Rokhlin's Lemma for Non-Invertible Maps

Stefan-M. Heinemann[*][†]
Oliver Schmitt[‡][§]
Institut für Mathematische Stochastik der Universität Göttingen
Lotzestraße 13, 37083 Göttingen
Germany

5th May 2000

Abstract

We present an elementary proof of Rokhlin's Lemma for measurable endomorphisms of separable spaces which are aperiodic with respect to an invariant Borel probability measure. As an application we explicitely compute the Rokhlin sets for classes of interval maps and Bernoulli shifts.

1 Introduction

Rokhlin's Lemma which was originally (cf. [Rokhlin, 1963]) proved for bi-measurable aperiodic automorphisms T of Polish spaces X is an important tool in ergodic theory. It states that for an automorphism of the above type which is invariant with respect to a Borel probability measure μ, any $n \in \mathbb{N}^*$ and $\varepsilon > 0$ one finds a measurable set F (a so-called (n, ε)-Rokhlin set) such that, for $j = 0, \ldots, n-1$, the sets $T^{-j}(F)$ are pairwise disjoint and exhaust X with exception of a remainder set whose mass is smaller then ε. In particular, Rokhlin's Lemma is indispensable for the canonical construction of generators. Here the most natural setting is that of a measurable endomorphism T which is not necessarily forward measurable. Since the usual proof ([Halmos, 1956, ps 70-72]) quoted in standard textbooks on ergodic theory (e.g. [Denker et al., 1976],[Friedman, 1970],[Petersen, 1989]) uses a Kakutani tower type construction and thus forward measurability[1], we felt

[*]`e-mail: sheinema@math.uni-goettingen.de`
[†]supported by the DFG Schwerpunkt 'DANSE'.
[‡]`e-mail: oschmitt@math.uni-goettingen.de`
[§]supported by the Graduiertenkolleg 'Strömungsinstabilitäten und Turbulenzen'.
[1]unfortunately, the only edition (1956) available to the authors even contained a misprint (definition of E on p. 72) and a slight gap (proof of lemma 1 on p. 70)

obliged to provide an elementary proof which does not rely on this assumption. In fact, our most 'sophisticated' tool is Poincaré's Recurrence Theorem. Moreover, the setting is generalised from Polish spaces (which allow to construct (local) measurable inverses [Royden, 1970, ch. 15.4]) to separable metric spaces, i.e. we consider a surjective measurable endomorphisms $T : X \to X$ of separable metric spaces such that the maps T are aperiodic with respect to invariant Borel probability measures μ.

In the second part we explicitely describe how to construct the Rokhlin sets for dynamical systems which are semi-conjugate to any full shift.

2 Rokhlin's Lemma

Throughout this paragraph we assume that X is a *separable metric* space. More precisely, we find a metric $d : X \times X \to \mathbb{R}_+$ on X which defines the topology and there exists a countable sequence $(x_i)_{i\in\mathbb{N}}$ of points of X which are dense in the space. We denote with $\mathcal{B}$ the *Borel σ-algebra* on X. Furthermore, let $T : X \to X$ be a *surjective measurable*, but not necessarily bi-measurable map with respect to $\mathcal{B}$ and let μ be a *T-invariant Borel probability measure* on $(X, \mathcal{B})$, i.e., for all $A \in \mathcal{B}$, we have that $\mu(T^{-1}(A)) = \mu(A)$. Finally, T is supposed to be *μ-a.e. aperiodic*, that is, for all $n \in \mathbb{N}^*$ the set of points $x \in X$ such that $T^n(x) = x$ has μ-measure 0.
We derive two lemmata which make use of the aperiodicity of T.

Lemma 2.1 (Disjoint Inverse Images)
Let X, $\mathcal{B}$, T, μ be as above. For any $n \in \mathbb{N}^*$, we find a measurable set F of positive measure such that

$$\mu(T^{-n}(F) \cap F) = 0. \tag{1}$$

Proof: Assume that we cannot determine a Borel set F of positive measure such that (1) holds. This implies that, in particular, for all ε-balls $B_\varepsilon(x_i)$ centred at the points x_i, we have that

$$\mu(T^{-n}(B_\varepsilon(x_i)) \setminus B_\varepsilon(x_i)) = 0.$$

As an immediate consequence we derive the following relation.

$$\mu(\{x : \ T^n(x) \in B_\varepsilon(x_i), \ d(x, T^n(x)) \geq 3 \cdot \varepsilon\}) = 0.$$

Note that, for any $\varepsilon > 0$, the union of the $B_\varepsilon(x_i)$ covers X. Thus, we obtain that

$$\begin{aligned}
&\mu(\{x:\ T^n(x) \in X,\ d(x,T^n(x)) \geq 3\cdot\varepsilon\}) \\
&= \mu\left(\bigcup_{i\in\mathbb{N}} \{x:\ T^n(x) \in B_\varepsilon(x_i),\ d(x,T^n(x)) \geq 3\cdot\varepsilon\}\right) \\
&\leq \sum_{i\in\mathbb{N}} \mu(\{x:\ T^n(x) \in B_\varepsilon(x_i),\ d(x,T^n(x)) \geq 3\cdot\varepsilon\}) \\
&= 0.
\end{aligned}$$

Letting ε tend to 0 we deduce that

$$\mu(\{x:\ T^n(x) \in X,\ d(x,T^n(x)) > 0\}) = 0,$$

hence

$$\mu(\{x:\ T^n(x) \in X,\ x = T^n(x)\}) = 1,$$

in contradiction to the assumption that T is aperiodic. $\square$

Definition 2.2 (Measurable n-Chains)
A measurable set F of *strictly positive μ-measure* which fulfils relation (1) for $n = 1,\ldots,m-1$ induces an *m-chain* $(\mathcal{F})_m$, which we define to be the m-tuple

$$(\mathcal{F})_m := \left\langle F, T^{-1}(F), \ldots, T^{-(m-1)}(F)\right\rangle.$$

Lemma 2.3 (Existence of Measurable n-Chains)
Let X, $\mathcal{B}$, T, μ be as above. For any $n \in \mathbb{N}^*$, there exists a set F which induces an n-chain.
Proof: For $n = 1$ there is nothing to prove, we can take $F = X$. We proceed by induction. Let us assume that we found $F \in \mathcal{B}$ with $\mu(F) > 0$ which induces an $(n-1)$-chain. There must be a measurable subset $G \subseteq F$ of positive μ-measure such that $\mu(T^{-(n-1)}(G) \setminus G) > 0$. If this is not the case, then the proof of lemma **2.1** implies that on F we have that μ-a.e. $T^{n-1}(x) = x$ in contradiction to the aperiodicity of T. Clearly, $G \setminus T^{-(n-1)}(G)$ induces an n-chain. $\square$

From now on we shall skip the attribute 'μ-a.e.' and neglect μ-null-sets. Furthermore, we use the symbol $\biguplus$ for *disjoint* unions.

Remark 2.4
If $F \in \mathcal{B}$ induces an n-chain, then, for any $k \in \mathbb{N}$, also the inverse image $T^{-k}(F)$ induces an n-chain.

Theorem 2.5 (Rokhlin's Lemma)
Let X, $\mathcal{B}$, T, μ be as above. For any $n \in \mathbb{N}^*$ and any $\varepsilon > 0$, there exists a measurable set $E \in \mathcal{B}$ which induces an n-chain $(\mathcal{E})_n$[2] such that

$$\mu((\mathcal{E})_n) := \mu\left(\biguplus_{\ell=0}^{n-1} T^{-\ell}(E)\right) > 1 - \varepsilon.$$

Proof: We choose $m \in \mathbb{N}$ such that $m \geq n$ and

$$1/m < \varepsilon/(n-1).$$

According to lemma **2.3**, the set of m-chains $(\mathcal{F})_m$ is not empty. We equip this set, the family of inducing sets, resp., with a partial ordering, by defining

$$F <_\mu F' \;:\Longleftrightarrow\; (F \subset F') \wedge (\mu(F) < \mu(F'))\,.$$

Zorn's Lemma then provides maximal elements. Let F be such a maximal inducing set and denote with $(\mathcal{F})_m$ the induced m-chain. We use the following notation. For $k \in \mathbb{N}$, we denote the k-th backward image of F with $F_k := T^{-k}(F)$. Note that $\mu(F_k) = \mu(F) \leq 1/m$.
From Poincaré's Recurrence Theorem (see e.g. [Walters, 1978]) we deduce that

$$\bigcup_{k=0}^{\infty} F_k = X.$$

Namely, the complement U of $\bigcup_{k=0}^{\infty} F_k$ is a measurable set which is forward invariant, i.e. $T(U) \subseteq U$. It is also backward invariant, because, according to Poincaré's Recurrence Theorem, any point z from $T^{-1}(U) \cap \bigcup_{k=0}^{\infty} F_k$ is mapped to F_0 for infinitely many times under forward iteration. But we have that $T(z) \in U$, hence all forward iterates stay in U. Then, because of the maximality of $(\mathcal{F})_m$, we conclude that $\mu(U) = 0$. If this did not hold, then we would find an n-chain $(\mathcal{G})_m$ in U, whose union with $(\mathcal{F})_m$ would yield an m-chain strictly bigger than $(\mathcal{F})_m$, which was assumed to be maximal.

In particular, we have that each point $x \in X$ eventually lands in F_0 under forward iteration of T.
We define the measurable sets

$$F^k := F_k \setminus \bigcup_{j=0}^{k-1} F_j,$$

which fulfil the following relation

$$z \in F^k \iff k = \min_{\ell \in \mathbb{N}} \left\{T^\ell(z) \in F_0\right\}.$$

[2]also called an (n, ε)-Rokhlin tower

Thus, we deduce that

$$F^{k+1} = T^{-1}(F^k) \setminus F_0$$

and, for $j \in \mathbb{N}$,

$$F^{k+j} \subseteq T^{-j}(F^k) \subseteq F^{k+j} \dot{\cup} \biguplus_{\ell=0}^{j-1} F^\ell. \tag{2}$$

From the definition of the F^k we derive that the F^k are pairwise disjoint and

$$\biguplus_{k=0}^{\infty} F^k = X.$$

Let us show that

$$E := \biguplus_{k=1}^{\infty} F^{kn-1}$$

induces an n-chain. For this, it is sufficient to prove that, for $j = 0, \dots, n-1$, we have that

$$E \cap T^{-j}(E) = \emptyset.$$

If we assume that, for $p, q \in \mathbb{N}^*$,

$$z \in F^{pn-1} \cap T^{-j}(F^{qn-1})$$

then the fact that the F^k are pairwise disjoint implies that

$$z \in T^{-j}(F^{qn-1}) \setminus F^{qn-1+j}.$$

Due to (2) we obtain that

$$z \in \bigcup_{i=0}^{j-1} F_i \subseteq \bigcup_{i=0}^{n-2} F_i = \bigcup_{i=0}^{n-2} F^i.$$

Then, by definition of E, we have that

$$z \in \bigcup_{i=0}^{n-2} F^i \cap E = \emptyset.$$

Thus, E induces an n-chain $(\mathcal{E})_n$. Its measure is estimated by

$$\begin{aligned}
\mu\left(\bigcup_{j=0}^{n-1} T^{-j}(E)\right) &\geq \mu\left(\biguplus_{i=n-1}^{\infty} F^i\right) \\
&= 1 - \mu\left(\biguplus_{i=0}^{n-2} F_i\right) \\
&\geq 1 - (n-1)/m \\
&> 1 - \varepsilon.
\end{aligned}$$

□

Setting $G = E_{n-1}$ in the above theorem, we obtain the classical form of Rokhlin's Lemma.

Corollary 2.6 (Classical Rokhlin's Lemma)
Let X, $\mathcal{B}$, T, μ be as above. For any $n \in \mathbb{N}^*$ and any $\varepsilon > 0$, there exists a measurable set $G \in \mathcal{B}$ such that the sets $G, T^1(G), \dots, T^{n-1}(G)$ are measurable and pairwise disjoint, moreover, we have that

$$\mu\left(\biguplus_{\ell=0}^{n-1} T^{\ell}(G)\right) > 1 - \varepsilon.$$

□

3 Canonical Rokhlin-Partitions for Circle Maps

In this paragraph we use Rokhlin's lemma in order to investigate the dynamics of an important class of examples. More precisely, we consider the following three families of mappings which are equivalent from the measure theoretical point of view.

3.1 Definition of Circle Maps

Example 3.1
For an integer $d > 1$, we consider the map

$$\widetilde{T}_d : S^1 \to S^1$$

which is given by

$$\exp(2\pi i t) \mapsto \exp(2\pi i d t), \tag{3}$$

for $t \in \mathbb{R}$. Clearly, together with the normalised Lebesgue measure λ on S^1, $\widetilde{T}_d$ induces an aperiodic measurable endomorphism.

By consideration of (3) we obtain a dynamical system on the half open unit interval.

Example 3.2
We define a piecewise linear interval map $T_d : [0,1) \to [0,1)$ by setting

$$T_d(t) = T_{d,k}(t) := d(t - k/d), \text{ for } t \in [k/d, (k+1)/d).$$

Its inverse branches are given by

$$T_{d,k}^{-1}(t) = t/d + k/d. \tag{4}$$

The corresponding invariant measure is the Lebesgue measure μ on $[0,1)$.

Recall that the d-adic coding for $t \in [0,1)$ is unique (with the exception of the null set of d-rationals) and induces a measurable isomorphism between $[0,1)$ and the shift space Σ_d defined below. Thus we get a third equivalent representation.

Example 3.3
Let Σ_d be the space of one-sided infinite sequences in the symbols $\{0,\ldots,d-1\}$, i.e. $x=(x_1x_2\ldots)$ with $x_i \in \{0,\ldots,d-1\}$, and denote with σ_d the shift on Σ_d

$$\sigma_d((x_1x_2x_3\ldots)) := (x_2x_3x_4\ldots).$$

As invariant measure ν_d we have the $(1/d,\ldots,1/d)$ Bernoulli measure on Σ_d (cf. [Denker et al., 1976]) which assigns mass $1/d^k$ to k-cylinders

$$Z(x_1^*,\ldots,x_k^*) := \{x \in \Sigma_d : x_1 = x_1^*, \ldots, x_k = x_k^*\}.$$

In the following we use the representation given in example **3.2** in order to construct n-chains for the systems introduced above. Thus, we consider (cf. theorem **2.5**) $X=[0,1)$, $T=T_d$ and μ the Lebesgue measure on $[0,1)$.

We construct a maximal m-chain by taking as $F=F_0$ an open interval associated to the minimal periodic point $\varphi_m := 1/(d^m-1)$ of prime period m. More precisely, we calculate maximal values α_m, β_m such that the interval $I_m := (\varphi_m - \alpha_m, \varphi_m + \beta_m)$ induces an m-chain.

In order to do so it is sufficient to ensure that I_m does not intersect any of its pre-images of order 1 to $m-1$. The pre-images which come closest to I_m are $T_{d,0}^{-(m-2)}T_{d,1}^{-1}(I_m)$ (from the right) and $T_{d,0}^{-1}(I_m)$ (from the left). This leads to the following equations for the maximal values α_m, β_m.

$$\begin{aligned}\varphi_m + \beta_m &= (\varphi_m - \alpha_m + 1)/(d^{m-1})\\ \varphi_m - \alpha_m &= (\varphi_m + \beta_m)/d.\end{aligned}$$

The solution of this pair of equations is given by $\alpha_m = 0$ and $\beta_m = (d-1)/(d^m-1)$. To conclude that I_m induces a maximal m-chain $(\mathcal{F})_m$, we consider the closure of

$$\Delta_m := \bigcup_{\ell=0}^{m-1} T_{d,0}^{-\ell} I_m \cup T_{d,0}^{-(m-2)} T_{d,1}^{-1}(I_m)$$

which is a closed interval of length

$$\frac{1}{d^{m-1}} + \frac{\beta_m}{d^{m-1}}.$$

Noting that T_d^{m-1} corresponds to multiplication by d^{m-1} we see that $T_d^{m-1}(\Delta_m)$ covers X, hence,

$$T_d^{m-1}\left(\bigcup_{\ell=0}^{m-1} F_\ell\right) = X.$$

Thus, we cannot find E such that

$$T_d^{-(m-1)}(E) \cap \bigcup_{\ell=0}^{m-1} F_\ell = \emptyset.$$

Also, $F_0 := I_m$ and its first $m-1$ forward images

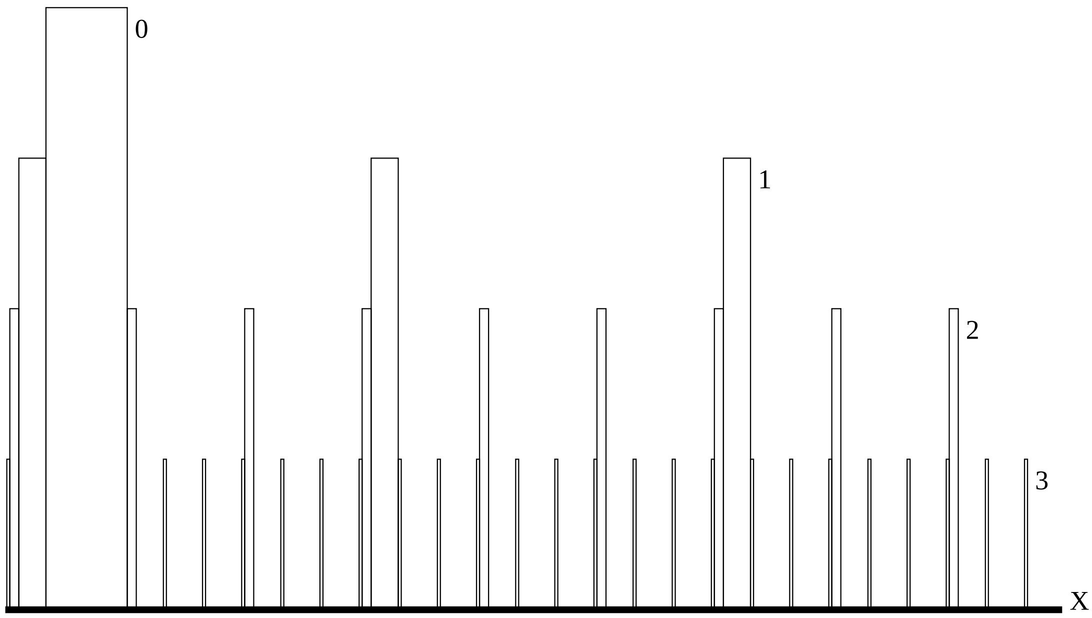

Figure 1: $F^0, \ldots, F^m$ *for* $d=3$ *and* $m=3$.

$$T_d^k(F_0) = (d^k/(d^m-1), d^{k+1}/(d^m-1)),$$

for $k = 1, \ldots, m-2$ and

$$T_d^{m-1}(F_0) = [0, 1/(d^m-1)) \dot\cup (d^{m-1}/(d^m-1), 1)$$

are pairwise disjoint. Moreover, we have that $\mu(F_0) = (d-1)/(d^m-1) \leq 1/m$.

3.2 Properties of the sequence $(\boldsymbol{F^k})$

We exploit the topological structure of our class of maps in order to obtain two refinements of relation (2).

Lemma 3.4
For $j \in \mathbb{N}$, let $F^* = (a,b) := T_{d,k_j}^{-1} \circ \ldots \circ T_{d,k_1}^{-1}(F^0)$ be a connected component of $T_d^{-j}(F^0)$. Then we have that either F^* is contained in F^0 or the intersection of F^* and F^0 is empty.

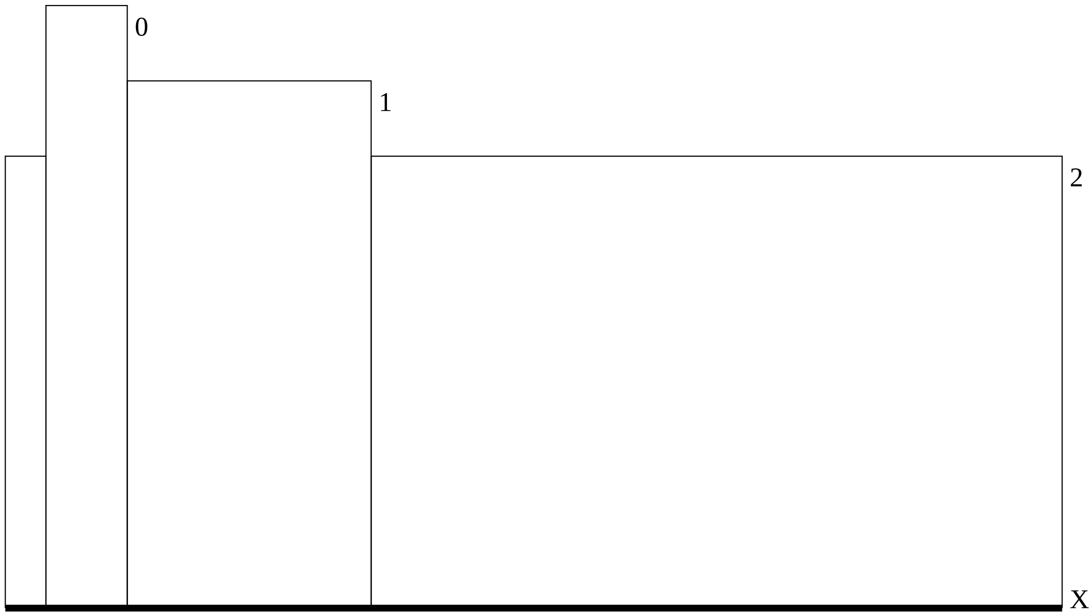

Figure 2: $T^0(F^0)$, ... , $T^{m-1}(F^0)$ *for* $d = 3$ *and* $m = 3$.

Proof: Let us assume that F^* and F^0 have non-empty intersection but that F^* is not contained in F^0. This either implies that $a < 1/(d^m - 1) < b$ or that $a < d/(d^m - 1) < b$. Note that under T_d^j the interval F^* is mapped to F^0. As T_d^j is order-preserving, it follows that either

$$1/(d^m - 1) < T^j(1/(d^m - 1)) < d/(d^m - 1)$$

or

$$1/(d^m - 1) < T^j(d/(d^m - 1)) < d/(d^m - 1).$$

This cannot be the case because $T^j(1/(d^m - 1)) = d^{(j \mod m)}/(d^m - 1)$. $\square$

Lemma 3.5
Let F^* be as in the preceding lemma. Denote with $\mathbb{T}_d^m(F^*)$ the disjoint union of the intervals in $\bigcup_{j=0}^{m-1} T_d^{-j}(F^*)$. Then there is a unique interval $G^* \in \mathbb{T}_d^m(F^*)$ such that $G^* \subseteq F^0$. We refer to this interval as the *hit marker* of F^*.
Proof: Due to the maximality of $(\mathcal{F})_m$, for any interval F^* we have that $\mathbb{T}_d^m(F^*) \cap F^0 \neq \emptyset$. Let us assume that we can find two different sets G_1, G_2 in $\mathbb{T}_d^m(F^*)$ such that $G_1 \subseteq T_d^{-j}(F^*) \cap F^0$ and $G_2 \subseteq T_d^{-k}(F^*) \cap F^0$, where we have that $0 \leq j, k \leq m - 1$. The fact that $G_1 \subseteq F_0$ implies that $F^* \subseteq T_d^j(F_0)$, the condition on G_2 gives that $F^* \subseteq T_d^k(F_0)$. According to the remark at the end of paragraph **3.1**, this can only happen if $j = k$. For this case we note that equation (4) implies that the minimal distance of two intervals in $T_d^{-j}(F^*)$ is $1/d^j - (d - 1)/(d^j(d^m - 1)) \geq \beta_m$. $\square$

We note that F^k consists of finitely many intervals of length β_m/d^k. In the following paragraph we use this fact to determine the mass of $(\mathcal{E})_n$.

3.3 The mass of $(\mathcal{E})_n$

Let us denote the number of intervals in F^k with ω_k. We have that

$$\mu(F^k) = \omega_k \cdot \beta_m/d^k.$$

Thus, in order to compute the mass of $(\mathcal{E})_n$, it is sufficient to know the sequence (ω_k). Clearly, as F^0 induces an m-chain, we have that, for $k = 0, \dots, m-1$,

$$\omega_k = d^k.$$

The cardinalities ω_{k+m-1}, for $k \geq 1$, can be computed applying lemma **3.5**.

Proposition 3.6
The number ω_{k+m-1} of intervals in F^{k+m-1}, for $k \geq 1$ is equal to

$$\omega_{k+m-1} = (d-1) \cdot \sum_{j=0}^{m-2} \omega_{k+j}. \tag{5}$$

Proof: We make use of relation (2) with $j = m-1$. The set $T_d^{-(m-1)}(F^k)$ consists of $d^{m-1} \cdot \omega_k$ intervals. We have to remove all the intervals which are contained in $\biguplus_{\ell=0}^{m-2} F^\ell$. This is equivalent to eliminating the predecessors of the hit markers in $\mathbb{T}_d^m(F^*)$ for all $F^* \in F^k$. The number of eliminated intervals depends on the position of the hit markers G^* in the $\mathbb{T}_d^m(F^*)$. More precisely, the fact that $G^* \subseteq T_d^{-\ell}(F^*)$ enforces the subtraction of $d^{m-1-\ell}$ intervals.

By lemma **3.5** we know that, for each of the ω_k intervals $F^* \subset F^k$, the collection $\mathbb{T}_d^m(F^*)$ contains a unique hit marker G^*. First, let us assume that all hit markers G^* are contained in $T_d^{m-1}(F^*)$ for the corresponding F^*. This gives the estimate $(d^{m-1}-1) \cdot \omega_k$ for ω_{k+m-1}. In order to calculate the exact value we have to determine the numbers of hit markers which are contained in $T^{-\ell}F^*$, for $1 \leq \ell \leq m-2$. For each of these markers we have to correct our estimate by $-d^{m-1-\ell}$ for the number of intervals which are actually 'shadowed' by the hit marker and by $+1$ for the one interval which we had subtracted assuming that the hit marker was contained in $T^{-(m-1)}(F^*)$. The number of these hit markers is easily seen to be equal to $d \cdot \omega_{k+\ell-1} - \omega_{k+\ell}$. Thus we obtain that

$$\begin{aligned}
\omega_{k+m-1} \quad = \quad & (d^{m-1} - 1) \cdot \omega_k \\
& -(d^{m-2} - 1) \cdot (d \cdot \omega_k - \omega_{k+1}) \\
& -(d^{m-3} - 1) \cdot (d \cdot \omega_{k+1} - \omega_{k+2}) \\
& - \ldots \\
& -(d^2 - 1) \cdot (d \cdot \omega_{k+m-4} - \omega_{k+m-3}) \\
& -(d^1 - 1) \cdot (d \cdot \omega_{k+m-3} - \omega_{k+m-2}) \\
= \quad & (d-1) \cdot \sum_{j=0}^{m-2} \omega_{k+j}.
\end{aligned}$$

□

With the aid of the generating function $\Omega(z) := \sum_{\ell=0}^{\infty} \omega_\ell z^\ell$ it is possible to transform the recursive representation of (ω_ℓ) given in (5) into an explicite form.

Proposition 3.7
The generating function for the sequence (ω_ℓ) is given by

$$\Omega(z) = \frac{1 - z^m}{(d-1) \cdot z^m - (d\,z - 1)}. \tag{6}$$

Proof: We take equation (5) and multiply it by the monomial z^{k+m-1}. Summing up over all $k \geq 1$ yields

$$\sum_{k=1}^{\infty} (d-1) \cdot \sum_{j=0}^{m-2} \omega_{k+j} z^{k+m-1} = \sum_{k=1} \omega_{k+m-1} z^{k+m-1}.$$

If we express this relation in terms of $\Omega(z)$ then we obtain the following equation

$$(d-1) \cdot \sum_{j=0}^{m-2} z^{m-1-j} \left(\Omega(z) - \sum_{\ell=0}^{j} d^\ell z^\ell \right) = \Omega(z) - \sum_{j=0}^{m-1} d^j z^j,$$

whose solution yields (6). □

For $m = 2$ and $m = 3$ we calculate the ω_k explicitely.

Example 3.8
For $m = 2$, we have that

$$\Omega(z) = \frac{1+z}{1-(d-1) \cdot z}.$$

The geometric series expansion for the denominator yields

$$(1-(d-1)\cdot z)^{-1} = \sum_{k=0}^{\infty}(d-1)^k z^k,$$

thus, we have that

$$\begin{aligned}\Omega(z) &= (1+z)\cdot\sum_{k=0}^{\infty}(d-1)^k z^k \\ &= 1+\sum_{k=1}^{\infty} d\cdot(d-1)^{k-1}z^k\end{aligned}$$

For the mass of the chain $(\mathcal{E})_n$ we obtain from

$$\mu\left(F^0\right) = 1/(d+1)$$

and, hence, for $k \geq 1$,

$$\mu\left(F^k\right) = \left(\frac{d-1}{d}\right)^{k-1} \Bigg/ (d+1)$$

that

$$\begin{aligned}\mu((\mathcal{E})_n) &= 2\cdot\sum_{k=1}^{\infty}\mu\left(F^{2k}\right) \\ &= \frac{2}{d+1}\cdot\sum_{k=1}^{\infty}\left(\frac{d-1}{d}\right)^{2k-1} \\ &= \frac{2\cdot(d-1)}{d\cdot(d+1)}\cdot\sum_{k=0}^{\infty}\left(\frac{d-1}{d}\right)^{2k} \\ &= \frac{d\cdot(d-1)}{(d+1)\cdot(d-1/2)}.\end{aligned}$$

Example 3.9
For $m=3$, we have that

$$\Omega(z) = \frac{1+z+z^2}{d-(d-1)\cdot(1+z+z^2)}.$$

The zeros of the denominator are

$$\zeta_{\pm} = \frac{(d-1)\pm\sqrt{(d-1)\cdot(d+3)}}{2\cdot(1-d)},$$

thus, we have that

$$
\begin{aligned}
\Omega(z) &= \frac{1+z+z^2}{1-d}\cdot\frac{1}{(z-\zeta_+)\cdot(z-\zeta_-)}\\
&= \frac{1+z+z^2}{(d-1)\cdot(\zeta_+-\zeta_-)}\cdot\left(\frac{1}{\zeta_+}\cdot\frac{1}{1-z/\zeta_+}-\frac{1}{\zeta_-}\cdot\frac{1}{-z/\zeta_-}\right)\\
&= \frac{1+z+z^2}{(d-1)\cdot(\zeta_+-\zeta_-)}\cdot\sum_{k=0}^{\infty}\left(\zeta_+^{-(k+1)}-\zeta_-^{-(k+1)}\right)\cdot z^k\\
&= 1+d\,z+\sum_{k=2}^{\infty}\left(\frac{\zeta_+^{-(k+1)}-\zeta_-^{-(k+1)}+\zeta_+^{-k}-\zeta_-^{-k}+\zeta_+^{-(k-1)}-\zeta_-^{-(k-1)}}{(d-1)\cdot(\zeta_+-\zeta_-)}\right)\cdot z^k.
\end{aligned}
$$

With

$$\mu\left(F^0\right)=1/(d^2+d+1)$$

and, for $k\geq 2$,

$$\mu\left(F^k\right)=\frac{\zeta_+^{-(k+1)}-\zeta_-^{-(k+1)}+\zeta_+^{-k}-\zeta_-^{-k}+\zeta_+^{-(k-1)}-\zeta_-^{-(k-1)}}{(d^3-1)\cdot d^k\cdot(\zeta_+-\zeta_-)}$$

we obtain that

$$
\begin{aligned}
\mu((\mathcal{E})_n) &= 3\cdot\sum_{k=1}^{\infty}\mu(F^{3k})\\
&= \frac{3}{d^3(d^3-1)\cdot(\zeta_+-\zeta_-)}\\
&\quad\cdot\left((\zeta_+^{-2}+\zeta_+^{-3}+\zeta_+^{-4})\cdot\sum_{k=0}^{\infty}\left(\frac{\zeta_+^{-3}}{d^3}\right)^k\right.\\
&\quad\left.-(\zeta_-^{-2}+\zeta_-^{-3}+\zeta_-^{-4})\cdot\sum_{k=0}^{\infty}\left(\frac{\zeta_-^{-3}}{d^3}\right)^k\right).\\
&= \frac{3}{(d^3-1)\cdot(\zeta_+-\zeta_-)}\cdot\left(\frac{\zeta_++1+\zeta_+^{-1}}{d^3\zeta_+^3-1}-\frac{\zeta_-+1+\zeta_-^{-1}}{d^3\zeta_-^3-1}\right)\\
&= \frac{3\cdot(d-1)\cdot d\cdot(d^4+d^3+d-1)}{3\,d^6+3\,d^4-3\,d^3+d^2-2\,d+1}.
\end{aligned}
$$

For the reader's convenience we show how to construct the F^k if one uses the point of view of example **3.3**. We define *m-words* as finite sequences $(x_1\dots x_m)$ of m symbols from $\{0,\dots,d-1\}$ and use the following notation. For each $p\in\{0,\dots,d^m-1\}$ let $\mathbf{p}$ be the uniquely determined m-word $(x_1\dots x_m)$ such that $p=\sum_{i=1}^{m}x_i d^{m-i}$. The fact that $t\in I_m$, i.e.

$$1/(d^m-1)<t<d/(d^m-1)$$

then corresponds to the following condition on $x \in \Sigma_d$.

$$(\mathbf{111}\ldots\mathbf{1}\ldots) < x < (\mathbf{ddd}\ldots\mathbf{d}\ldots),$$

where we use the lexicographical order on Σ_d. Thus we have that

$$x = \begin{cases} (\underbrace{\mathbf{1}\ldots\mathbf{1}}_{k \text{ times}}\mathbf{p}\ldots), & \text{where } k \geq 1 \text{ and } 2 \leq p \leq d^m - 1; \\ (\mathbf{p}\ldots), & \text{where } 2 \leq p \leq d-1; \\ (\underbrace{\mathbf{d}\ldots\mathbf{d}}_{k \text{ times}}\mathbf{p}\ldots), & \text{where } k \geq 1 \text{ and } 0 \leq p \leq d-1. \end{cases}$$

It turns out that we can describe the sets F^k entirely by conditions on the leading d-adic digits of the x. From (2) we derive the following lemma which gives the equivalent to the hit markers in proposition **3.6**.

Lemma 3.10
For $\ell \geq 1$ we have that $t \in T^{-1}(F^{\ell-1}) \setminus F^\ell \subseteq F_0$ if and only if the corresponding x fulfils the following condition

$$(\mathbf{10}\ldots) \leq x \leq (\mathbf{d}[\mathbf{d-1}]\ldots)\,.$$

□

Corollary 3.11
Let $T : X \to X$ be a dynamical system such that the d-shift (Σ_d, σ_d) is a measurable factor. Then there exist (n, ε)-Rokhlin sets with respect to the invariant measure which is induced by the pullback of ν_d to X. □

The transfer to the setting of example **3.1** is easily done using the measurable map $z \mapsto \exp(2\pi i \cdot z)$.

An important special case is the *invariant harmonic measure* supported on the connected Julia-set $J(f)$ of a monic polynomial f of degree $d \geq 2$. On obtains a conjugation (by the Riemann map of $\overline{\mathbb{C}} \setminus K(f)$) between $\widetilde{T_d}$ extended to $\overline{\mathbb{C}} \setminus \overline{B_1(0)}$ and f on $\overline{\mathbb{C}} \setminus K(f)$, where $K(f)$ denotes the set of points with bounded forward orbit under iteration of f. The harmonic measure then is defined as the pullback of λ (cf. [Przytycki et al., 1989]).

Corollary 3.12
There exist (n, ε)-Rokhlin sets with respect to the harmonic measure of the connected Julia sets of polynomials of degree $d \geq 2$. □

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