Projective Connections, AGD Manifold and Integrable Systems

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To appear in Reviews in Mathematical Physics

dedicated to Professor N. Mukunda on his 60th birthday

Abstract

If u_is are periodic function on the line, the operator $\frac{d^n}{dx^n} + u_{n-1}\frac{d^{n-1}}{dx^{n-1}} + u_{n-2}\frac{d^{n-2}}{dx^{n-2}} + \ldots + u_1\frac{d}{dx} + u_0$, acting on periodic functions, is called a Adler-Gelfand-Dikii (or AGD) operator. In this paper we consider a projective connection as defined by this n-th order operator on the circle. In particular, projective connection as defined by a second order operator can be identified with the dual of Virasoro algebra, and it is well known that the KdV equation as a Euler-Arnold equation in the coadjoint orbit of the Bott-Virasoro group. In this paper we study (formally) the evolution equation of the Adler-Gelfand-Dikii operator, $\Delta^{(n)}$, (at least for $n \leq 4$), under the action of $Vect(S^1)$. This yields a single generating equation for periodic function u. We also establish a connection between the projective vector field, a vector field leaves fixed a given (extended) projective connection, and the C. Neumann system using the idea of Knörrer and Moser. We show that certain quadratic function of a projective field satisfies C. Neumann system.

1 Introduction

Projective connections on the circle were classified from a geometric point of view by Kuiper [16]. Lazutkin and Pankratova [17] were the first persons to formulate this analytically. A projective connection on the circle is a linear second order differential operator, $\frac{d^2}{dx^2} + u(x)$, acting on a periodic functions, known as Hill's operator. This is a dual space of Virasoro algebra. It has a sequence of eigenvalues tending to infinity. If the eigenvalues are fixed, then the possible function u form an infinite dimensional torus, one can imagine this to be a product of one circle for each pair of consecutive eigenvalues. The Korteweg-de Vries equation evolves in a straight line on one of the above tori.

The connection between the geodesics equation of the Virasoro-Bott group and the periodic Korteweg-de Vries (KdV) equation follows from the work of Arnold [4], Kirillov [12,13,14], Segal [23,24] and others [22,25]. The connection between the C. Neumann system and the Bott-Virasoro group has been studied in [9, 10].

In late seventies Adler [1] defined a family of second Hamiltonian structures with respect to which generalized KdV equation can be written as Hamiltonian system. The Adler-Gelfand-Dikii brackets are defined on the space of n-th order scalar differential operators of the form

$$\frac{d^n}{dx^n} + u_{n-1}\frac{d^{n-1}}{dx^{n-1}} + u_{n-2}\frac{d^{n-2}}{dx^{n-2}} + \ldots + u_1\frac{d}{dx} + u_0, \tag{1}$$

where the coefficients are smooth and periodic. This space of differential operators are called Adler-Gelfand-Dikii (AGD) manifold. It is known that the extended classical conformal algebras, that is, $n > 2$ spin algebras are obtained from the second Hamiltonian structure of Lax equations for a Lax operators of type (1) [5].

Let X_f denote the formal sum

$$X_f = \sum_{i=1}^{n} \partial^{-1}\left(\frac{\delta f}{\delta u_{i-1}}\right). \tag{2}$$

Then all the classical W-algebras are derived from the following Lie-Poisson bracket:

$$\{f, g\}_L = \int \text{res } (V_{X_f}(L)X_g), \tag{3}$$

where

$$V_{X_f}(L) = L(X_f L)_+ - (LX_f)_+ L. \tag{4}$$

Here L is differential operator of type (1) the positive sign denotes the projection to differential part.

This is a generalization of the relation between KdV equation and the Virasoro algebra ($n = 2$ case).

The space of differential operators of type (1) is connected to $gl(n, R)$ current algebra, i.e. algebra of loops $C^\infty(S^1, gl(n, R))$. This automatically reduces to $sl(n, R)$ current algebra when $u_{n-1} = 0$. Gelfand and Dikii established the relation between dual spaces of Kac-Moody algebras on the circle and the AGD space. Later is a Poisson subspace of former [6]. They coincide only for $n = 2$ case.

Earlier physicists have studied the AGD operators and its connection to extended classical conformal algebras. Pierre Mathieu [19] has listed several extended conformal operators. These are denoted by Δ^{2k-1}, and some of members of this family are:

$$\Delta^{(0)} = 1,$$

$$\Delta^{(1)} = \partial_x,$$

$$\Delta^{(2)} = \partial_x^2 + u,$$

$$\Delta^{(3)} = \partial_x^3 + 4u\partial_x + 2u',$$

$$\Delta^{(4)} = \partial_x^4 + 9u^2 + 3u'' + 10u'\partial_x + 10u\partial_x^2,$$

etc. etc.

They define the Poisson bracket of the spin-k field. All these scalar differentail operators are covariant operators, in other words, these transform covariantly under the coadjoint action of $Diff(S^1)$. As shown by Mathieu at least for $k \leq 3$, $\Delta^{(2k-1)}$, these operators $\Delta^{(2k-1)}$ depend only on u.

In this paper we will study the relation between AGD space and projective structure on circle, and its connection to integrable systems.

This paper is organized as follows:
In the next section we present a brief description of projective connection on the circle, basically all the necessary definitions are given here. We will describe the action of $Diff(S^1)$ on the Adler-Gelfand-Dikii manifold in section 3. In section 4 we discuss the evolution equation of the extended projective connection. This equation is closely related to Alber and Alber generating equation, this reduces to KdV equation for a special choice of function. Section 5 is devoted to the stabilizer set of the action of $Vect(S^1)$ on the space of Adler-Gelfand-Dikii manifolds and its connection to C. Neumann system. In section 6 we discuss few things about Kirillov's super algebra. This is connected to the solution of C. Neumann systems.

We now state our main results:

Proposition 1.1 *The action of $Vect(S^1) = T_e Diff^+(S^1)$ on Adler-Gelfand-Dikii operator $\Delta^{(n)}$ (for $n \leq 2$) yields an evolution equation*

$$\frac{\partial \Delta^{(n)}}{\partial t} + [\mathcal{L}_v, \Delta^{(n)}] = 0 \quad n < 5$$

This generates a nonlinear evolution equation for periodic functions u

$$\frac{\partial u}{\partial t} = \frac{1}{2} f''' + 2f'u + fu'.$$

In this paper we argue that the action of $Vect(S^1)$ on the AGD manifold, $[\mathcal{L}_v, \Delta^{(n)}] = \mathcal{L}_v \circ \Delta^{(n)} - \Delta^{(n)} \circ \mathcal{L}_v$, introduced in [18], (notation will be explained later) can be considered as a coadjoint action.

Theorem 1.2 *The Euler equation gives the KdV equation, it is a Hamiltonian flow on the "coadjoint orbits" in $\Delta^{(n)}$ for the Hamiltonian function $H(u) = \frac{1}{2}u^2$.*

Theorem 1.3 *Let $v = f(x)\frac{d}{dx} \in Vect(S^1) = T_e Diff^+(S^1)$ be a vector field, this acts on $\Delta^{(n)}$ by*

$$[\mathcal{L}_v, \Delta^{(n)}] := \mathcal{L}_v^{-(n+1)/2} \circ \Delta^{(n)} - \Delta^{(n)} \circ \mathcal{L}_v^{(n-1)/2},$$

where $\mathcal{L}_v$ is the Lie derivative with respect to $v = f(x)\frac{d}{dx}$ vector field.

(a) The stabilizer set of the action of $f(x)\frac{d}{dx}$ on the space of Adler-Gelfand-Dikii operators $\Delta^{(n)}$is a projective vector and satisfies

$$f''' + 4f'u + 2fu' = 0.$$

(b) If $f =< \chi, A^{-1}\chi >$, then χ satisfies

$$\chi'' = -A\chi - u\chi$$

where $< \chi, \chi >= 1$ and

$$u = - < \chi, A\chi > + < \chi', \ \chi' >$$

which is the system of C. Neumann equation.

(c) Let f be a projective vector field, the square root of f is the element of scalar density of weight -$\frac{1}{2}$. Then the pair satisfies Lie super algebra.

Acknowledgement: It is my pleasant duty to acknowledge gratefully for several stimulating conversations with Professors Mitya Alekseevsky, Sasha Kirillov and Richard Montgomery. I would like to thank very much IHES (Bures-sur-Yvette)- where this project was started. This work is partially supported by the S. Chandrasekhar memorial ICSC-World Laboratory fellowship.

2 Projective Structures

Let Ω denote the cotangent bundle of the circle, this is a trivial real line bundle on S^1. Let Ω^{-1} and Ω^m be the tangent bundle and the m-fold tensor product of Ω. The section of Ω^m is locally given by

$$s = g(x)dx^m,$$

where $g(x) = g(x + 2\pi)$. There is a natural action of $Diff(S^1)$ on the sections of Ω^m. We will compute the infinitesimal action of $Vect(S^1)$ on Ω^m. Let $h_\epsilon(x)$ be a one-parameter family of diffeomorphisms of the circle, such that

$$h_\epsilon(x) = x + \epsilon f + O(\epsilon^2).$$

Hence,

$$g(h_\epsilon(x))(d(h_\epsilon(x)))^{\otimes m} = g(x+\epsilon f + O(\epsilon^2))d(x+\epsilon f + O(\epsilon^2))^{\otimes m}$$
$$= (g(x) + \epsilon g_x f + O(\epsilon^2))(dx + \epsilon f_x dx + O(\epsilon^2)^{\otimes m}$$
$$= g(x)dx^{\otimes m} + \epsilon(g_x f + m f_x g)dx^{\otimes m} + O(\epsilon^2).$$

Hence,

$$\delta g = g_x f + m f_x g)dx^{\otimes m}.$$

We will express this action by

$$\mathcal{L}_v s \;=\; (fg' + mf'g)dx^m, \tag{5}$$

where $\mathcal{L}_v$ is the Lie derivative of the vector field $v = f\frac{d}{dx}$. Suppose $w \;=\; g\frac{d}{dx} \in \Omega^{-1}$, then we get

$$\mathcal{L}_v w \;=\; [v,w] \;=\; (gf' - g'f)\frac{d}{dx}. \tag{6}$$

2.1 Projective Connection on the Circle

Let us denote $\Omega^{\pm 1/2}$ be the square root of the tangent and cotangent bundle respectively.

Definition 2.1 (11) *A projective connection on the circle is a linear second order differential operator*

$$\Delta : \Gamma(\Omega^{-\frac{1}{2}}) \longrightarrow \Gamma(\Omega^{\frac{3}{2}})$$

such that the symbol of
(1) Δ is the identity and

$$(2) \quad \int_{S^1} (\Delta s_1)s_2 \;=\; \int_{S^1} s_1(\Delta s_2)$$

for all $s_i \in \Gamma(\Omega^{-\frac{1}{2}})$.

Let us take $s = \psi(x)dx^{-\frac{1}{2}} \in \Gamma(\Omega^{-\frac{1}{2}})$, then $\Delta s \in \Gamma(\Omega^{3/2})$ is locally described by

$$\Delta s \;=\; (a\psi'' + b\psi' + c\psi)dx^{\frac{3}{2}}.$$

As discussed in [7] any differential equation of the form

$$\frac{d^2y}{dx^2} = p_3(\frac{dy}{dx})^3 + p_2(\frac{dy}{dx})^2 + p_1(\frac{dy}{dx}) + p_0$$

defines a projective structure.

From the definition of the projective connection condition (1) implies $a \;=\; 1$ and condition (2) implies $b \;=\; 0$, hence projective connection can be identified with the Hill operator

$$\Delta^{(2)} \equiv \Delta \;=\; \frac{d^2}{dx^2} + u(x).$$

Using the equation $\mathcal{L}_v \Delta^{(2)} = \Delta^{(2)} \mathcal{L}_v$, we obtain

Proposition 2.2 *A projective vector field $v = f\frac{d}{dx} \in \Gamma(\Omega^{-1})$ satisfies*

$$f''' + 4f'u + 2fu' \;=\; 0. \qquad (7)$$

2.2 Extended Projective Connections

In this section we will explicitly compute the equations satisfied by projective vector fields.

Definition 2.3 (Extended Projective Connection) *An extended projective connection on the circle is a class of differential (conformal) operators*

$$\Delta^{(n)} : \Gamma(\Omega^{-\frac{n-1}{2}}) \longrightarrow \Gamma(\Omega^{\frac{n+1}{2}})$$

such that
(1) the symbol of Δ^n is the identity,

$$(2) \int_{S^1} (\Delta s_1) s_2 \;=\; \int_{S^1} s_1(\Delta s_2)$$

for all $s_i \in \Gamma(\Omega^{-\frac{n-1}{2}})$.

It is known that the symbol of a n-th order operator from a vector bundle U to V is a section of $\mathrm{Hom}\ (U, W \otimes Sym^n T)$, where

$$U = \Omega^{-\frac{(n-1)}{2}} \; V = \Omega^{\frac{n+1}{2}}.$$

Since $T = \Omega^{-1}$, then we get

$$V \otimes Sym^n T \cong U,$$

giving an invariant meaning to the first condition.
If $s_2 \in \Gamma(\Omega^{-\frac{n-1}{2}})$ then $\Delta^n s_2 \in \Gamma(\Omega)$ is a one form to integrate.

The consequence of the *first condition* is that all the differential operators are monic, that is the coefficient of the highest derivative is always one, and the *second condition* says that the term $u_{n-1} = 0$.

Definition 2.4 *A vector field is called projective vector field which keeps fixed a given projective connection Δ*

$$\mathcal{L}_v \Delta^{(n)} s \;=\; \Delta^{(n)}(\mathcal{L}_v s),$$

for all $s \in \Gamma(\Omega^{-\frac{n-1}{2}})$.

n = 3:

In this case the projective connection is defined by

$$\Delta^{(3)} = \partial_x^3 + 4u\partial_x + 2u',$$

and $s \in \Gamma(\Omega^{-1})$.

After computing

$$\mathcal{L}_v \Delta^{(3)} = \Delta^{(3)} \mathcal{L}_v, \tag{8}$$

we obtain,

$$f((\phi''' + 4\phi' u + 2\phi u')' + 2f'(\phi''' + 4\phi' u + 2\phi u') = (\partial_x^3 + 4u\partial_x + 2u')(f\phi' - f'\phi). \tag{9}$$

This yields the equation for v

$$(f''' + 4f'u + 2fu')'\phi + (f''' + 4f'u + 2fu')\phi' = 0.$$

Therefore we say

Proposition 2.5 *A projective vector field* $v = f\frac{d}{dx} \in \Gamma(\Omega^{-1})$ *corresponding to* $\Delta^{(3)}$ *also satisfies*

$$(f''' + 4f'u + 2fu') \ = \ 0. \tag{10}$$

This only shows that the expression defines a linear skew-symmetric third order operator from $\Gamma(\Omega^{-1})$ to $\Gamma(\Omega^2)$

n = 4:

The 4-th order conformal operator is given by

$$\Delta^{(4)} = \partial_x^4 + 9u^2 + 3u'' + 10u'\partial_x + 10u\partial_x^2, \tag{11}$$

and

$$\Delta^{(4)} : \Gamma(\Omega^{-\frac{3}{2}}) \longrightarrow \Gamma(\Omega^{\frac{5}{2}}).$$

In this case,

$$\mathcal{L}_v s = (f\phi' - \frac{3}{2}f'\phi)dx^{-\frac{3}{2}}. \tag{12}$$

Thus the equation

$$\mathcal{L}_v(\Delta^{(4)} s) = \Delta^{(4)} \mathcal{L}_v s \tag{13}$$

yields the equation for v in this case

$$\begin{aligned}&(f''''' + 10uf''' + 10u'f'' + 8u''f' + 24u^2 f' + 12uu'f + 2fu''')\phi + \\ &5(f''' + 4f'u + 2fu')'\phi' + (f''' + 4f'u + 2fu')\phi'' = 0.\end{aligned}$$

Thus we obtain

Proposition 2.6 *A projective vector field $v = f\frac{d}{dx}$ corresponding to 4-th order conformal operator $\Delta^{(4)}$ satisfies*

$$f''''' + 10uf''' + 10u'f'' + 8u''f' + 24u^2f' + 12uu'f + 2fu''' = 0 \tag{14}$$

and

$$f''' + 4f'u + 2fu' = 0.$$

Although, apparantly it seems we get a new set equations for $\Delta^{(4)}$ operator, however the first equation can be expressed as

$$\begin{aligned}&(\partial_x^2 + 6u)(f''' + 4f'u + 2fu' = \\ &f''''' + 10uf''' + 10u'f'' + 8u''f' + 24u^2f' + 12uu'f + 2fu'''.\end{aligned}$$

Hence we obtain the same equation (7) for all the extended conformal operators $\Delta^{(n)}$ (for $n \leq 4$).

2.3 Generalised Bracket and Projective Vector Field

Let us switch back to earlier section.

Consider a one parameter family of $Vect(S^1)$ action on the space of smooth function $a(x) \in C^\infty$

$$\mathcal{L}_v^\lambda a(x) := f(x)a'(x) - \lambda f'(x)a(x), \tag{15}$$

where $\mathcal{L}_v^\lambda$ is the Lie derivative with respect to $v = f(x)\frac{d}{dx} \in Vect(S^1)$, given by

$$\mathcal{L}_v^\lambda := f(x)\frac{d}{dx} - \lambda f'(x). \tag{16}$$

Definition 2.7 *The action of $Vect(S^1)$ on the space of Hill's operator $\Delta \equiv \Delta^{(2)}$ is defined by the commutator with the Lie derivative*

$$[\mathcal{L}_v, \Delta] := \mathcal{L}_v^{-3/2} \circ \Delta - \Delta \circ \mathcal{L}^{1/2}. \tag{17}$$

This action is given in equation (7), and can be identified with the coadjoint action of Virasoro algebra on its dual. Here we discuss this briefly.

Similarly we can generalizes this action on $\Delta^{(n)}$

Definition 2.8 *The $Vect(S^1)$ action on Δ^n is defined by*

$$[\mathcal{L}_v, \Delta^{(n)}] := \mathcal{L}_v^{-(n+1)/2} \circ \Delta^{(n)} - \Delta^{(n)} \circ \mathcal{L}_v^{(n-1)/2}. \tag{18}$$

Hence the equation (18) can be considered as some coadjoint action of $Vect(S^1)$ on some "dual".

For example, $[\mathcal{L}_v, \Delta^{(3)}](s) = \mathcal{L}_v(\Delta^{(3)}s) - \Delta^{(3)}\mathcal{L}_v s$ and $[\mathcal{L}_v, \Delta^{(4)}](s) = \mathcal{L}_v(\Delta^{(4)}s) - \Delta^{(4)}\mathcal{L}_v s$ are identified with the equations (9) and (14) respectively.

2.4 Hill's operator and dual of Virasoro algebra

The algebra $Vect(S^1)$ has a unique non-trivial central extension

$$0 \longrightarrow \mathbf{R} \longrightarrow Vir \longrightarrow Vect(S^1)$$

described by the Gelfand-Fuks cocycle [14]

$$(\xi \frac{d}{dx}, \eta \frac{d}{dx}) \longmapsto \frac{1}{2} \int_{S^1} \xi' \eta'' dx.$$

The elements of the Virasoro algebra Vir can be identified with the pairs (2π periodic function , real number). The commutator in Vir takes the form

$$[(\beta(x), a), (\gamma(x), b)] = (\beta\gamma' - \gamma\beta', \int_{S^1} \beta'\gamma'').$$

The dual space Vir^* can be identified to the set

$$\{(\mu, u) |\ \mu \in \mathbf{R} \text{ and } u \text{ is a quadratic differential } \}.$$

A pairing between a point $(\lambda, \xi \frac{d}{dx}) \in Vir$ and a point $(\mu, u dx^{\otimes 2})$ is given by

$$\lambda\mu + \int_{S^1} \xi u dx.$$

Next theorem follows from the work of Lazutkin and Pankratova [17]:

Theorem 2.9 *The space Vir^* can be identified with the Vir-module of Hill's operators*

$$\{\mu \partial_x^2 + u\}$$

acting on distributions of weight $-\frac{1}{2}$ as $\widehat{D}iff_+(S^1)$ modules.

This is verifiable by the direct computaion of the action of Vir^* on its dual. We will sketch it in the next section.

It is known from the work of Kirillov [12,14] that the dual Lie algebra $\Omega\mathcal{G}^*$ is a $Diff_+(S^1)$ module.

Lemma 2.10

$$Ad^*(\lambda, \phi)(\mu, u) = (\mu, Ad^*(\phi)u + \lambda\Lambda(\phi))$$

where $\Lambda(\phi) = \mathcal{S}(\phi) \circ \phi^{-1}$, $\mathcal{S}(\phi)$ is called Schwarzian derivative of an analytic function ϕ and it is defined by

$$\frac{1}{2}(\frac{\phi'''}{\phi'} - \frac{3}{2}(\frac{\phi''}{\phi'})^2)$$

Hence the infinitesimal action is given by:

Lemma 2.11

$$ad^*_{(\lambda,\xi)}(\mu, u) \ = \ (\mu, \frac{1}{2}\mu\xi_{xxx} + 2\xi' u + 2\xi u') \equiv (\mu, \tilde{u}). \qquad (19)$$

Proof: It follows from the definition

$$
\begin{aligned}
< ad^*_{(\lambda,\xi)}(\mu,q),(\nu,\eta) > &= < (\mu,u), ad_{(\lambda,\xi)}(\nu,\eta) > \\
&= < (\mu,u),(\frac{1}{2}\int_{S^1}\xi'\eta''dx,[\xi,\eta]) > \\
&= \mu\frac{1}{2}\int_{S^1}\xi'\eta''dx + \int_{S^1}[\xi,\mu]udx.
\end{aligned}
$$

□

3 $Diff(S^1)$ Action on AGD Manifold

In this section we will describe the transformations on the AGD manifold under the action of $Diff(S^1)$. This transformation of scalar differential operators has been known since last century. The action of $Diff(S^1)$ induces a change of variable in the independent parameter x.

Suppose $\Delta^{(n)}$ be a scalar differential operator. The action of $Diff(S^1)$ transform the solutions of

$$\Delta^{(4)}\psi = 0.$$

as densities of degree $\frac{n-1}{2}$

Let us consider the action of $Diff(S^1)$:

$$Diff(S^1)\times AGD \longrightarrow AGD$$

$$(x,\Delta^{(4)}) \longmapsto (\sigma(x),\Delta^{(4)})$$

where $\sigma(x)\Delta^{(4)}$ is the unique scalar operator of the form $\Delta^{(4)}$ having u_{n-1}. If μ and ξ be the solutions of $\sigma(x)\Delta^{(4)}$ and $\Delta^{(4)}$, then their solutions are connected by

$$\mu = (x'^{\frac{n-1}{2}})\circ x.$$

In the case for $n=2$, this coincides with the action of Virasoro group on the space of Hill operators, dual space of Virasoro algebra.

Let us demonstrate for $n=2$ case.

Suppose ξ_1 and ξ_2 are two linearly independent solutions of Hill's equation

$$(\partial_x^2 + u)\xi = 0,$$

where

$$u = \mathcal{S}(\frac{\xi_1}{\xi_2}).$$

Let us perform a change of variables $x \longrightarrow \sigma(x)$. Then using the property of the Schwarzian

$$\mathcal{S}(f\circ g) = (\mathcal{S}(f)\circ g)(g_x)^2 + \mathcal{S}(g), \tag{20}$$

we obtain that $\xi \circ \sigma(x)$ satisfies

$$(\partial_x^2 + \tilde{u})\xi \;=\; 0,$$

where

$$\tilde{u}(x) = \mathcal{S}(\frac{\xi_1(\sigma(x))}{\xi_2(\sigma(x)}))$$
$$= u \circ \sigma(x)((\sigma(x))_x)^2 + \mathcal{S}(\sigma(x)).$$

The equation (20) means $\mathcal{S}$ is a one cocycle on $Diff(S^1)$ with values in the space of quadratic differentials. It is a projectively invariant.

It should be noted that the operators Δ^n do not preserve their form under the action of $Diff(S^1)$, $x \to \sigma(x)$, due to the appearance of the $(n-1)$-th term $-\frac{1}{2}n(n-1)(\sigma''/\sigma'^{n+1}$. Hence we should think the operators are acting on *densities of weight* $-1/2(n-1)$ rather than on scalar functions, in this case we can always find $u_{n-1} = 0$ as a reparametrization invariant. Therefore, the action of $Diff(S^1)$ on Δ^n is given by

$$\partial_x^n + u_{n-2}(x)\partial_x^{n-2} + \cdots + u_0(x) \longrightarrow \sigma'^{-(n+1)/2}(\partial_x^n + \tilde{u}_{n-2}\partial_x^{n-2} + \cdots + \tilde{u}_0)\sigma'^{-(n-1)/2}, \quad (21)$$

where

$$\tilde{u}_{n-2} = \sigma'^2 u_{n-2}(\sigma(x)) + \frac{1}{12}n(n-1)(n+1)\mathcal{S}'(x).$$

In particular for $\mathbf{n} = \mathbf{3}$ we find

$$\tilde{u}_1(x) = \sigma'^2 u_1(\sigma(x)) + 2\mathcal{S}(x)$$

$$\tilde{u}_0(x) = \sigma'^3 u_0(\sigma(x)) + \sigma'\sigma'' u_1(\sigma(x)) + \mathcal{S}'(x).$$

For $\mathbf{n} = \mathbf{4}$ we obtain:

$$\tilde{u}_2(x) = \sigma'^2 u_2(\sigma(x)) + 5\mathcal{S}(x)$$
$$\tilde{u}_1(x) = \sigma'^3 u_1(\sigma(x)) + 2\sigma'\sigma'' u_2(\sigma(x)) + 5\mathcal{S}'(x).$$
$$\tilde{u}_0(x) = \sigma'^4(\sigma(x)) + \frac{3}{2}\sigma'^2\sigma'' u_1(\sigma(x)) + \frac{3}{2}\sigma''^2 u_2(\sigma(x)) + \frac{3}{2}\sigma'^2 u_2(\sigma(x))\mathcal{S}(x) + \frac{3}{2}\mathcal{S}''(x) + \frac{3}{2}\mathcal{S}^2(x).$$

4 Generating equation and Euler-Arnold equation

In the preceding section we have already remarked that the action $[\mathcal{L}_v, \Delta] = \mathcal{L}_v \circ \Delta - \Delta \circ \mathcal{L}_v$ exactly coincides with the coadjoint action of Virasoro on its dual. So it is quite likely to get KdV equation from the following equation:

$$\frac{\partial \Delta}{\partial t} + [\mathcal{L}_v, \Delta] = 0. \quad (22)$$

This type of equation has been considered by M. Alber and S. Alber [2,3] of course, in a different settings.

They considered the spectral problem for an associated Schrdinger operator

$$L = -\frac{\partial^2}{\partial x^2} + V(x,t,E),$$

where E is a parameter and

$$V(x,t,E) = \sum_{j=-l}^{m} V_j(x,t)E^j.$$

In the case of KdV equation E appears as an eigenvalue and one ultimately equates the potential with a solution of the nonlinear equation itself.

We want to investigate the analogous equation for the extended projective connection on the circle, i.e.

$$\frac{\partial \Delta^{(n)}}{\partial t} + [\mathcal{L}_v, \Delta^{(n)}] = 0, \tag{23}$$

where $[\mathcal{L}_v, \Delta^{(n)}]$ is given in (!!). It should be noted that one single function u.

Proposition 4.1 *The equation*

$$\frac{\partial \Delta^{(n)}}{\partial t} + [\mathcal{L}_v, \Delta^{(n)}] = 0 \quad n < 5$$

generates only one single equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}f''' + 2f'u + fu'. \tag{24}$$

Proof: We prove it by direct computation.
□

For a special choice of $f = u$ we obtain KdV equation.

Geometrically this follows from the following argument:

We have argued that the action $[\mathcal{L}_v, \Delta^{(n)}]$, can be considered as a coadjoint action of $Vect(S^1)$ on $\Delta^{(n)}$. Using the equation (23) we know

$$\tilde{u} = \frac{1}{2}f''' + 2f'u + fu' = (\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x)f.$$

The operator $(\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x)$ is called symplectic operator. The Euler equation is the Hamiltonian flow on the coadjoint orbits in $\widehat{\Omega\mathcal{G}}^*$ generated by the Hamiltonian

$$H(u) = \frac{1}{2}u^2(x),$$

given by

$$\frac{d}{dt}u(t) = -ad^*{}_{\mathrm{u(t)}}\ u(t). \tag{25}$$

Thus by applying Euler-Arnold equation we prove the first theorem.

5 Stabilizer orbit and Neumann equation

The Neumann system deals with the motion of a particle on a sphere under the influence of a quadratic potential. This system is completely integrable and the solutions are given by hyperelliptic theta function. In fact the geometrical realization of this system is very interesting. Moser [11] observed that the integrals of the Hamiltonian system describing the motion of Neumann system have a very close similarity with the integrals of the Hamiltonian system describing the geodesics on a quadric. Later Knörrer [8] showed that the Neumann problem can be recasted through the Gauss map as the geodesic motion problem on a quadric. In fact the relationship between the C. Neumann system and the Jacobi system is clarified by making use of this Gauss map. The Jacobi system is another integrable system which describes the geodesic flow on an ellipsoid.

C. Neumann System: Here we give a very brief description of C. Neumann system. Let us consider a point moving on the $(n-1)$ dimensional sphere

$$\xi_1^2 + \cdots + \xi_n^2 = 1$$

under the influence of a quadratic potential $v(\xi) =< \xi, A\xi >$ where A is a symmetric $n \times n$ matrix. The differential equation describing this motion is called C. Neumann system, given by

$$\xi'' = -A\xi + u\xi, \quad \sum_{i=1}^{n} \xi_i^2 = 1 \quad u =< \xi, A\xi > - < \xi', \xi' > .$$

By Sylvesters theorem, we can assume that A is a diagonal matrix. Algebraic integrals for C. Neumann problem were first found by Karen Uhlenbeck in an unpublished note in 1975 [see for details, pp180, 21].

Now we will give the proof our theorem.

This follows from the properties of the Virasoro action for $n = 2$ case. Let us confine our attention to a specific hyperplane $\mu = 1$ in the coadjoint orbit. The action in this hyperplane will be

$$\tilde{u} = fu' + 2f'u + \frac{1}{2}f'''.$$

The stabilizer of the action of $(f(x)\frac{d}{dx}, 1)$ on the dual $(u(x)dx^2, 1)$, it means $(f\frac{d}{dx}, 1) \in Stab(u, 1)$ if and only if

$$f''' + 2u'f + 4uf' = 0.$$

Thus we say if $u \in \mathcal{O}_{stab}(\Delta)$, stabilizer space of Hill's operator, then u satisfies equation (!).

Now we will prove for $\mathbf{n} > \mathbf{2}$ case. We will consider now the stabilizer set of the action of $Vect(S^1)$ on the space of Adler-Gelfand-Dickey operators. It is clear from the equation (23) that this set is given by

$$[\mathcal{L}_v, \Delta^{(n)}] = \mathcal{L}_v^{-(n+1)/2} \circ \Delta^{(n)} - \Delta^{(n)} \circ \mathcal{L}_v^{(n-1)/2} = 0.$$

One could readily see that the equation satisfied for $n = 3$ is the derivative of $f''' + 2u'f + 4uf' = 0$. Similarly, for $n = 4$ one can express the equation (14)

$$(\partial_x{}^2 + 6u)(f''' + 2u'f + 4uf') = 0$$

Therefore we conclude

Lemma 5.1 *The stabilizer space of the action of $f\frac{d}{dx} \in vect(S^1)$ on the space differential operators of type (1) (for $n < 5$) is given by*

$$f''' + 2q'f + 4qf' = 0.$$

Now we seek to characterize the stabilizer space $\mathcal{O}_{stab}$.

We wish to connect the projective vector field and the Neumann system. Let us define a function

$$f \ := \ < \chi, A^{-1}\chi >,$$

where A is a constant diagonal matrix.

Proposition 5.2 *If $f =< \chi, A^{-1}\chi >$ and satisfies*

$$f''' + 2fu' + 4f'u = 0$$

then χ satisfies

$$\chi'' = -A\chi - u\chi$$

where $< \chi, \chi >= 1$ and $u = - < \chi, A\chi > + < \chi', \chi' >$ which is the system of Neumann equations.

Proof :: We know

$$f =< \chi, A^{-1}\chi >$$

hence

$$f' = 2 < \chi', A^{-1}\chi >$$

.

Then after using the condition of the Neumann equation, we obtain

$$f'' = -2 - 2uf + 2 < \chi', A^{-1}\chi' > .$$

Taking one more derivative we get

$$f''' + 4uf' + 2u'f = 0.$$

□

Thus we obtain the theorem (1.3b) part.

6 Super Algebra

The first characteristic special property of super algebra is that all the additive groups of its basic and derived structures are $\mathbb{Z}_2$. A vector superspace is a $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_0 + V_1$. An element v of V_0 (resp. V_1) is said to be even (resp. odd). The super commutator of a pair of elements $v, w \in V$ is defined to be the element

$$[v, w] = vw - (-1)^{\tilde{v}\tilde{w}} wv,$$

where $\tilde{v}$ and $\tilde{w}$ are the gradings of v and w respectively.

Lemma 6.1 *The equation* $f''' + 2u'f + 4uf' = 0$ *traces out a three dimesional spaces of solutions.*

Proof: If ϕ_1 and ϕ_2 are the solutions of

$$\Delta\phi \;=\; (\frac{d^2}{dx^2} + u)\phi \;=\; 0, \tag{26}$$

then it is easy to see that $\phi_i\phi_j \in \Gamma(\Omega^{-1})$ satisfies the above equation. Hence the solutions space is spanned by ϕ_1^2, ϕ_2^2 and $\phi_1\phi_2$.

□

Let us recall that the tangent bundle over S^1 has two square roots, one is trivial and other is a Möbius strip, which is denoted by $\Omega^{-\frac{1}{2}}$.

The sections of $\Gamma(\Omega^{-\frac{1}{2}})$ which satisfy the equation (26) are not functions but the square root of a *projective* vector field, since $\phi \in \Omega^{-1/2}$, the space of scalar densities of weight $-1/2$, square root of $f \in Vect(S^1)$.

Let us define

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$$

, where we denote $\mathcal{G}_0 \equiv Vect(S^1)$ and $\mathcal{G}_1 \equiv \Omega^{-1/2}(S^1)$. $\mathcal{G}$ forms a super Lie algebra on $S^{1,1}$ and $\mathcal{G}_1$ is the super-partner of $\mathcal{G}_0$. This is asserted since $\mathcal{G}_1$ is the $\mathcal{G}_0$ module and it is compatible with the structure of $\mathcal{G}_0$ module and satisfies $\mathcal{G}_1 \times \mathcal{G}_1 \longrightarrow \mathcal{G}_0$. A typical element of $\mathcal{G}$ would be

$$f(x)\frac{d}{dx} + \phi(x)\frac{d}{dx}^{1/2},$$

and the super Lie bracket is given by

$$[(f_1, \phi_1), (f_2, \phi_2)] = ([f_1, f_2] + \phi_1\phi_2, \{f_1, \phi_2\} + \{\phi_1, f_2\}).$$

In this realization $f(x)\frac{d}{dx} \oplus \phi(x)\frac{d}{dx}^{1/2}$ i.e. $(f(x), \phi(x))$ forms a super Lie algebra. $(f(x), \phi(x))$ satisfies

$$f(x + 2\pi) = f(x)$$

$$\phi(x + 2\pi) = \pm\phi(x)$$

When it is in the '+ ' sector, it is called Ramond sector super Lie algebra and '$-$' sector is known as Neveu-Schwarz sector.

The cocycle may be extended to this superalgebra via

$$c(\phi_1, \phi_2) = \int_{S^1} \phi_1' \phi_2' dx. \tag{27}$$

Thus we can propose:

Proposition 6.2 *If f is a projective vector field, satisfies (7), then the square root of f is an element of scalar density of weight -$\frac{1}{2}$, ϕ. Then (f, ϕ) satisfies super algebra.*

The stabilizer of the action of this superalgebra on pairs $\Delta^{(n)}$ is given by (26) and (27). This gives us a geometrical explanation that if ϕ_1, ϕ_2 satisfy (27) then $\phi_1\phi_2$ satisfies (26).

7 References

1. M. Adler, On a trace functional for formal pseudo-differential operators and the symplectic structure of the KdV. Inventiones Math. 50, 219-248, 1979.
2. M.S. Alber and S.J. Alber, Hamiltonian formalism for finite-zone solutions of integrable equations. C.R. Acad. Sci. Paris 301, 777-781, 1985.
3. M.S. Alber and S.J. Alber, Hamiltonian formalism for nonlinear Schrdinger equations and sine-Gordon equations. J. Lond. Math. Soc. 36, 176-192
4. V.I. Arnold, *Mathematical methods of classical mechanics*, second edition, Graduate Texts in Mathematics, vol. 60, Springer-Verlag, 1989.
5. I. Bakas, The Hamiltonian Structure of the Spin-4 Operator Algebra. Phys. Lett. 213B, 313-318, 1988.
6. G.M. Beffa, On the Poisson geometrry of the Adler-Gelfand-Dikii brackets. Jour. Diff. Geom. 1998.
7. E. Cartan, *Lecons sur la Theorie des Espaces a connection Projective,* Gauthier-Villars, Paris, 1937.
8. I.M. Gelfand and L.A. Dikii, A family of Hamiltonian structures connected with integrable nonlinear differentail equations, in I.M. Gelfand, collected papers, vol. 1, Springer-Verlag, 1987.
9. P. Guha, Diffeomorphism, Periodic KdV and C. Neumann System. IHES/M/98/47, to appear in Diff. Geom. and its Appl.
10. P. Guha Diffeomorphism with some Sobolev metric, geodesic flow and integrable systems. IHES/M/98/69.
11. N. Hitchin, Vector fields on the circle, in *Mechanics, Analysis and Geometry: 200 years after Lagrange.* Edited by M. Francaviglia, Elsevier Science Publishers B.V., 1991.

12. A. Kirillov, Infinite dimensional Lie groups; their orbits, invariants and representations. The Geometry of Moments, in *Twistor Geometry and Non-linear Systems*, edited by A. Dold and B. Eckmann Springer Lecture Notes in Mathematics 970, 1980.

13. A. Kirillov, Kähler structure on k-orbits of the group of diffeomorphisms of a circle. Func. Anal. Appl.21, 122-125, 1987.

14. A. Kirillov, The orbit method, I and II : Infinite-dimensional Lie groups and Lie algebras, Contemporary Mathematics, Volume 145, 1993.

15. H. Knörrer, Geodesic on quadrics and a mechanical problem of C.Neumann, Jour. R. Ang. math. 334, 1982, 69 -78.

16. N.H. Kuiper, Locally projective spaces of dimension one, Michigan Math. 2, 95-97, 1954.

17. V.F. Lazutkin and T.F. Pankratova, Normal forms and the versal deformations for Hill's equation, Funct. Anal. Appl. 9(4), 41-48, 1975.

18. P. Marcel, V. Ovsienko and C. Roger, Extension of the Virasoro algebras and generalized Sturm-Liouvile operatos. Lett. Math. Phys.

19. P. Mathieu, Extended Classical Conformal Algebras and the Second Hamiltonian Structure of Lax Equations. Phys. Lett. 208B, 101-106, 1988.

20. P. Michor and T. Ratiu, On the geometry of the Virasoro-Bott group, DG/9801115, to appear in J. Lie Theory.

21. J. Moser, Geometry of quadrics and spectral theory, in The Chern Symposium 1979, Berline-Heidelberg-New York 147-188, 1980.

22. V. Yu. Ovsienko and B. A. Khesin, KdV superequation as an Euler equation, Funct. Anal. Appl. 21, 329-331, 1987.e

23. G. Segal, Unitary representations of some infinite dimensional groups, Comm. Math. Phys. 80, 301-342, 1981.

24. G. Segal, The geometry of the KdV equation, Int. J. Mod. Phys A 16 , 2859-2869, 1991.

25. E. Witten, Coadjoint orbits of the Virasoro group, Comm. Math. Phys. 114, 1-53, 1988.