{1-\lambda_i}$ and hence

$$\lim_{\beta\to+\infty} \left\| \mathcal{T}^{-}_{\beta;\underline{\lambda}} - \mathcal{T}_{\lambda} \right\| = 0,$$

where

$$\mathcal{T}_{\underline{\lambda}} h\left(\underline{z}\right) = h(-\mathbf{\Lambda}\underline{\lambda} - \mathbf{\Lambda}\underline{z}),$$

is nuclear of order zero on the Banach space $\mathcal{B}\left(D_{\underline{\widetilde{R}}}\right)$. The spectrum of $\mathcal{T}_{\underline{\lambda}}$, however, is given by the numbers $(-\underline{\lambda})^{\underline{\alpha}}$ with $\underline{\alpha} \in \mathbb{N}_0^m$ and hence the leading even eigenvalues $\varrho_{\underline{\alpha}}$ of the operator $\mathcal{L}_\beta$ behave for $\beta \to -\infty$ like $(-1)^{|\underline{\alpha}|}\, \underline{\lambda}^{\underline{\alpha}} \exp\left(-\beta\underline{J}\cdot(1+\mathbf{\Lambda})^{-1}\underline{\lambda}\right)$.

In exactly the same way one shows that the leading odd eigenvalues $\varrho_{\underline{\alpha}}$ of $\mathcal{L}_\beta$ behave for $\beta \to -\infty$ like $-(-1)^{|\underline{\alpha}|}\, \underline{\lambda}^{\underline{\alpha}} \exp\left(-\beta\underline{J}\cdot(1+\mathbf{\Lambda})^{-1}\underline{\lambda}\right)$. □

Theorem 6.2 shows that both for $\beta \to \pm\infty$ infinitely many eigenvalues of $\mathcal{L}_\beta$ tend to $+\infty$. Therefore the determinants $\det\left(1 - \underline{\lambda}^{\underline{\alpha}}\mathcal{L}_\beta\right)$ with $\underline{\alpha} \in \{0,1\}^m$ have infinitely many zeros in the real variable β. Therefore for generic $\underline{\lambda}$ with $0 < \lambda_i < \frac{1}{2}$ for $1 \leqslant i \leqslant m$ the Ruelle zeta function has infinitely many poles and zeros on the real axis.

For special values of the parameters $\underline{\lambda}$ some eigenfunctions and their eigenvalues for the Ruelle operator $\mathcal{L}_\beta$ are explicitly known. If $\underline{\lambda} = \underline{\lambda}_0$ with $\lambda_{0,i} = \lambda_0 = \frac{1}{2}$ for all $1 \leqslant i \leqslant m$, then the identity

$$\sinh\left(2\beta\underline{J}\cdot\left(\frac{1}{2}(\underline{z}\pm\underline{1})\right)\right) = \frac{1}{2}\left(e^{\beta\underline{J}\cdot\underline{z}\pm\beta\underline{J}\cdot\underline{1}} - e^{-\beta\underline{J}\cdot\underline{z}\mp\beta\underline{J}\cdot\underline{1}}\right)$$

and the calculation

$$\begin{aligned}
&e^{\beta\underline{J}\cdot\underline{z}}\sinh\left(2\beta\underline{J}\cdot\left(\frac{1}{2}(\underline{z}+\underline{1})\right)\right) + e^{-\beta\underline{J}\cdot\underline{z}}\sinh\left(2\beta\underline{J}\cdot\left(\frac{1}{2}(\underline{z}-\underline{1})\right)\right) = \\
&\quad= \frac{1}{2}e^{\beta\underline{J}\cdot\underline{z}}\left(e^{\beta\underline{J}\cdot\underline{z}+\beta\underline{J}\cdot\underline{1}} - e^{-\beta\underline{J}\cdot\underline{z}-\beta\underline{J}\cdot\underline{1}}\right) + \frac{1}{2}e^{\beta\underline{J}\cdot\underline{z}}\left(e^{\beta\underline{J}\cdot\underline{z}-\beta\underline{J}\cdot\underline{1}} - e^{-\beta\underline{J}\cdot\underline{z}+\beta\underline{J}\cdot\underline{1}}\right) \\
&\quad= \frac{1}{2}\left(e^{2\beta\underline{J}\cdot\underline{z}+\beta\underline{J}\cdot\underline{1}} - e^{-\beta\underline{J}\cdot\underline{1}}\right) + \frac{1}{2}\left(e^{-\beta\underline{J}\cdot\underline{1}} - e^{-2\beta\underline{J}\cdot\underline{z}+\beta\underline{J}\cdot\underline{1}}\right) \\
&\quad= \frac{1}{2}e^{\beta\underline{J}\cdot\underline{1}}\left(e^{2\beta\underline{J}\cdot\underline{z}} - e^{-2\beta\underline{J}\cdot\underline{z}}\right) \\
&\quad= e^{\beta\underline{J}\cdot\underline{1}}\sinh(2\beta\underline{J}\cdot\underline{z})
\end{aligned}$$

shows that the functions

$$f_{1,n}(\underline{z}) = P_n(\underline{z})\sinh(2\beta\underline{J}\cdot\underline{z}), \tag{20}$$

with $P_n(\underline{z})$ a polynomial homogeneous of degree n in all the variables z_l which is invariant under all translations of the form $z_l \to z_l + c$ for real c, are indeed eigenfunctions of $\mathcal{L}_\beta$ with eigenvalue

$$\varrho_{1,n} = e^{\beta\sum_{l=1}^m J_l}\lambda_0^n := \varrho_1\lambda_0^n.$$

The dimension of the space $\mathcal{B}(D)_n$ of eigenfunctions defined by (20) for fixed n is the dimension of the above space of polynomials and will be calculated in the following proposition.

Proposition 6.3 *Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. The dimension of the space $V_{m,n,\underline{c}}$ of homogeneous polynomials $f(z_1,\ldots,z_m) \in \mathbb{K}[z_1,\ldots,z_m]$ of degree n in m variables which are invariant under a fixed non-zero translation $\underline{z} \mapsto \underline{z}+\underline{c}$ with $\underline{c} \in \mathbb{K}^n$ is*

$$\begin{cases} \begin{pmatrix} m+n-2 \\ n \end{pmatrix} & \text{for } \ m \geq 2 \\ 1 & \text{for } \ m = 1 \text{ and } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $f \in V_{m,n,\underline{c}}$. For fixed $\underline{z}$ consider first the polynomial $t \mapsto f(\underline{z}-t\underline{c}) - f(\underline{z})$. It is of degree less or equal n and has infinitely many zeros (for $t \in \mathbb{Z}$), so it is zero. Thus we have

$$f(\underline{z}+\mathbb{K}\underline{c}) = f(\underline{z}) \quad \forall \underline{z} \in \mathbb{K}^m.$$

Changing coordinates we may now assume that $\underline{c} = (1,0,\ldots,0)$, i.e. f depends only on the variables $z_2,\ldots,z_m$. Thus for $m=1$ only the constant polynomials satisfy the required invariance, whereas for $m \geq 2$ the dimension of $V_{m,n,\underline{c}}$ is equal to the dimension of the space of homogeneous polynomials of degree n in $m-1$ variables and that is $\begin{pmatrix} m+n-2 \\ n \end{pmatrix}$. In fact, if $d(m,n)$ denotes the dimension of the space of homogeneous polynomials of degree n in m variables we obtain the following recursion formula:

$$d(m,n) = \sum_{j=0}^{n} d(m-1,n-j).$$

Twice induction (first on n, then on m) yields

$$\sum_{j=0}^{n} \begin{pmatrix} m+j-2 \\ j \end{pmatrix} = \begin{pmatrix} m+n-1 \\ n \end{pmatrix}$$

and

$$d(m,n) = \sum_{j=0}^{n} d(m-1,j) = \sum_{j=0}^{n} \begin{pmatrix} m+j-2 \\ j \end{pmatrix} = \begin{pmatrix} m+n-1 \\ n \end{pmatrix}.$$

□

The Ruelle zeta function $\zeta_R(\beta) := \zeta_R(1,\beta)$ for our special choice of the parameter $\underline{\lambda} = \underline{\lambda}_0 = (\frac{1}{2},\ldots,\frac{1}{2})$ has the form

$$\zeta_R(\beta) = \prod_{k=0}^{m} \left(\det\left(1 - \lambda_0^k \mathcal{L}_\beta\right)^{\binom{m}{k}} \right)^{(-1)^{k+1}}$$

as one easily checks. According to Proposition 6.3 the contribution of the eigenspaces $\mathcal{B}(D)_n$ for the eigenvalues $\varrho_{1,n}$ to the Ruelle function $\zeta_R(\beta)$ is given by

$$\prod_{k=0}^{m} \prod_{r=0}^{\infty} \left(\left(1 - \lambda_0^{k+r} \varrho_1\right)^{\binom{m}{k}\binom{m+r-2}{r}} \right)^{(-1)^{k+1}} \tag{21}$$

with $k+r=n$.

Remark 6.4 Note that for any β the $\mathcal{B}(D)_n$ represent the entire eigenspaces for the eigenvalues $\varrho_{1,n}$. This is true for $\beta=0$ by Remark 6.1 and Proposition 6.3. Indeed the eigenspace of the eigenvalue $\varrho=2\lambda_0^r$ of the operator $\mathcal{L}_0$ has dimension $\binom{m+r-1}{m-1}$ as one can easily check. On the other hand we have

$$\binom{m+r-1}{m-1}=\sum_{k=0}^{r}\binom{m+k-2}{k}$$

where $\binom{m+k-2}{k}$ is just the dimension of the eigenspace of the eigenvalue $\varrho_{j,k}(\beta)=\varrho_j(\beta)\lambda_0^k$. But for $\beta=0$ the eigenvalues $\varrho_j(\beta)$ are given by $2\lambda_0^j$.

Since the eigenvalue $\varrho_{1,k}(\beta)$ is holomorphic in the entire β-plane the dimension of its eigenspace does not depend on β (see [Kt66], p.68) and is given by $\binom{m+k-2}{m-2}$. $\square$

We will check below that (21) can be reduced to the expression

$$\frac{1-\lambda_0\varrho_1}{1-\varrho_1}. \tag{22}$$

To see this one needs the following result on binomial coefficients.

Proposition 6.5 *For all $m\geqslant 2$, all $r\geqslant 0$ and all $l\leqslant m-1$ the following identity holds*

$$\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}\binom{m-2k+r}{l}=\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m}{2k+1}\binom{m-2k+r-1}{l},$$

where $\lfloor r\rfloor$ is the largest integer less or equal than r.

Proof. The proof is by induction on m, r and l. For $m=2$, $r\geqslant 0$ and $l\in\{0,1\}$ one finds for the left respectively right hand side: $LHS=\binom{r+2}{l}+\binom{r}{l}$ respectively $RHS=2\binom{r+1}{l}$ and hence the two sides of the identity coincide for $l\in\{0,1\}$. Next we show that it suffices to show the identity for $r\geq 0$. In fact, assume it holds for r. We show that then it holds for $r+1$ and hence for all r . Since $\binom{n+1}{s}=\binom{n}{s}+\binom{n}{s-1}$ we find

$$\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}\binom{m-2k+r+1}{l}=\sum_{k=0}^{[\frac{m}{2}]}\binom{m}{2k}\left(\binom{m-2k+r}{l}+\binom{m-2k+r}{l-1}\right).$$

But the right hand side of this equation is equal to

$$\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m}{2k+1}\left(\binom{m-2k+r-1}{l}+\binom{m-2k+r-1}{l-1}\right)=\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m}{2k+1}\binom{m-2k+r}{l}$$

which proves our claim.

Assume next the identity holds up to some m and all $l \leqslant m-1$. Then we show that the identity of Proposition 6.5 holds for $m+1$ and $0 \leqslant l \leqslant m$. Indeed for the LHS of the identity one finds

$$\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m+1}{2k}\binom{m-2k+1}{l} = \\
&= \sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} \left(\binom{m}{2k} + \binom{m}{2k-1}\right)\binom{m-2k+1}{l} \\
&= \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k}\binom{m-2k+1}{l} + \sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m}{2k-1}\binom{m-2k+1}{l} \\
&= \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1}\binom{m-2k}{l} + \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1}\binom{m-2k-1}{l} \\
&= \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1}\binom{m-2k}{l} + \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k}\binom{m-2k}{l}.
\end{aligned}$$

But this is just

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2k+1}\binom{m-2k}{l}$$

and hence the two sides of the identity coincide. The proposition therefore holds for $m+1$, $0 \leqslant l \leqslant m-1$. We have still to show that it holds also for $l = m$, that means

$$\sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m+1}{2k}\binom{m+1-2k}{m} = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2k+1}\binom{m-2k}{m}.$$

But in this case we get $LHS = \binom{m+1}{m}$ respectively $RHS = m+1$ for this last equation, and hence the two sides agree. This proves the proposition. $\square$

The contribution (21) of the eigenspace $\mathcal{B}(D)_n$ can be rewritten first as

$$(1-\lambda_0^n \varrho_1)^{\sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1}\binom{m+n-2k-3}{n-2k-1} - \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k}\binom{m+n-2k-2}{n-2k}}$$

or

$$(1-\lambda_0^n \varrho_1)^{\sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1}\binom{m+n-2k-3}{m-2} - \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k}\binom{m+n-2k-2}{m-2}}.$$

For the cases $n=0$ and $n=1$ one finds the factors $\frac{1}{1-\varrho_1}$ and $1-\lambda_0\varrho_1$. For $n \geqslant 2$, however, Proposition 6.5 shows that all the other eigenvalues $\varrho_{1,n}$ contribute the trivial factor 1 to the zeta function. This proves

$$\prod_{k=0}^{m}\prod_{r=0}^{\infty}\left((1-\lambda_0^{k+r}\varrho_1)^{\binom{m}{k}\binom{m+r-2}{r}}\right)^{(-1)^{k+1}} = \frac{1-\lambda_0\varrho_1}{1-\varrho_1},$$

i.e. the reduction of (21) to (22). This factor leads to "trivial" zeros β_n of the Ruelle zeta function with $\beta_n = \frac{\log 2 + 2\pi i n}{J}$ where $J = \sum_{l=1}^{m} J_l$. This conclusion can be drawn only if there are no cancellations with other eigenvalues, which at the moment we cannot exclude.

For the Kac-Baker model with this special choice of parameters $\underline{\lambda} = \underline{\lambda}_0$ there exists another Ruelle transfer operator which in a certain sense is simpler than $\mathcal{L}_\beta$. These parameters namely describe a Kac-Baker model with an interaction consisting of one exponentially decreasing term only and whose strength is just $J = \sum_{l=1}^m J_l$. The Ruelle transfer operator for this model is

$$\widetilde{\mathcal{L}}_\beta g(z) = e^{\beta J z} g(\tfrac{1}{2} + \tfrac{1}{2} z) + e^{-\beta J z} g(-\tfrac{1}{2} + \tfrac{1}{2} z),$$

where g is now holomorphic in a disc D in the complex plane $\mathbb{C}$. The Ruelle zeta function in terms of the Fredholm determinants of this operator has the form

$$\zeta_R(\beta) = \frac{\det(1 - \frac{1}{2}\widetilde{\mathcal{L}_\beta})}{\det(1 - \widetilde{\mathcal{L}_\beta})}.$$

It is easy to see that the function $g_1(z) = \sinh(2\beta J z)$ is an eigenfunction of the operator $\widetilde{\mathcal{L}}_\beta$ with eigenvalue $\varrho_1 = \exp(\beta J)$. This eigenvalue leads exactly to the zero of the Ruelle function we discussed above. Indeed any eigenfunction $f = f(\underline{z})$ of the Ruelle operator $\mathcal{L}_\beta$ determines an eigenfunction $g = g(z)$ of the operator $\widetilde{\mathcal{L}}_\beta$ through $g(z) := f(z, \ldots, z)$ with the same eigenvalue as long as this function does not vanish identically. On the other hand given an eigenfunction $g_k = g_k(z)$ of the operator $\widetilde{\mathcal{L}}_\beta$ with eigenvalue ϱ_k we know that g_k is holomorphic in the entire β-plane. Consider then the function $f_k = f_k(\underline{z})$ defined as

$$f_k(\underline{z}) := g_k\left(\frac{\sum_{l=1}^m J_l z_l}{J}\right)$$

with $J = \sum_{l=1}^m J_l$. It is entire in $\mathbb{C}^m$ and since

$$f_k\left(\pm\underline{\lambda}_0 + \mathbf{\Lambda_0}\underline{z}\right) = g_k\left(\pm\lambda_0 + \lambda_0 \frac{\sum_{l=1}^m J_l z_l}{J}\right)$$

it is easy to see that the function f_k is an eigenfunction of the operator $\mathcal{L}_\beta$ with eigenvalue ϱ_k. But with f_k also the functions $f_{k,n}(\underline{z}) := P_n(\underline{z}) f_k(\underline{z}), n = 0, 1, \ldots$ with P_n a polynomial homogeneous of degree n in m variables and invariant under translations $z_l \to z_l + c$ for $c \in \mathbb{R}$ are eigenfunctions of $\mathcal{L}_\beta$ with eigenvalue $\varrho_{k,n} = \varrho_k \lambda_0^n$ as we have seen already for the eigenfunction $f_{1,n}$. Their degree of degeneracy is again given by $\binom{m+n-2}{n}$. The eigenfunctions $g_1(z)$ and $f_{1,n}(\underline{z})$ are only special cases of this quite general connection between the eigenfunctions and eigenvalues of the two operators $\mathcal{L}_\beta$ and $\widetilde{\mathcal{L}}_\beta$. An argument similar to the case $k = 1$ shows that among all the eigenvalues $\varrho_{k,n} = \varrho_k \lambda_0^n$ only $\varrho_{k,n}$ with $n \in \{0, 1\}$ give nontrivial contributions to the Ruelle zeta function, which coincide exactly with the contributions of the eigenvalues ϱ_k of the operator $\widetilde{\mathcal{L}}_\beta$ to the Ruelle zeta function for the model with $\underline{\lambda} = \underline{\lambda}_0$.

Our discussion shows that there exist infinitely many trivial zeros of the Ruelle zeta function for the Kac model at least for this special parameter $\underline{\lambda} = \underline{\lambda}_0$ as long as there are not an infinite number of cancellations occuring among the eigenvalues, which one would certainly not expect. The structure of the zeros of the dynamical zeta functions of this family of Kac-Baker models of statistical mechanics hence seems to be very similar to the one well known for arithmetic zeta functions like the Riemann function. It would be interesting to determine their nontrivial zeros and poles and their distribution at least numerically.

7 Matrix elements of the Kac-Gutzwiller operator $\widetilde{\mathcal{G}}_\beta$

For the numerical determination of the zeros of the Ruelle zeta function $\zeta_R(\beta) = \zeta_R(1,\beta)$ the modified Kac-Gutzwiller operator $\widetilde{\mathcal{G}}_\beta$ seems to be best suited. This was realized already in the case $m = 1$ by Gutzwiller in (see [Gu82, §7]). For a numerical study of the eigenvalues and eigenfunctions of a closely related Ruelle operator see [Thr94]. It turns out that also for general m the matrix elements $\mathbb{G}_{\underline{\alpha},\underline{\delta}}$ of the operator $\widetilde{\mathcal{G}}_\beta$ can be determined explicitly in the basis of $L^2(\mathbb{R}^m, d\underline{\xi})$ given by the Hermite functions $\{h_{\underline{\alpha}}\}$. We start with the identity (see [Gu82, (41)])

$$\begin{aligned}
&\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\prod_{i=1}^m \frac{1}{(4\pi\sinh\gamma_i)^{\frac12}} \exp\left(-\frac14\sum_{i=1}^m\left(\left(\xi_i^2+\eta_i^2\right)\tanh\frac{\gamma_i}{2}+\frac{\left(\xi_i-\eta_i\right)^2}{\sinh\gamma_i}\right)\right)\cdot\\
&\quad\cdot\exp\left(\pm\underline{R}\cdot\underline{\eta}-\frac{\underline{\xi}^2}{4}-\frac{\underline{\rho}^2}{2}+\underline{\rho}\cdot\underline{\xi}-\frac{\underline{\eta}^2}{4}-\frac{\underline{\sigma}^2}{2}+\underline{\sigma}\cdot\underline{\eta}\right)d\underline{\xi}d\underline{\eta}=\\
&\quad=(2\pi)^{\frac m2}\exp\left(\frac{\underline{R}^2}{2}+\underline{\rho}\cdot\mathbf{\Lambda}\underline{\sigma}\pm\underline{R}\cdot\underline{\sigma}\pm\underline{R}\cdot\mathbf{\Lambda}\underline{\rho}-\frac12\sum_{i=1}^m\gamma_i\right),
\end{aligned}$$

where as before $\mathbf{\Lambda}$ denotes the diagonal matrix with entries $\mathbf{\Lambda}_{i,j} = \lambda_i\delta_{i,j}$. Adding the two identities for $\pm\underline{R}$ one gets

$$\begin{aligned}
&2\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\prod_{i=1}^m \frac{1}{(4\pi\sinh\gamma_i)^{\frac12}} \exp\left(-\frac14\sum_{i=1}^m\left(\left(\xi_i^2+\eta_i^2\right)\tanh\frac{\gamma_i}{2}+\frac{\left(\xi_i-\eta_i\right)^2}{\sinh\gamma_i}\right)\right)\cdot\\
&\quad\cdot\cosh(\underline{R}\cdot\underline{\eta})\exp\left(-\frac{\underline{\xi}^2}{4}-\frac{\underline{\rho}^2}{2}+\underline{\rho}\cdot\underline{\xi}-\frac{\underline{\eta}^2}{4}-\frac{\underline{\sigma}^2}{2}+\underline{\sigma}\cdot\underline{\eta}\right)d\underline{\xi}\,d\underline{\eta}=\\
&\quad=2\,(2\pi)^{\frac m2}\cosh\left(\underline{R}\cdot\underline{\sigma}+\underline{R}\cdot\mathbf{\Lambda}\underline{\rho}\right)\exp\left(\frac{\underline{R}^2}{2}+\underline{\rho}\cdot\mathbf{\Lambda}\underline{\sigma}-\frac12\sum_{i=1}^m\gamma_i\right).
\end{aligned}$$

But (see [Gu82, (37)])

$$\exp\left(-\frac{\underline{\xi}^2}{4}-\frac{\underline{\rho}^2}{2}-\underline{\rho}\cdot\underline{\xi}\right)=(2\pi)^{\frac m4}\sum_{\underline{\alpha}\in\mathbb{N}_0^m}\frac{\underline{\rho}^{\underline{\alpha}}}{\sqrt{\underline{\alpha}!}}h_{\underline{\alpha}}(\underline{\xi})$$

and hence

$$\begin{aligned}
&\sum_{\underline{\alpha}\in\mathbb{N}_0^m}\sum_{\underline{\delta}\in\mathbb{N}_0^m}2\prod_{i=1}^m \frac{1}{(4\pi\sinh\gamma_i)^{\frac12}}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \exp\left(-\frac14\sum_{i=1}^m\left(\left(\xi_i^2+\eta_i^2\right)\tanh\frac{\gamma_i}{2}+\frac{\left(\xi_i-\eta_i\right)^2}{\sinh\gamma_i}\right)\right)\cdot\\
&\quad\cdot\cosh(\underline{R}\cdot\underline{\eta})\frac{\underline{\rho}^{\underline{\alpha}}}{\sqrt{\underline{\alpha}!}}\frac{\underline{\sigma}^{\underline{\delta}}}{\sqrt{\underline{\delta}!}}h_{\underline{\alpha}}(\underline{\xi})\,h_{\underline{\delta}}(\underline{\eta})\,d\underline{\xi}\,d\underline{\eta}=\\
&\quad=2\exp\left(\frac{\underline{R}^2}{2}+\underline{\rho}\cdot\mathbf{\Lambda}\underline{\sigma}-\frac12\sum_{i=1}^m\gamma_i\right)\cosh\left(\underline{R}\cdot\left(\underline{\sigma}+\mathbf{\Lambda}\underline{\rho}\right)\right).
\end{aligned}$$

Comparing this for $\underline{R} = \underline{R_0} := \sqrt{\beta}(\sqrt{J_1},\ldots,\sqrt{J_m})$ with the kernel $\widetilde{\mathcal{G}}_\beta\left(\underline{\xi},\underline{\eta}\right)$ of the modified Kac-Gutzwiller operator in Theorem 5.12 we hence find

$$\begin{aligned}
&\sum_{\underline{\alpha}\in\mathbb{N}_0^m}\sum_{\underline{\delta}\in\mathbb{N}_0^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\widetilde{\mathcal{G}}_\beta\left(\underline{\xi},\underline{\eta}\right)\;h_{\underline{\alpha}}(\underline{\xi})\,h_{\underline{\delta}}(\underline{\eta})\frac{\underline{\rho}^{\underline{\alpha}}}{\sqrt{\underline{\alpha}!}}\frac{\underline{\sigma}^{\underline{\delta}}}{\sqrt{\underline{\delta}!}}\,d\underline{\xi}\,d\underline{\eta}=\\
&\quad=\;2\exp\left(\underline{\rho}\cdot\mathbf{\Lambda}\underline{\sigma}\right)\cosh\left(\underline{R_0}\cdot\left(\underline{\sigma}+\mathbf{\Lambda}\underline{\rho}\right)\right).
\end{aligned}$$

In terms of the matrix elements $\widetilde{\mathbb{G}}_{\underline{\alpha},\underline{\delta}} := \left(\widetilde{\mathcal{G}}_\beta h_{\underline{\alpha}}, h_{\underline{\delta}}\right)$ this reads

$$\sum_{\underline{\alpha}\in\mathbb{N}_0^m}\sum_{\underline{\delta}\in\mathbb{N}_0^m} \widetilde{\mathbb{G}}_{\underline{\alpha},\underline{\delta}} \frac{\underline{\rho}^{\underline{\alpha}}}{\sqrt{\underline{\alpha}!}}\frac{\underline{\sigma}^{\underline{\delta}}}{\sqrt{\underline{\delta}!}} = 2\exp\left(\underline{\rho}\cdot\mathbf{\Lambda}\underline{\sigma}\right)\cosh\left(\underline{R}_0\cdot\left(\underline{\sigma}+\mathbf{\Lambda}\underline{\rho}\right)\right).$$

To determine from this the matrix elements $\widetilde{\mathbb{G}}_{\underline{\alpha},\underline{\delta}} = \widetilde{\mathbb{G}}_{\underline{\alpha},\underline{\delta}}(\underline{R}_0,\underline{\lambda})$ one first solves the problem

$$\sum_{\underline{\alpha}\in\mathbb{N}_0^m}\sum_{\underline{\delta}\in\mathbb{N}_0^m} \mathbb{A}_{\underline{\alpha},\underline{\delta}} \frac{\underline{\rho}^{\underline{\alpha}}}{\sqrt{\underline{\alpha}!}}\frac{\underline{\sigma}^{\underline{\delta}}}{\sqrt{\underline{\delta}!}} = \exp\left(\underline{\rho}\cdot\mathbf{\Lambda}\underline{\sigma}\right)\exp\left(\underline{R}_0\cdot\left(\underline{\sigma}+\mathbf{\Lambda}\underline{\rho}\right)\right).$$

Obviously the right hand side factorizes into a product of exponentials depending only on the i-th coordinates of the different variables. Hence it suffices to solve the equation

$$\sum_{\alpha_i\in\mathbb{N}_0}\sum_{\beta_i\in\mathbb{N}_0} \mathbb{A}_{\alpha_i,\beta_i}\frac{\rho_i^{\alpha_i}}{\sqrt{\alpha_i!}}\frac{\sigma_i^{\beta_i}}{\sqrt{\beta_i!}} = \exp\left(\rho_i\lambda_i\sigma_i\right)\exp\left(R_{0,i}\left(\sigma_i+\lambda_i\rho_i\right)\right).$$

Expanding the exponentials on the right hand side in σ_i and ρ_i leads to

$$\mathbb{A}_{\alpha_i,\beta_i}\left(R_{0,i},\lambda_i\right) = \sqrt{\alpha_i!\beta_i!}\,\lambda_i^{\alpha_i}\,R_{0,i}^{\mu_i}\sum_{k_i=1}^{M_i-\mu_i}\frac{R_{0,i}^{2k_i}}{(M_i-\mu_i-k_i)!\,k_i!\,(\mu_i+k_i)!}$$

where $M_i = \max\{\alpha_i,\beta_i\}$ and $\mu_i = |\alpha_i-\beta_i|$. It is not too difficult to see that

$$\sqrt{\alpha_i!\beta_i!}\sum_{k_i=0}^{M_i-\mu_i}\frac{R_{0,i}^{2k_i}}{(M_i-\mu_i-k_i)!\,k_i!\,(\mu_i+k_i)!} = \frac{1}{\sqrt{\alpha_i!\beta_i!}}\frac{M_i!}{\mu_i!}\,\Phi\left(\mu_i-M_i,\mu_i+1;-R_{0,i}^2\right),$$

where $\Phi(\cdot,\cdot;\cdot)$ denotes the confluent hypergeometric function. Therefore the matrix elements $\mathbb{A}_{\underline{\alpha},\underline{\delta}} = \prod_{l=1}^m \mathbb{A}_{\alpha_l,\delta_l}$ are given by

$$\mathbb{A}_{\underline{\alpha},\underline{\delta}}\left(\underline{R}_0,\underline{\lambda}\right) = \frac{1}{\sqrt{\underline{\alpha}!\underline{\delta}!}}\,\underline{\lambda}^{\underline{\alpha}}\,\underline{R}_0^{\underline{\mu}}\frac{\underline{M}!}{\underline{\mu}!}\,\Phi\left(\underline{\mu}-\underline{M},\underline{\mu}+\underline{1};-\underline{R}_0^2\right),$$

where $\Phi\left(\underline{\mu}-\underline{M},\underline{\mu}+\underline{1};-\underline{R}_0^2\right) := \prod_{i=1}^m \Phi\left(\mu_i-M_i,\mu_i+1;-R_{0,i}^2\right)$.

Finally we then find for the matrix elements $\widetilde{\mathbb{G}}_{\underline{\alpha},\underline{\delta}}$ of the modified Kac-Gutzwiller operator $\widetilde{\mathcal{G}}_\beta$

$$\widetilde{\mathbb{G}}_{\underline{\alpha},\underline{\delta}}\left(\underline{R}_0,\underline{\lambda}\right) = \frac{1}{\sqrt{\underline{\alpha}!\underline{\delta}!}}\,\underline{\lambda}^{\underline{\alpha}}\,\underline{R}_0^{\underline{\mu}}\left(1+(-1)^{|\underline{\mu}|}\right)\frac{\underline{M}!}{\underline{\mu}!}\Phi\left(\underline{\mu}-\underline{M},\underline{\mu}+\underline{1};-\underline{R}_0^2\right).$$

This shows that $\widetilde{\mathbb{G}}_{\underline{\alpha},\underline{\delta}} \neq 0$ only iff $|\underline{\alpha}+\underline{\delta}| = 0 \mod 2$ which generalizes the result for $m=1$ by Gutzwiller to arbitrary m. Since $\left(\widetilde{\mathcal{G}}_\beta h_{\underline{\alpha}}, h_{\underline{\delta}}\right) = 0$ if $|\underline{\alpha}| \mod 2 \neq |\underline{\delta}| \mod 2$ also for general m the Hilbert space $L^2(\mathbb{R}^m, d\underline{\xi})$ can be decomposed into two subspaces invariant under the operator $\widetilde{\mathcal{G}}_\beta$ spanned by the Hermite functions $\left\{h_{\underline{\alpha}}\right\}$ with $|\underline{\alpha}| = 0 \mod 2$ respectively $|\underline{\alpha}| = 1 \mod 2$. Obviously this property of the operator $\widetilde{\mathcal{G}}_\beta$ corresponds to the fact that the Ruelle operator $\mathcal{L}_\beta$ leaves invariant the subspaces of the Banach space $\mathcal{B}(D)$ spanned by the functions $F = F(\underline{z})$ which are even, respectively odd, under the transformation $\underline{z} \to -\underline{z}$. This follows from the fact that the Segal-Bargmann transform B maps the Hermite functions $h_{\underline{\alpha}}$ to the functions $\zeta_{\underline{\alpha}}$ in (17) which under the above transformation $\underline{z} \to -\underline{z}$ have parity $|\underline{\alpha}| \mod 2$ as one checks easily. One can use the representation of the operator $\widetilde{\mathcal{G}}_\beta$ in terms of the matrix $\widetilde{\mathbb{G}}_\beta$ to calculate its traces. For instance one finds

$$\operatorname{trace}\widetilde{\mathbb{G}}_\beta = \sum_{\underline{\alpha}\in\mathbb{N}_0^m} \widetilde{\mathbb{G}}_{\underline{\alpha},\underline{\alpha}} = 2\sum_{\underline{\alpha}\in\mathbb{N}_0^m} \underline{\lambda}^{\underline{\alpha}}\,\Phi(-\underline{\alpha},\underline{1};-\beta\underline{J}).$$

Because the confluent hypergeometric function $\Phi(-n,1;x)$ is identical to the Laguerre polynomial $L_n(x) = L_n^0(x)$ we get by using the generating function for these polynomials the result

$$\operatorname{trace}\widetilde{\mathbb{G}}_\beta = \frac{2}{\prod_{i=1}^m(1-\lambda_i)}\exp\left(\sum_{i=1}^m \frac{\beta J_i\lambda_i}{1-\lambda_i}\right)$$

which coincides with $\operatorname{trace}\mathcal{L}_\beta$.

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References

[ArMa65] M. Artin, B. Mazur. On periodic points. Ann. Math. 81, 82–99 (1965).

[AtBo67] M. Atiyah, R. Bott. A Lefschetz fixed point formula for elliptic complexes I. Ann. Math. 86, 374–407 (1967).

[Ba00] V. Baladi. *Positive Transfer Operators and Decay of Correlations* World Scientific, Singapore, 2000.

[Ba61] G. Baker. One dimensional order-disorder model which approaches a second order phase transition. Phys. Rev. 122, 1477–1484 (1961).

[Be86] M. Berry. Riemann's zeta function, a model of quantum chaos. Lect. Notes in Physics 263, Springer Verlag, Berlin, 1986.

[BoLa75] R. Bowen, O.E. Lanford. Zeta functions of restrictions of the shift transformation. In *Global Analysis*, Proc. Symp. Pure Math. vol. 14, Amer. Math. Soc., Providence, 1975.

[Co96] A. Connes. Formule de trace en geometrie non commutative et hypothese de Riemann. C.R. Acad. Sci. Paris Ser I 323, 1231–1236 (1996).

[Cr46] H. Cramér. *Mathematical Methods in Statistics.* Princeton Univ. Press, 1946.

[De99] C. Deninger. On dynamical systems and their possible significance for arithmetic geometry. Progr. Math. 171, 29–87 (1999).

[Dw60] B. Dwork. On the rationality of the zeta function of an algebraic variety. Amer. J. Math. 82, 631–648 (1960).

[Fo89] G.B. Folland. *Harmonic Analysis in Phase Space.* Princeton University Press, 1989.

[Gr55] A. Grothendieck. Produits Tensoriels Topologiques et Éspaces Nucléaires. Mem. Amer. Math. Soc. 16, (1955).

[Gu82] M. Gutzwiller. The quantization of a classically ergodic system. Physica $\underline{5D}$, 183–207 (1982).

[HiMa01] J. Hilgert, D. Mayer. The dynamical zeta function and transfer operators for the Kac-Baker model. MPIM-Bonn preprint 2001-86.

[Ka59] M. Kac. On the partition function of a one-dimensional lattice gas. Phys. Fluids 2, 8–12 (1959).

[Ka66] M. Kac. The mathematical mechanism of phase transitions. In *Brandeis University Summer Institute in Theoretical Physics*, vol 1, 245–305 eds.: H. Chretien et al. Gordon & Breach, New York, 1966.

[Kt66] T. Kato. *Perturbation theory for linear operators.* Springer Verlag, Berlin, 1966.

[Ma80] D. Mayer. The Ruelle-Araki transfer operator in classical statistical mechanics. Lect. Notes in Physics 123(1), Springer Verlag, Berlin, 1980.

[Ma91] D. Mayer. The thermodynamic formalism approach to Selberg's zeta function for $PSL(2,\mathbb{Z})$. Bull. Am. Math. Soc. 25, 55–60 (1991).

[Mo89] B. Moritz. *Die Transferoperator–Methode in der Behandlung des Kacschen Spinmodells.* Diplomarbeit, RWTH Aachen, 1989.

[Ro86] P. Robba. Une introduction naive aux cohomologies de Dwork. Mem. Soc. Math. France 23, 61–105 (1986).

[Ru68] D. Ruelle. Statistical mechanics of a one dimensional lattice gas. Commun. Math. Phys. 9, 267–278 (1968).

[Ru78] D. Ruelle. *Thermodynamic Formalism.* Addison-Wesley, London, 1978.

[Ru92] D. Ruelle. Dynamical zeta functions: where do they come from and what are they good for? In *Proc. Internat. Congress of Math. Phys. X (Leipzig 1991)*, 43–51, Springer Verlag, Berlin, 1992.

[Ru94] D. Ruelle. *Dynamical Zeta Functions for Piecewise Monotone Maps of the Interval.* Amer. Math. Soc., 1994.

[Thr94] C. Thron. Taylor series expansions for eigenvalues and eigenfunctions of parametrized composition operators. J. Math. Phys. 35, 2024–2035 (1994).

[ViMa77] K. Vishwanathan, D. Mayer. Statistical mechanics of one-dimensional Ising and Potts models with exponential interactions. Physica $\underline{89A}$, 97–112 (1977).

[We49] A. Weil. Numbers of solutions of equations in finite fields. Bull. Amer. Math. Soc. 55, 497–508 (1949).

[Zi00] M. Zinsmeister. *Thermodynamic Formalism and Holomorphic Dynamical Systems.* Amer. Math. Soc., 2000.