Geometric Exponents for Hyperbolic Julia Sets

Stefan-M. Heinemann[*†]
Bernd O. Stratmann[‡]
Institut für Theoretische Physik der TU Clausthal
Arnold-Sommerfeld-Straße 6,
D-38678 Clausthal-Zellerfeld, Germany
Mathematical Institute, University of St Andrews
St Andrews KY16 9SS, Scotland

27th July 2000

Abstract

For hyperbolic rational maps we show that the Hausdorff dimension of the associated Julia set is bounded away from 2, where the bounds depend exclusively on certain intrinsic geometric exponents. This result is derived via lower estimates for the iterate-counting-function and for the dynamical Poincaré series. Subsequently, we deduce some interesting consequences, such as upper bounds for the decay of the area of parallel-neighbourhoods of the Julia set, and lower bounds for the Lyapunov exponents with respect to the measure of maximal entropy.

1 Introduction

We consider Julia sets $J(T) \subsetneq \overline{\mathbb{C}}$ of hyperbolic holomorphic endomorphisms $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Riemann sphere. It is well-known that these maps form a large open set inside the set of all holomorphic endomorphisms, and it is conjectured that they are in fact dense ('Fatou-conjecture'). Furthermore, for hyperbolic maps there already exists an affirmative answer to the 'analogue of the Ahlfors-conjecture for Kleinian groups', which asserts that the Julia set has always vanishing 2-dimensional Lebesgue measure. That this conjecture is true for

[*]e-mail: Stefan.Heinemann@TU-Clausthal.DE
[†]supported by the DFG Schwerpunkt 'DANSE' and by the DFG Forschergruppe 'Zetafunktionen und Lokalsymmetrische Räume.'
[‡]e-mail: bos@maths.st-and.ac.uk

hyperbolic rational maps is of course an immediate consequence of the well-known fact that in this case the Julia set is porous, and hence its Hausdorff dimension h is strictly less than 2 (see e.g. [13]).

In this paper we add to the latter result an estimate for h from above in terms of certain intrinsic parameters. Our estimate clearly reveals the geometric obstacles which prevent a hyperbolic Julia set from being of Hausdorff dimension 2. In order to state the estimate, let T have *critical distance* c (the distance of $J(T)$ to the forward orbit of the critical points of T), *core exponent* κ (the inverse of the maximal distortion of T on $J(T)$), and *inner lacunarity exponent* λ (that is roughly, the area of $U(J(T))/\langle T_*^{-1}\rangle$, for $U(J(T))$ denoting a suitable neighbourhood of $J(T)$ and $\langle T_*^{-1}\rangle$ the semi-group generated by the holomorphic inverse branches of T). Our main result relates these three geometric exponents to the Hausdorff dimension of $J(T)$. Namely with $U_c(J(T))$ denoting the c-neighbourhood of $J(T)$ we derive the formula (see theorem 4.1)

$$h < 2 - \frac{2\,\lambda\,\kappa^{10}}{\mathrm{area}(U_c(J(T)))}.$$

Originally, this type of formula arose from attempts to clarify the relationship between the spectrum of the Laplacian and certain intrinsic geometric quantities such as volume and length-spectrum for hyperbolic manifolds. For geometrically finite, infinite-volume hyperbolic manifolds a lower bound for the bottom of the Laplace-spectrum was obtained in [5] in terms of the convex core of the manifold (namely, its volume and the area of its boundary), which in certain cases leads to upper bounds for the Hausdorff dimension of the associated Kleinian limit sets. Recently, we derived by purely geometric means a similar type of formula for all convex cocompact Kleinian groups ([12]).

In this paper we adopt the geometric method of [12] and show how to adjust it to the setting of hyperbolic rational maps. For this we require the existence of geometrically well-behaved coverings of $J(T)$ (lemma 2.2) and packings of the Fatou set $F(T) := \overline{\mathbb{C}} \setminus J(T)$ (lemma 2.3). This allows us to introduce the concepts 'iterate-counting-function' and 'dynamical Poincaré series' (representing the natural analogues of the 'orbital counting function' and 'Poincaré series' for Kleinian groups). We derive a finer estimate for the h-conformal measure (lemma 2.5), and apply a folklore-lemma in elementary number theory, which in turn leads to some lower estimates for the iterate-counting-function (lemmata 2.2 and 3.1) and the dynamical Poincaré series (proposition 3.3). Subsequently, we interpret these estimates in terms of the fractality of $J(T)$ and the lacunarity of $F(T)$, which then gives our main result, the afore-stated formula for the Hausdorff dimension of $J(T)$.

2 Hyperbolic Julia sets

2.1 The geometry of hyperbolic Julia sets

Throughout the paper, let $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic holomorphic endomorphism of the Riemann sphere. We assume that $\mathbb{C}$ is equipped with the Euclidean metric d. Let $J = J(T)$ denote the Julia set of T, and $F = F(T) := \overline{\mathbb{C}} \setminus J(T)$ the Fatou set. Furthermore, let $c = c(T)$ denote the *critical distance* of T, that is c is the distance (with respect to d) between the Julia set J and the forward orbit $\bigcup_{n=1}^{\infty} T^{\ell}(\mathrm{Crit})$ of the critical points Crit of T. Then it is well-known that the hyperbolicity of T is equivalent to the fact that c is strictly positive (cf. [2]). Furthermore, the periodic components of F are basins of (super-)attracting periodic points. Without loss of generality, we assume that ∞ is such a periodic point, thus, in particular, J is contained in $\mathbb{C}$, and that the holomorphic inverse branches $\{T_*^{-1}\}$ of T are strictly contracting on J, i.e. $\max_{z\in J}\{|(T_*^{-1})'(z)|\} < 1$.

Definition 2.1 (Core Exponent) *The constant* $\kappa := \min_{z\in J}\{|(T_*^{-1})'(z)|\}$ *will be referred to as the* core exponent *of* T.

We now define the sequences $(\mathcal{C}_n)$ of coverings of J as follows. Fix some optimal initial cover $\mathcal{C}_0$ of J by open discs $D_{0,i}$ of diameter cr_0 centred at points z_i of J, where

$$r_0 = r_0(\kappa) := (1 - \sqrt[4]{\kappa})/(1 + \sqrt[4]{\kappa}),$$

and where optimal refers to that we have chosen this cover to be of minimal cardinality. For $n \in \mathbb{N}$, let $\mathcal{C}_n = \{D_{n,i}\}$ denote the n-th lexicographical cover of J, which is given by the family of all n-th inverse images of the elements in $\mathcal{C}_0$. Put $\mathcal{C} := \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$, and order the elements of $\mathcal{C}$ according to their dynamical sizes, which are defined by $\|D\| := |(T_*^{-n})'(z_i)|$ for $D = T_*^{-n}(D_{0,i}) \in \mathcal{C}_n$. More precisely, let $\mathcal{C} = \{D_0, D_1, D_2, \cdots\}$ such that $\|D_n\| \geq \|D_{n+1}\|$, for all $n \in \mathbb{N}$.

Lemma 2.2 *For all positive* $t \leq 1$*, the set* $\{D \in \mathcal{C} : \kappa^2 t < \|D\| < t\}$ *covers the Julia set* J*.*

Proof: We define $r_1 := (1 - r_0)^3/(1 + r_0)$ and $r_2 := (1 + r_0)^3/(1 - r_0)$. By definition of r_0 it follows that $r_1/r_2 = \kappa$. For $z \in J(T)$, there exists $n(z) \in \mathbb{N}$ such that

$$r_1 \cdot \kappa t < |(T^{n(z)})'(z)|^{-1} \leq r_1 \cdot t.$$

Let $D_{0,i}$ be an element of $\mathcal{C}_0$ such that $T^{n(z)}(z) \in D_{0,i}$. For $T_z^{-n(z)}$ denoting the inverse branch such that $T_z^{-n(z)}(T^{n(z)}(z)) = z$, and using Koebe's theorem (see [11]), we derive for the centre z_i of $D_{0,i}$ that

$$|(T_z^{-n(z)})'(T^{n(z)}(z))|/r_2 < |(T_z^{-n(z)})'(z_i)| < |(T_z^{-n(z)})'(T^{n(z)}(z))|/r_1.$$

This shows that

$$\kappa^2 t = r_1 \cdot \kappa t / r_2 < \|T_z^{-n(z)}(D_{0,i})\| < r_1 \cdot t/r_1 = t.$$

□

Let us make the following conventions. For every $D_{0,i} \in \mathcal{C}_0$, fix a disc $F_{0,i} \subset D_{0,i} \cap F(T)$. We associate to each disc D_n an inverse image $F_n \subset D_n$ of one of the discs $F_{0,i}$ obtained by the same backward iterate as D_n. More precisely, if $D_n = T_*^{-i}(D_{0,j})$ then $F_n := T_*^{-i}(F_{0,j})$.

The following lemma shows that we can choose the family $\{F_{0,i}\}$ such that the induced family $\{F_n\}$ serves as an inner approximation for the orbit of some kind of fundamental domain for the 'almost-discontinuous action' of the inverse branches of T. Recall that a *packing* of a given set is a collection of non-empty open subsets which are pairwise disjoint.

Lemma 2.3 *There exists a choice of $\{F_{0,i}\}$ with the following properties. Here, $r_1 = (1 - r_0)^3/(1 + r_0)$ as in the proof of lemma 2.2.*

(i) The set $\mathcal{F} := \{F_n : n \in \mathbb{N}\}$ is a packing of the Fatou set F.

(ii) There exists a positive constant $\lambda_{\mathcal{F}} = \lambda_{\mathcal{F}}(T)$ such that

$$\text{area}(F_n) \geq \lambda_{\mathcal{F}} r_1^2 \cdot \|D_n\|^2 \quad \textit{for all } n \in \mathbb{N}.$$

Proof: The hyperbolicity of T implies that the periodic components of F are basins of (super-)attracting periodic points. We fix such a periodic point ω. Let p be the prime period of ω. Note that if T is a polynomial then ω may be chosen to be equal to ∞, and consequently $p = 1$. Thus, there exists a simply connected neighbourhood $V \ni \omega$ such that V is mapped into itself by T^p. Furthermore, for some $0 < s < 1$ and $B_s := \{z : |z| < s\}$, the map $T^p|_V$ is conjugate to $z \mapsto (T^p)'(\omega) \cdot z|_{B_s}$, for ω an attracting fixed point, or it is conjugate to $z \mapsto z^\nu|_{B_s}$, for ω a super-attracting fixed point (thus $(T^p)'$ has a zero of order $\nu - 1$ at ω). Let R denote the ring-domain which is defined as the pre-image (under conjugation) of $\{z : |(T^p)'(\omega)|s \leq |z| < s\}$, for the attracting case, or else as the pre-image (under conjugation) of $\{z : s^\nu \leq |z| < s\}$ in the super-attracting case. Let G denote a maximal disc in R disjoint from the forward orbit of Crit. Clearly, all inverse branches of T are well-defined on G. Since $T^i(R) \cap T^j(R) = \emptyset$ for $i \neq j$, we deduce for any two different inverse branches T_*^{-k}, $T_+^{-\ell}$ that

$$T_*^{-k}(G) \cap T_+^{-\ell}(G) = \emptyset.$$

If E denotes a relatively compact set whose closure does not contain periodic points in F, then it is well-known that $T^{-n}(E)$ converges to J in the Hausdorff topology (this is an immediate consequence of [3, Th. 6.1]). In particular, this holds for $E = G$. Thus, there exists n_1 such that $T^{-n}(G) \subset \bigcup_i D_{0,i}$ for all

$n \geq n_1$. Furthermore, there exist n_2 such that for each $n \geq n_2$ the diameter of every element of $T^{-n}(G)$ is less than the Lebesgue number of the cover $\mathcal{C}_0$. Finally, since $\mathcal{C}_0$ was chosen to be optimal, it follows that each $D_{0,i}$ contains an open subset $D^*_{0,i}$ intersecting J with the property that $D^*_{0,i} \cap D_{0,j} = \emptyset$, for $j \neq i$. Now, there exists n_3 such that $T^{-n}(G) \cap D^*_{0,i} \neq \emptyset$, for each i and for every $n \geq n_3$.

With $n_0 := \max\{n_1, n_2, n_3\}$, we define $F_{0,i}$ as a maximal disc which is contained in an element of $T^{-n_0}(G)$ and which intersects $D^*_{0,i}$. Clearly, the induced sets F_n fulfil the first assertion of the lemma, in particular they form a packing of the Fatou set in the sense of the remark preceding lemma 2.3. For the second assertion, note that for $F_n = T_*^{-j}(F_{0,i})$ we have that

$$r_1 \cdot \|D_n\| \leq \inf_{z \in F_{0,i}} |(T_*^{-j})'(z)|.$$

Thus, we obtain that

$$\text{area}(F_n) \geq \lambda_{\mathcal{F}} r_1^2 \cdot \|D_n\|^2,$$

where have set $\lambda_{\mathcal{F}} := \min_i\{\text{area}(F_{0,i})\}$. □

Definition 2.4 (Inner Lacunarity Exponent) *Let λ denote the supremum of all the $\lambda_{\mathcal{F}}$ from lemma 2.3, where the supremum is taken over all possible choices of families $\mathcal{F}$. The number λ will be referred to as the* inner lacunarity exponent *of T.*

2.2 Conformal measures

Recall from [1], [6] and [13] that for a hyperbolic rational map with Julia set of Hausdorff dimension h there exists a unique h-conformal measure m supported on J, i.e. a probability measure with the property that for each Borel set $E \subset J$ on which T is injective, we have that

$$m(T(E)) = \int_E |T'(\xi)|^h dm(\xi).$$

It was shown in [7] that h is the least real number s for which there exists an s-conformal measure. It is known that m is a non-atomic measure which is in the same measure class as the h-dimensional Hausdorff measure on J. In particular, we have that $0 < h < 2$.

The following estimate will be crucial in the sequel. It gives a refinement of the fact that the h-conformal measure is absolutely continuous with respect to the h-dimensional Hausdorff measure.

Lemma 2.5 *For for each $D \in \mathcal{C}$ we have that*

$$m(D) < r_2^2 \kappa^{-4} \|D\|^h.$$

Proof: Without loss of generality (by construction we have that $0 < \|D\| \leq 1$) we assume that $\kappa^{n+2} < \|D\| \leq \kappa^n$, for some $n \in \mathbb{N}$. By construction we have that $D = T_*^{-j}(D_{0,i})$ for some $i, j \in \mathbb{N}$. Note that $\|D\|$ is given by $|(T_*^{-j})'(z_i)|$. Hence, using Koebe's theorem as in the proof of lemma 2.2 we immediately have that

$$|(T^j)'(\xi)| > 1/(r_2 \cdot \|D\|) \geq \kappa^{-n}/r_2$$

for all $\xi \in D$. Using the h-conformality of m, it follows that

$$1 \geq m(T^j(D)) = \int_D |(T^j)'(\xi)|^h dm(\xi) > (\kappa^{-nh}/r_2^h) \cdot m(D).$$

Since $h < 2$, we deduce that

$$m(D) < r_2^2 \kappa^{-4} \|D\|^h.$$

$\square$

3 Estimates from below for the iterate-counting-function and the dynamical Poincaré series

Lemma 3.1 *For the 'iterate-counting-function'* $N(t) := \operatorname{card}\{D \in \mathcal{C} : t \leq \|D\|\}$ *we have for each* $t \in (0, 1]$, *with* $K_\kappa := \kappa^8/r_2^2$, *that*

$$N(t) > K_\kappa \, t^{-h}.$$

Proof: For $t \in (0, 1]$ define $A(t) := \{D \in \mathcal{C} : t \leq \|D\| < t/\kappa^2\}$. By lemma 2.2, the sets in $A(t)$ form a covering of J. Hence, using lemma 2.5, it follows that

$$\begin{aligned}
1 &= m\Big(\bigcup_{D \in A(t)} D\Big) \\
&\leq \sum_{D \in A(t)} m(D) \\
&\leq r_2^2 \kappa^{-4} (t/\kappa^2)^h \operatorname{card}(A(t)) \\
&< r_2^2 \kappa^{-8} t^h \cdot N(t) \\
&= K_\kappa^{-1} t^h \cdot N(t).
\end{aligned}$$

$\square$

Definition 3.2 (Dynamical Poincaré series) *For* $s \geq 0$, *the* dynamical Poincaré series *for the hyperbolic rational map* T *is given by*

$$\sum_{D \in \mathcal{C}} \|D\|^s.$$

It is well-known that h is the exponent of convergence of the Poincaré series for T ([7]). The following proposition gives a refinement of this fact. In particular, it gives an estimate for the rate with which the Poincaré series approaches infinity as s tends to h. Also, note that the following proposition has the immediate implication that hyperbolic rational maps are of 'h-divergence-type', that is for $s = h$ the dynamical Poincaré series diverges.

The proof is essentially an adaption of the 'lemma of partial summation' which is standard for instance in elementary number theory (cf. [4]).

Proposition 3.3 *For each $s > h$, we have that*

$$\sum_{D \in \mathcal{C}} \|D\|^s > K_\kappa \, s/(s-h).$$

Proof: For $n \in \mathbb{N}$, we define the increasing sequence $t_n := \|D_n\|^{-1}$ and let $g(t) := t^{-s}$. We have that $t_0 = 1$, and for $x \in [x, \infty)$ that

$$N(1/x) = \operatorname{card}\{t_n : t_n \leq x\}.$$

Thus, partial summation and lemma 3.1 yield that

$$\begin{aligned}
\sum_{\substack{n \\ \|D_n\| \geq 1/x}} \|D_n\|^s &= \sum_{\substack{n \\ t_n \leq x}} g(t_n) \\
&= N(1/x)g(x) - \int_{t_1}^{x} N(1/t)g'(t)dt \\
&= N(1/x)g(x) + s\int_{1}^{x} N(1/t)/t^{s+1}dt \\
&> N(1/x)g(x) + K_\kappa \, s \int_{1}^{x} t^{h-s-1}dt \\
&= N(1/x)g(x) + K_\kappa \, s \left(x^{-(s-h)}/(h-s) + 1/(s-h)\right).
\end{aligned}$$

Letting x tend to infinity then proves the assertion of the proposition. □

4 Upper bounds for the Hausdorff dimension

Theorem 4.1 *Let T be a hyperbolic rational map with Julia set of Hausdorff dimension h. If T has critical exponent c, core exponent κ and lacunarity exponent λ, then*

$$h < 2 - \frac{2\,\lambda\,\kappa^{10}}{\operatorname{area}(U_c(J(T)))}.$$

Proof: Recall that $r_0 := (1 - \sqrt[4]{\kappa})/(1 + \sqrt[4]{\kappa}) < 1$. Using lemma 2.3 and proposition 3.3, we have for any family $\mathcal{F}$ of the type introduced in section 2.1, with $U_\varrho(J) := \{z : d(z, J) < \varrho\}$ for $\varrho > 0$, that

$$\begin{aligned}\text{area}(U_c(J)) \geq \text{area}(U_{cr_0}(J)) &> \sum_{n\in\mathbb{N}} \text{area}(F_n) \\ &\geq \lambda_{\mathcal{F}} r_1^2 \sum_{n\in\mathbb{N}} \|D_n\|^2 \\ &\geq \lambda_{\mathcal{F}} r_1^2 K_\kappa \cdot \frac{2}{2-h}.\end{aligned}$$

Taking the supremum over all $\mathcal{F}$, we have

$$\text{area}(U_c(J)) \geq \lambda\, r_1^2 K_\kappa \cdot \frac{2}{2-h}.$$

An elementary rearrangement then gives the theorem. □

5 Some applications

5.1 Upper bounds for the decay of area

Theorem 4.1 may be applied to derive some lower bounds for the decay of the area of ε-neighbourhoods of the Julia set.

In order to obtain this type of application, consider some compact set $E \subset\subset \mathbb{R}^n$, and let $\mathcal{N}_\varepsilon(E)$ denote the smallest number of sets of diameter at most ε that cover E. For the n-dimensional volume (vol_n) of the ε-neighbourhood of E, we then have the following well-known formula (see e.g. [8]).

$$\lim_{\varepsilon\to 0} \frac{\log \mathcal{N}_\varepsilon(E)}{-\log \varepsilon} = n - \lim_{\varepsilon\to 0} \frac{\log\ \text{vol}_n(U_\varepsilon(E))}{\log \varepsilon}.$$

Combining theorem 4.1 with this general fact from fractal geometry, we obtain the following result.

Proposition 5.1 *Let $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic rational map with Julia set $J(T) \subset \overline{\mathbb{C}}$, critical distance c, core exponent κ and lacunarity exponent λ. For the area of ε-neighbourhoods of $J(T)$ we have that*

$$\lim_{\varepsilon\to 0} \frac{\log\ \text{area}(U_\varepsilon(J(T)))}{\log \varepsilon} \geq \frac{2\,\lambda\,\kappa^{10}}{\text{area}(U_c(J(T)))}.$$

Proof: For hyperbolic Julia sets it is well-known that the Hausdorff dimension and the box-counting dimension coincide and are equal to h. In particular, we hence have that

$$\lim_{\varepsilon\to 0} \frac{\log \mathcal{N}_\varepsilon(J(T))}{-\log \varepsilon} = h.$$

Combining this fact, theorem 4.1 and the afore-mentioned general observation from fractal geometry, the assertion of the proposition follows. □

5.2 Lower bounds for the Lyapunov exponent

Theorem 4.1 may be applied to derive some lower bounds for the Lyapunov exponent with respect to the measure of maximal entropy for T.

For this recall that there exists a unique measure $\mu = \mu(T)$ of maximal entropy $\log(d)$ for T, where d denotes the degree of T (see [10]). By a result of Ledrappier [9], it follows for the Lyapunov exponent χ_μ with respect to μ that

$$\log d = h\,\chi_\mu.$$

Combining this equality with the estimate for h in theorem 4.1, we immediately have the following proposition.

Proposition 5.2 *Let $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic rational map with Julia set $J(T) \subset \overline{\mathbb{C}}$, critical distance c, core exponent κ and lacunarity exponent λ. For the Lyapunov exponent χ_μ we have that*

$$\chi_\mu \;\geq\; \frac{1}{2} \cdot \frac{\log(d)\cdot \mathrm{area}(U_c(J(T)))}{\mathrm{area}(U_c(J(T))) - \lambda\,\kappa^{10}}.$$

□

5.3 A class of examples

Due to the fractal nature of the Julia set, it is in general a highly non-trivial task to compute the numerical values of the geometric exponents c, κ, λ. In particular, the explicit determination of $\lambda_{\mathcal{F}}$ may turn out to be difficult. In this section we give for a certain easy class of examples an outline of how our general method works in practice.

For $\tau \in \mathbb{R}$, $\tau < -(5 + 2\sqrt{5})/4$, we consider the family of quadratic maps $T = T_\tau$ given by

$$T : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \quad \text{such that} \quad z \mapsto z^2 + \tau.$$

It is well-known that $J(T)$ is a Cantor set which is a subset of the real line. Let $\Gamma := 1/2 + \sqrt{1/4 - \tau}$ denote the outer escape radius and $\gamma := \sqrt{-\tau - \Gamma}$ the inner escape radius. It is easy to see that $|z| > \Gamma$ implies $|T(z)| > |z|$, and that $|z| < \gamma$ implies $|T(z)| > \Gamma$. Hence, it follows that $J(T)$ is contained in the closed annulus $A_{\gamma,\Gamma}$ centred at the origin with inner radius γ and outer radius Γ. Also, note that for the critical point 0 we have that $T(0) = \tau$, $T^2(0) = \tau^2 + \tau$, and that $|T^n(0)| > |\tau^2 + \tau|$ for all $n > 2$. Hence, the critical distance c of T is equal to γ, and the core exponent κ of T turns out to be equal to $(2\Gamma)^{-1}$. Consequently, here in this example the constant r_0 of our construction in section 2 is equal to $(\sqrt[4]{2\Gamma}-1)/(\sqrt[4]{2\Gamma}+1)$.

In order to determine a lower bound for λ, the following observation will be helpful. The proof of this statement is straight-forward and we omit it.

Lemma 5.3 *For $z_0 \in \mathbb{R}, z \in \mathbb{C}$ we have that*

$$\begin{aligned} &|2\,T(z) - T(z_0 + |z-z_0|) - T(z_0 - |z-z_0|)| \\ &\qquad \geq 2\,|T(z_0 + |z-z_0|) - T(z_0 - |z-z_0|)|. \end{aligned}$$

$\square$

For a closed interval $I \subset \mathbb{R}$, let $B(I)$ denote the ball centred at the real line such that $B(I) \cap \mathbb{R} = I$. It follows that for each interval $I \subset \mathbb{R}$ and any holomorphic inverse branch T_*^{-1} it holds that $T_*^{-1}(B(I)) \subset B(T_*^{-1}(I))$.

The structure of $J(T)$ as a Cantor set is described as follows. $J(T)$ is contained in the two intervals $[-\Gamma, -\gamma]$ and $[\gamma, \Gamma]$ of order 1. The inverse images of these two intervals give rise to four intervals of order 2. By carrying on in this way, we obtain inductively 2^n intervals of order n. Clearly, $(\Gamma-\gamma)/(2\gamma)^{(n-1)}$ is an upper bound for the length of an interval of order n. Hence, we find n_0 such that $2cr_0$ is greater than the maximal length of an interval of order n_0. Obviously, we have that

$$n_0 \leq \left[\frac{\log((\Gamma-\gamma)/(2cr_0))}{\log(2\gamma)}\right] + 2.$$

Now, we can construct by induction a covering of $J(T)$ consisting of balls of radius cr_0 as follows. Denote the intervals of order n_0 from left to right by I_1 up to $I_{2^{n_0}}$, and choose $D_{0,1}$ to be a ball of radius cr_0 centred at $J(T)$ such that $D_{0,1}$ contains I_1 up to I_{i_1}, where i_1 is chosen to be maximal with respect to this property. Next, choose $D_{0,2}$ to be a ball of radius cr_0 centred at $J(T)$ such that $D_{0,2}$ contains I_{i_1+1} up to I_{i_2}, where i_2 is chosen to be maximal with respect to this property. Continuing in this way, we eventually derive $D_{0,k}$ such that $i_k = 2^{n_0}$. For $\ell = 1, \ldots, k$, it is clear that only $D_{0,\ell}$ fully contains the interval I_{i_ℓ}. Now, define the simply connected set G as follows

$$G := B_{\Gamma-\tau}(\tau) \setminus \left(\overline{B_\Gamma(0)} \cup \overline{B_{-(\Gamma+\tau)}(\tau)}\right).$$

By lemma 5.3, we have that the two inverse images of G contain the set

$$\tilde{G} := B_\Gamma(0) \setminus \left(\overline{B_{(\Gamma-\gamma)/2}(-(\Gamma+\gamma)/2)} \cup \overline{B_{(\Gamma-\gamma)/2}((\Gamma+\gamma)/2)}\right).$$

We have that $\text{area}(\tilde{G}) = \pi(\Gamma^2 - (\Gamma-\gamma)^2/2)$. Clearly, to each interval I_{i_ℓ} of order n_0 there corresponds an inverse image of order n_0 of $\tilde{G}$ which is contained

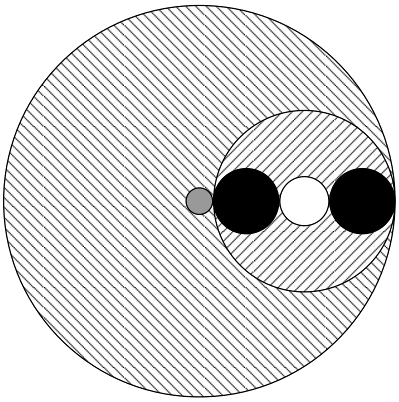

Figure 1.
Sketch of the regions G and $\tilde{G}$ for $\tau = -2,5$. The shading corresponds to G, the shading to $\tilde{G}$, the white ball is $B_\gamma(0)$, the gray ball its forward image under T. The black regions are $B([-\Gamma, -\gamma])$ and $B([\gamma, \Gamma])$.

in $B(I_{i_\ell})$ and whose area is at least $\pi(\Gamma^2 - (\Gamma-\gamma)^2/2)/(2\Gamma)^{2n_0}$. Thus, it follows that

$$\lambda \geq \pi(\Gamma^2 - (\Gamma-\gamma)^2/2)/(2\Gamma)^{2n_0}.$$

Since it holds that $\mathrm{area}(U_c(J(T))) \leq 4\gamma(\Gamma-\gamma) + 2\pi\gamma^2$, we conclude for the Hausdorff dimension h of $J(T)$ that

$$h < 2 - \frac{2\pi(\Gamma^2 - (\Gamma-\gamma)^2/2)/(2\Gamma)^{2n_0+10}}{4((\Gamma-\gamma) + 2\pi\gamma)\gamma}.$$

References

[1] J. Aaronson, M. Denker, and M. Urbański. Ergodic theory of Markov fibred systems and parabolic rational maps. *Transactions American Mathematical Society*, 337:495–548, 1993.

[2] A. F. Beardon. *Iteration of Rational Functions.* Number 132 in Graduate Texts in Mathematics. Springer, 1991.

[3] H. Brolin. Invariant sets under iteration of rational functions. *Arkiv för Matematik*, 6:103–144, 1965.

[4] P. Bundschuh. *Einführung in die Zahlentheorie.* Hochschultext. Springer, 1988.

[5] M. Burger and R.D. Canary. A lower bound on λ_0 for geometrically finite hyperbolic n-manifolds. *Journal für reine und angewandte Mathematik*, 454:37–57, 1994.

[6] M. Denker and M. Urbański. Hausdorff measures and conformal measures on Julia sets with a rationally indifferent periodic point. *Journal London Mathematical Society*, 43:107–118, 1991.

[7] M. Denker and M. Urbański. On Sullivan's conformal measures for rational maps of the Riemann sphere. *Nonlinearity*, 4(2):365–384, 1991.

[8] K. J. Falconer. *Fractal Geometry*. Wiley, 1990.

[9] F. Ledrappier. Quelques proprietes ergodiques des applications rationelles. *Comptes Rendus d'Academie des Seances Paris*, 299:37–40, 1984.

[10] M. Yu. Lyubich. The maximum-entropy measure of a rational endomorphism of the Riemann sphere. *Functional Analysis and its Applications*, 16:309–311, 1983.

[11] C. Pommerenke. *Univalent Functions*, volume XXV of *Studia Mathematica*. Vandenhoek & Ruprecht, 1975.

[12] B.O. Stratmann. A note on geometric upper bounds for the exponent of convergence of convex cocompact Kleinian groups. *Mathematica Gottingensis*, 18:1–13, 1999.

[13] D. P. Sullivan. Conformal dynamical systems. In *Geometric dynamics*, number 1007 in Lecture Notes in Mathematics, pages 725–752, Rio de Janeiro, 1982. Springer.

Acknowledgement: *We should like to thank the Mathematisches Institut and the Institut für Mathematische Stochastik at the University of Göttingen for their warm hospitality.*

St.-M. Heinemann,
Institut für Theoretische Physik der TU Clausthal
Arnold-Sommerfeld-Straße 6
D-38678 Clausthal-Zellerfeld
Germany
e-mail: Stefan.Heinemann@TU-Clausthal.DE

B.O. Stratmann,
Mathematical Institute,
University of St Andrews,
St Andrews KY16 9SS,
Scotland.
e-mail: bos@maths.st-and.ac.uk