Projective structure and Integrable Geodesic Flows on the extension of Bott-Virasoro Group

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Dedicated to the memory of Professor Jürgen Moser

Abstract

This is a sequel to my paper IHES/M/00/23, triggered from a question posed by Marcel-Ovsienko-Roger in their paper (Lett. Math. Phys. 40 (1997) 31-39). In this paper we show that the multicomponent (or vector) Ito equation, modified dispersive water wave equation and modified dispersionless long wave equation are the geodesic flows with respect to an L^2 metric on the semidirect product space $\widehat{Diff^s(S^1) \bigodot C^\infty(S^1)^k}$, where $Diff^s(S^1)$ is the group of orientation preserving Sobolev H^s diffeomorphisms of the circle. We also study the projective structure associated with the matrix Strum-Liouville operators on the circle.

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1 Introduction

It is known that the periodic Korteweg-de Vries (KdV) and the Camassa-Holm equation [5] can be interpreted as geodesic flow of the right invariant metric on the Bott-Virasoro group, which at the identity is given by the L^2 and the Sobolev metric H^1 - inner product respectively [22, 23, 25].

With the advancement of the integrable systems, Physicists and mathematicians discovered many new multicomponent version of the existing integrable nonlinear p.d.e.s [1, 2, 19]. In the theory of integrable systems these multicomponent generalizations have been sporadically used. There are several 2 component generalization of KdV equation, viz., Hirota-Satsuma equation [14], Wilson equation [26] and Ito equation [12] are notables among them. All most all the integrable systems of these classes share a common property. These are all bihamiltonian systems, enjoys a compatible pair of Hamiltonian structure. These systems belong to an infinite dimensional hierarchy of bihamiltonian systems. The resulting Hamiltonian flows can be mapped into each other by the recursion operator, which is formally defined as the "quotient" of the two Hamiltonian structure. Several of the well known bihamiltonian systems fall into two component case are actually trihamiltonian. There are a few disparate examples of trihamiltonian systems. The classical dispersive water equation and the long waves equation [19] belong to this class of integrable systems.

Earlier [10], we have shown that the Ito equation, modified dispersive water wave equation and modified dispersionless long waves equation arise in a unified geometric setting, all of them are integrable systems which describe geodesic flows. Thus we unify the Ito equation, the dispersive water wave equation and the Long-wave equation through a common construction, all are integrable systems which describe geodesic flows with respect to L^2 on the extension of the Bott-Virasoro group.

Recently there has been a growing interest in the multicomponent integrable systems. These systems are far less known than compare to ordinary integrable systems. It was presented in [1, 2] that an N component system has a remarkable property of possessing $(N+1)$ local Hamiltonian structure. Antonowicz and Fordy showed that these systems were isospectral to an energy dependent Schrödinger operator and gave a systematic derivation of the Hamiltonian from knowledge of the generalized Lax representation.

Using the extension of the Bott-Virasoro group with an L^2 metric, we will show how the multicomponent KdV type equation arising from a geodesic flow.

Following Ebin-Marsden [8], we enlarge $Diff(S^1)$ to a Hilbert manifold $Diff^s(S^1)$, the diffeomorphism of Sobolev class H^s. This is a topological space. If $s > n/2$, it makes sense to talk about an H^s map from one manifold to another. Using local charts, one can check whether the derivation of order $\leq s$ are square integrable.

The Lie algebra of $Diff^s(S^1) \bigodot C^\infty(S^1)$ has a three dimensional extension (explained in the next section), viz.

$$Vect^s(S^1) \bigodot C^\infty(S^1) \oplus \mathbf{R}^3.$$

Then a typical element of this algebra would be

$$(f\frac{d}{dx}, u(x), \alpha) \quad \text{where } f\frac{d}{dx} \in Vect(S^1), \quad u(x) \in C^\infty(S^1) \quad \alpha \in \mathbf{R}^3.$$

The $Diff^s\widehat{(S^1) \bigodot} C^\infty(S^1)$ is the non-trivial extension of $Diff^s(S^1) \bigodot C^\infty(S^1)$.

In this paper we will extend our previous results [10]. We will study the geodesic flows on $Diff^s\widehat{(S^1) \bigodot} C^\infty(S^1)^k$. The Lie algebra of $Diff^s\widehat{(S^1) \bigodot} C^\infty(S^1)^k$ also has a three dimensional extension

$$Vect(S^1) \bigodot C^\infty(S^1)^k \oplus \mathbf{R}^3.$$

Motivation of the paper : My motivation is to understand how one can incorporate the multidimensional integrable systems as a geodesic flow on some extended Bott-Virasoro groups. Thus to unify several multicomponent integrable systems through a common geometrical construction.

Organization of the paper:
In section 2, we will study the Lie algebra $Vect(S^1) \bigodot C^\infty(S^1)^k$ of the extended Bott-Virasoro group $Diff^s\widehat{(S^1) \bigodot} C^\infty(S^1)^k$. In section 3, we describe the projective connections on the circle, in particular, we introduce the matrix projective connection related to $Vect(S^1) \bigodot C^\infty(S^1)^k$. In section 4, we study the geodesic flows on the $Diff^s\widehat{(S^1) \bigodot} C^\infty(S^1)^k$ related to L^2 inner product, and this gives rise to several multicomponent integrable systems corresponding to the different hyperplanes in the dual space $C^\infty(S^1) \oplus \underbrace{C^\infty(S^1) \oplus \cdots C^\infty(S^1)}_{k} \oplus \mathbf{R}^3$. In section 5, we study the geodesic flow of the right invariant inner metric on the $Diff^s\widehat{(S^1) \bigodot} C^\infty(S^1)^{k+1}$, which at the identity is given by the L^2 inner product.

The result of the paper:

Theorem 1 *Let $t \mapsto \hat{c}$ be a curve in the $Diff^s\widehat{(S^1) \bigodot} C^\infty(S^1)^k$. Let $\hat{c} = (e, \mathbf{e}, 0)$ be the initial point, directing to the vector $\hat{c}(0) = (u(x)\frac{d}{dx}, \vec{v}(x), \gamma_0)$, where $\gamma_0 \in \mathbf{R}^3$ and $\vec{v}(x) = (v_1(x), \cdots, v_k(x))^t$. Then $\hat{c}(t)$ is a geodesic of the L^2 metric*

- *(A) associted to a hyperplane* $\gamma_0^1 = -1, \gamma_0^2 = \gamma_0^3 = 0$ *if and only if* $(u(x,t)\frac{d}{dx}, \vec{v}(x,t), \gamma)$ *satisfies the multicomponent Ito type system.*
- *(B) associated to a hyperplane* $\gamma_0^2 = 1, \gamma_0^1 = \gamma_0^3 = 0$ *if and only if* $(u(x,t)\frac{d}{dx}, \vec{v}(x,t), \gamma)$ *satisfies modified multicomponent dispersive water wave equation.*
- *(C) associated to a hyperplane* $\gamma_0^1 = -1, \gamma_0^2 = 1, \gamma_0^3 = 0$ *if and only if* $(u(x,t)\frac{d}{dx}, \vec{v}(x,t), \gamma)$ *satisfies some generalized multicomponent integrable system (stated in section 4).*
- *(D) associated to a hyperplane* $\gamma_0^1 = \gamma_0^2 = \gamma_0^3 = 0$ *if and only if* $(u(x,t)\frac{d}{dx}, \vec{v}(x,t), \gamma)$ *satisfies modified multicomponent dispersionless long wave equations.*

Next, we derive the Euler-Arnold flows on the dual space of $Vect(S^1) \bigodot C^{\infty k+1}$. A typical element of this algebra is $\begin{pmatrix} f(x)\frac{d}{dx} \\ a(x) \\ \vec{p} \end{pmatrix}$, where $f(x)\frac{d}{dx} \in Vect(S^1)$, $a(x) \in C^\infty(S^1)$ and $\vec{p} \in C^\infty(S^1)^k$.

Theorem 2 *The geodesic flow of the right invariant inner product on the* $Diff(S^1) \bigodot C^{\infty k+1}$*, which at the identity is given by the* L^2 *metric, yields the multicomponent Ito equation.*

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2 Structure of $Vect(S^1) \bigodot C^\infty(S^1)^k$

Let $Diff^s(S^1)$ be the group of orientation preserving Sobolev H^s diffeomorphisms of the circle. It is known that the group $Diff^s(S^1)$ as well as its Lie algebra of vector fields on S^1, $T_{id}Diff^s(S^1) = Vect^s(S^1)$, have non-trivial one-dimensional central extensions, the Bott-Virasoro group $\widehat{D}iff^s(S^1)$ and the Virasoro algebra Vir respectively [17, 18, 25].

The Lie algebra $Vect^s(S^1)$ is the algebra of smooth vector fields on S^1. This satisfies the commutation relations

$$[f\frac{d}{dx}, g\frac{d}{dx}] := (f(x)g'(x) - f'(x)g(x))\frac{d}{dx}. \qquad (1)$$

One parameter family of $Vect^s(S^1)$ acts on the space of smooth functions $C^\infty(S^1)^k$ by

$$L^{(\mu)}_{f(x)\frac{d}{dx}} \vec{a}(x) = f(x)\vec{a}'(x) - \mu f'(x)\vec{a}(x), \tag{2}$$

where

$$L^{(\mu)}_{f(x)\frac{d}{dx}} = f(x)\frac{d}{dx} - \mu f'(x)$$

is the Lie derivative with respect to the vector field $f(x)\frac{d}{dx}$ and

$$\vec{a} = (a_1, \cdots, a_k)^t \in C^\infty(S^1)^k.$$

The Lie algebra of $Diff^s(S^1) \bigodot C^\infty(S^1)^k$ is the semidirect product Lie algebra

$$\mathcal{G} = Vect^s(S^1) \bigodot C^\infty(S^1)^k.$$

An element of $\mathcal{G}$ is a pair $(f(x)\frac{d}{dx}, \vec{a}(x))$, where $f(x)\frac{d}{dx} \in Vect^s(S^1)$.

It is known that this algebra has a three dimensional central extension given by the non-trivial cocycles

$$\omega_1((f\frac{d}{dx}, \vec{a}), (g\frac{d}{dx}, \vec{b})) = \int_{S^1} f'(x)g''(x)dx \tag{3}$$

$$\omega_2((f\frac{d}{dx}, \vec{a}), (g\frac{d}{dx}, \vec{b})) = \int_{S^1} f''(x)\vec{b}(x) - g''\vec{a}(x))dx \tag{4}$$

$$\omega_3((f\frac{d}{dx}, \vec{a}), (g\frac{d}{dx}, \vec{b})) = 2\int_{S^1} \vec{a}^t(x)\vec{b}'(x)dx. \tag{5}$$

The first cocycle ω_1 is the well known Gelfand-Fuchs cocycle. The Virasoro algebra is the unique non-trivial central extension of $Vect(S^1)$ via this ω_1 cocycle. Hence we define the Virasoro algebra

$$Vir = Vect^s(S^1) \oplus \mathbf{R}.$$

The space $C^\infty(S^1) \oplus \mathbf{R}$ is identified with a part of the dual space to the Virasoro algebra. It is called the *regular part*, and the pairing between this space and the Virasoro algebra is given by:

$$< (u(x), a), (f(x)\frac{d}{dx}, \alpha) >= \int_{S^1} u(x)f(x)dx + a\alpha.$$

Similarly, we consider an extension of $\mathcal{G}$. This extended algebra is given by

$$\hat{\mathcal{G}} = Vect^s(S^1) \bigodot C^\infty(S^1)^k \oplus \mathbf{R}^3. \tag{6}$$

The Lie algebra $\hat{\mathcal{G}}$, for $k = 1$, has been considered in various places [3, 13, 21]. It was shown in [24] that the cocycles define the universal central extension the Lie algebra $Vect^s(S^1) \bigodot C^\infty(S^1)$. This we generalizes to

$$H^2(Vect(S^1) \bigodot C^\infty(S^1)^k) = \mathbf{R}^3.$$

Definition 1 *The commutation relation in $\hat{\mathcal{G}}$ is given by*

$$[\begin{pmatrix} f(x)\frac{d}{dx} \\ \vec{a} \\ \alpha \end{pmatrix}, \begin{pmatrix} g(x)\frac{d}{dx} \\ \vec{b} \\ \beta \end{pmatrix}] = \begin{pmatrix} (fg' - f'g)\frac{d}{dx} \\ f\vec{b} - g\vec{a} \\ \omega \end{pmatrix} \tag{7}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbf{R}^3$, $\omega = (\omega_1, \omega_2, \omega_3)$ are the two cocycles.

The dual space of smooth functions $C^\infty(S^1) \oplus \underbrace{C^\infty(S^1) \oplus \cdots C^\infty(S^1)}_{k}$ is the space of distributions (generalized functions) on S^1. Of particular interest are the orbits in $\hat{\mathcal{G}}^*_{reg}$. In the case of current group, Gelfand, Vershik and Graev have constructed some of the corresponding representations.

Definition 2 *The regular part of the dual space $\hat{\mathcal{G}}^*$ to the Lie algebra $\hat{\mathcal{G}}$ as follows: Consider*

$$\hat{\mathcal{G}}^*_{reg} = C^\infty(S^1) \oplus \underbrace{C^\infty(S^1) \oplus \cdots C^\infty(S^1)}_{k} \oplus \mathbf{R}^3.$$

and fix the pairing between this space and $\hat{\mathcal{G}}$, $< \cdot, \cdot >: \hat{\mathcal{G}}^*_{reg} \otimes \hat{\mathcal{G}} \to \mathbf{R}$:

$$< \hat{u}, \hat{f} >= \int_{S^1} f(x)u(x)dx + \int_{S^1} \vec{v}^t(x)\vec{a}(x)dx + \alpha \cdot \gamma, \tag{8}$$

where $\hat{u} = (u(x), \vec{v}^t Œ, \gamma)$, $\hat{f} = (f\frac{d}{dx}, \vec{a}, \alpha)$.

Extend (8) to a right invariant metric on the semi-direct product space $\widehat{Diff^s(S^1) \bigodot C^\infty(S^1)^k}$ by setting

$$< \hat{u}, \hat{f} >_{\hat{\xi}} = < d_{\hat{\xi}} R_{\hat{\xi}^{-1}} \hat{u}, d_{\hat{\xi}} R_{\hat{\xi}^{-1}} \hat{f} >_{L^2} \tag{9}$$

for any $\hat{\xi} \in \hat{\mathcal{G}}$ and $\hat{u}, \hat{f} \in T_{\hat{\xi}}\widehat{\mathcal{G}}$, where

$$R_{\hat{\xi}} \; : \; \hat{\mathcal{G}} \longrightarrow \hat{\mathcal{G}}$$

is the right translation by $\hat{\xi}$.

3 Projective Connections on the Circle

In this section, we describe the matrix projective connection associated to $Vect^s(S^1) \bigodot C^\infty(S^1)^k$ algebra. Let us start with some definitions of projective connections [11, 12, 15].

Let us denote $\Omega^{\pm 1/2}$ be the square root of the tangent and cotangent bundle of S^1 respectively.

Let Δ is a second order differential operator.

Definition 3 *A projective connection on the circle is a linear second order differential operator*

$$\Delta : \Gamma(\Omega^{-\frac{1}{2}}) \longrightarrow \Gamma(\Omega^{\frac{3}{2}})$$

such that the symbol of
(1) Δ *is the identity and*

$$(2) \quad \int_{S^1} (\Delta s_1) s_2 \;=\; \int_{S^1} s_1 (\Delta s_2)$$

for all $s_i \in \Gamma(\Omega^{-\frac{1}{2}})$.

Let us take $s = \psi(x) dx^{-\frac{1}{2}} \in \Gamma(\Omega^{-\frac{1}{2}})$, then $\Delta s \in \Gamma(\Omega^{3/2})$ is locally described by

$$\Delta s \;=\; (a\psi'' + b\psi' + c\psi) dx^{\frac{3}{2}}.$$

As discussed in [6], any differential equation of the form

$$\frac{d^2 y}{dx^2} = p_3 (\frac{dy}{dx})^3 + p_2 (\frac{dy}{dx})^2 + p_1 (\frac{dy}{dx}) + p_0$$

defines a projective structure.

From the definition of the projective connection condition (1) implies $a = 1$ and condition (2) implies $b = 0$, hence projective connection can be identified with the Hill operator

$$\Delta^{(2)} \equiv \Delta \;=\; \frac{d^2}{dx^2} + u(x).$$

Let L_v be the Lie derivative with respect to the vector field $v = f(x) \frac{d}{dx}$ Using the equation $L_v \Delta^{(2)} = \Delta^{(2)} L_v$, we obtain

Proposition 1 *A projective vector field* $v = f \frac{d}{dx} \in \Gamma(\Omega^{-1})$ *satisfies*

$$f''' + 4f'u + 2fu' \;=\; 0.$$

If u_is are periodic function on the line, the operator

$$\frac{d^n}{dx^n} + u_{n-1}\frac{d^{n-1}}{dx^{n-1}} + u_{n-2}\frac{d^{n-2}}{dx^{n-2}} + u_1\frac{d}{dx} + u_0,$$

acting on periodic functions, is called a Adler-Gelfand-Dikii (or AGD) operator. Let us consider a projective connection as defined by this n-th order operator on the circle [11, 12].

Definition 4 (Extended Projective Connection) *An extended projective connection on the circle is a class of differential (conformal) operators*

$$\Delta^{(n)} : \Gamma(\Omega^{-\frac{n-1}{2}}) \longrightarrow \Gamma(\Omega^{\frac{n+1}{2}})$$

such that
(1) the symbol of $\Delta^{(n)}$ is the identity,

$$(2) \quad \int_{S^1} (\Delta^{(n)} s_1) s_2 \;=\; \int_{S^1} s_1 (\Delta^{(n)} s_2)$$

for all $s_i \in \Gamma(\Omega^{-\frac{n-1}{2}})$.

It is known that the symbol of a n-th order operator from a vector bundle U to V is a section of Hom $(U, W \otimes Sym^n T)$, where

$$U = \Omega^{-\frac{(n-1)}{2}} \; V = \Omega^{\frac{n+1}{2}}.$$

Since $T = \Omega^{-1}$, hence we get

$$V \otimes Sym^n T \cong U,$$

giving an invariant meaning to the first condition.
If $s_2 \in \Gamma(\Omega^{-\frac{n-1}{2}})$ then $s_1 \Delta^{(n)} s_2 \in \Gamma(\Omega)$ is a one form to integrate.

The consequence of the *first condition* is that all the differential operators are monic, that is the coefficient of the highest derivative is always one, and the *second condition* says that the term $u_{n-1} = 0$.

3.1 Matrix Projective Connection

Let us consider first the projective connections yield by the $Vect(S^1) \bigodot C^\infty(S^1)$. Consider the following matrix linear differential operators on $C^\infty(S^1) \oplus C^\infty(S^1)$:

$$\mathbf{\Delta} = \begin{pmatrix} \frac{d^2}{dx^2} + u(x) & \frac{d}{dx} + v(x) \\ \frac{d}{dx} + v(x) & c \end{pmatrix}$$

where $c \in \mathbf{R}$, $u(x) = u(x + 2\pi)$ and $v(x) = v(x + 2\pi)$.
It defines a matrix projective connection.

Definition 5 (Matrix Projective Connection) *An extended projective connection on the circle is a class of differential (conformal) operators*

$$\mathbf{\Delta} : \Gamma(\Omega^{-\frac{1}{2}}) \oplus \Gamma(\Omega^{\frac{1}{2}} \longrightarrow \Gamma(\Omega^{\frac{3}{2}}) \oplus \Gamma(\Omega^{\frac{1}{2}})$$

such that
(1) the symbol of $\mathbf{\Delta}$ *is the identity,*

$$(2) \int_{S^1} \vec{s}^t \mathbf{\Delta} \vec{s},$$

where $\vec{s}^t = (s_1\ s_2)$*, for* $s_1 \in \Gamma(\Omega^{-\frac{1}{2}})$ *and* $s_2 \in \Gamma(\Omega^{\frac{1}{2}})$*, is a one form which can be integrated over the circle.*

The second condition says that no choice of measure is necessary to make it invariant. Hence, the definition of a projective connection requires no structure on the circle other than a differentiable one. Thus the space of matrix differential operator on a circle considered as a module over the group of diffeomorphisms. In otherwords, the group of diffeomorphisms acts naturally on the projective structures.

Above construction can be easily generalized to the higher dimensional matrix linear differential operators acting on

$$C^\infty(S^1) \oplus \underbrace{C^\infty(S^1) \oplus \cdots \oplus C^\infty(S^1)}_{k}$$

$$\mathbf{\Delta_{k+1}} = \begin{pmatrix} \frac{d^2}{dx^2} + u(x) & c^t \frac{d}{dx} + \vec{v}^t(x) \\ c\frac{d}{dx} + \vec{v}(x) & c_1 \end{pmatrix}$$

where $c \in \mathbf{R}$, $u(x) = u(x + 2\pi)$ and $\vec{v}(x) = (v_1, \cdots, v_k)^t = \vec{v}(x + 2\pi)$.

3.2 $Vect(S^1) \bigodot C^\infty(S^1)^k$-module structure and Projective vector field

Let us define a 1-parameter family of $Vect(S^1) \bigodot C^\infty(S^1)^k$ modules on the space $C^\infty(S^1) \oplus \underbrace{C^\infty(S^1) \oplus \cdots \oplus C^\infty(S^1)}_{k}$:

$$T^\lambda_{(f(x)\frac{d}{dx}, \vec{a}(x))} \begin{pmatrix} m \\ \vec{n} \end{pmatrix} = \begin{pmatrix} L^\lambda_{f(x)\frac{d}{dx}} m \\ L^{\lambda-1}_{f(x)\frac{d}{dx}} \vec{n} - \lambda \vec{a}'(x) m(x) \end{pmatrix} \tag{10}$$

where $m(x) \in C^\infty(S^1)$ and $\vec{n}(x) \in C^\infty(S^1)^k$.

Definition 6 *The* $Vect(S^1) \bigodot C^\infty(S^1)^k$ *action on the space of operators* $\mathbf{\Delta_{k+1}}$ *is given by*

$$[T_{(f(x)\frac{d}{dx}, \vec{a}(x))}, \mathbf{\Delta_{k+1}}] = T^{-\frac{1}{2}}_{(f(x)\frac{d}{dx}, \vec{a}(x))} \circ \mathbf{\Delta_{k+1}} - \mathbf{\Delta_{k+1}} \circ T^{\frac{1}{2}}_{(f(x)\frac{d}{dx}, \vec{a}(x))}. \tag{11}$$

Claim 1

$$[T^{\lambda}_{f(x)\frac{d}{dx},\vec{a}(x)}, T^{\lambda}_{f(x)\frac{d}{dx},\vec{b}(x)}] = T^{\lambda}_{((fg'-f'g)\frac{d}{dx}, f\vec{b}-g\vec{a})}. \tag{12}$$

Sketch of the Proof: It is easy to compute from (10) that

$$T^{\lambda}_{g(x)\frac{d}{dx},\vec{b}(x)} T^{\lambda}_{f(x)\frac{d}{dx},\vec{a}(x)} \begin{pmatrix} m \\ \vec{n} \end{pmatrix} = \begin{pmatrix} L^{\lambda}_{g(x)\frac{d}{dx}} L^{\lambda}_{f(x)\frac{d}{dx}} m \\ L^{\lambda-1}_{g(x)\frac{d}{dx}} L^{\lambda-1}_{f(x)\frac{d}{dx}} \vec{n} - L^{\lambda-1}_{g(x)\frac{d}{dx}} (\lambda \vec{a}'(x) m(x)) - \lambda \vec{b}'(x) L^{\lambda}_{f(x)\frac{d}{dx}} m(x) \end{pmatrix} \tag{13}$$

$$T^{\lambda}_{f(x)\frac{d}{dx},\vec{a}(x)} T^{\lambda}_{g(x)\frac{d}{dx},\vec{b}(x)} \begin{pmatrix} m \\ \vec{n} \end{pmatrix} = \begin{pmatrix} L^{\lambda}_{f(x)\frac{d}{dx}} L^{\lambda}_{g(x)\frac{d}{dx}} m \\ L^{\lambda-1}_{f(x)\frac{d}{dx}} L^{\lambda-1}_{g(x)\frac{d}{dx}} \vec{n} - L^{\lambda}_{f(x)\frac{d}{dx}} (\lambda \vec{a}'(x) m(x)) - \lambda \vec{a}'(x) L^{\lambda}_{g(x)\frac{d}{dx}} m(x) \end{pmatrix} \tag{14}$$

Our result follows from the equations (13) and (14).
□

A projective vector field in this case is a vector field $v = f(x)\frac{d}{dx}$ which leaves the projective connection, i.e.

$$T_{(f(x)\frac{d}{dx},\vec{a}(x))} \mathbf{\Delta_{k+1}} \vec{s} = \mathbf{\Delta_{k+1}} T_{(f(x)\frac{d}{dx},\vec{a}(x))} \vec{s}$$

for all $\vec{s} = (s_1, s_2)^t \in \Gamma(\Omega^{-\frac{1}{2}}) \oplus \Gamma(\Omega^{\frac{1}{2}})$.

Proposition 2

$$[T_{f(x)\frac{d}{dx},\vec{a}(x)}, \mathbf{\Delta_{k+1}}] = \begin{pmatrix} fu' + 2f'u + f''' + \vec{v}^t \vec{a}' + \vec{a}'' & f\vec{v}' + f'\vec{v} + f'' + c_1 \vec{a}' \\ f\vec{v}' + f'\vec{v} - f'' + c_1 \vec{a}' & 0 \end{pmatrix} = \mathbf{0}. \tag{15}$$

Proof: By direct calculation.
□

4 Integrable Geodesic Flows on $\widehat{Diff(S^1) \bigodot C^\infty(S^1)^k}$

Given any three elements

$$\hat{f} = (f\frac{d}{dx}, \vec{a}, \alpha), \quad \hat{g} = (g\frac{d}{dx}, \vec{b}, \beta) \ \hat{u} = (u\frac{d}{dx}, \vec{v}, \gamma)$$

in $\hat{\mathcal{G}}$.

Lemma 1

$$ad^*_{\hat{f}}\hat{u} =$$
$$\begin{pmatrix} (2f'(x)u(x) + f(x)u'(x) + \vec{a}'v(x) - c_1 f''' + c_2\vec{a}'' \\ f'\vec{v}(x) + f(x)\vec{v}'(x) - c_2 f''(x) + 2c_3\vec{a}'(x) \\ 0 \end{pmatrix}$$

Proof: This follows from

$$< ad^*_{\hat{f}}\hat{u}, \hat{g} >_{L^2} = < \hat{u}, [\hat{f}, \hat{g}] >_{L^2}$$

$$= < (u(x)\frac{d}{dx}, \vec{v}(x), c), [(fg' - f'g)\frac{d}{dx}, f\vec{b}' - g\vec{a}', \omega) >_{L^2}$$

$$= -\int_{S^1} (fg' - f'g)u(x)dx - \int_{S^1} (f\vec{b}' - g\vec{a}')vdx - c_1 \int_{S^1} f'(x)g''(x)dx -$$

$$c_2 \int_{S^1} (f''(x)\vec{b}(x) - g''(x)\vec{a}(x))dx - 2c_3 \int_{S^1} \vec{a}^t(x)\vec{b}'(x)dx.$$

Since f, g, u are periodic functions, hence integrating by parts we obtain our result.

□

Case-1: The multicomponent Ito equation:

The coadjoint action leaves the parameter space invariant. Let us consider a hyperplane $c_1 = -1, c_2 = c_3 = 0$.

Lemma 2

$$ad^*_{\hat{f}}\hat{u} =$$
$$\begin{pmatrix} (2f'(x)u(x) + f(x)u'(x) + \vec{a}'(x)\vec{v}(x) + f''' \\ f'\vec{v}(x) + f(x)\vec{v}'(x) \\ 0 \end{pmatrix}$$

The Euler-Arnold equation [4] is the Hamiltonian flow on the coadjoint orbit in $\hat{\mathcal{G}}^*$ [3], generated by the Hamiltonian

$$H(\hat{u}) \equiv H(u,v) =< (u(x), \vec{v}(x)), (u(x), \vec{v}(x)) >, \tag{16}$$

given by

$$\frac{d\hat{u}}{dt} = -ad^*_{\hat{u}(t)}u(t). \tag{17}$$

Let V be a vector space and assume that the Lie group G acts on the left by linear maps on V, thus G acts on the left on its dual space V^* [for details, see for example, 7].

Proposition 3 *Let $G \bigodot V$ be a semidirect product space (possibly infinite dimensional), equipped with a metric $< \cdot, \cdot >$ which is right translation. A curve $t \to c(t)$ in $G \bigodot V$ is a geodesic of this metric if and only if $u(t) = d_{c(t)}R_{c(t)^{-1}}\dot{c}(t)$ satisfies the Euler-Arnold equation.*

Thus we obtain the multicomponent Ito equation

$$u_t + u_{xxx} + 6uu_x + 2\vec{v}^t\vec{v}_x = 0$$

$$\vec{v}_t + 2(u\vec{v})_x = 0.$$

The original Ito system [16] admits a tri-Hamiltonian structure, hence, this multicomponent Ito would exhibit the same property.

Case-II: Modified multicomponent dispersive water wave equation

When we restrict to a hyperplane $c_1 = 0, c_2 = 1, c_3 = 0$, we obtain

Lemma 3

$$ad^*_f\hat{u} = \begin{pmatrix} (2f'(x)u(x) + f(x)u'(x) + \vec{a}'(x)\vec{v}(x) + \vec{a}'' \\ f'\vec{v}(x) + f(x)\vec{v}'(x) - f'' \\ 0 \end{pmatrix}.$$

Thus, by applying the Euler-Arnold equation we obtain

$$u_t + 6uu_x + 2\vec{v}^t\vec{v}_x + \vec{v}_{xx} = 0$$

$$\vec{v}_t + 2(\vec{v}u)_x - u_{xx} = 0.$$

Case-III: The multicomponent new integrable system

Lemma 4

$$ad^*_{\hat{f}}\hat{u} =$$
$$\begin{pmatrix} (2f'(x)u(x) + f(x)u'(x) + a'v(x) + a'' + f''' \\ f'v(x) + f(x)v'(x) - f'' \\ 0 \end{pmatrix}.$$

Thus, by applying the Euler-Arnold equation we obtain another pair of integrable Hamiltonian system

$$u_t + 6uu_x + 2\vec{v}^t\vec{v}_x + \vec{v}_{xx} + u_{xxx} = 0$$

$$\vec{v}_t + 2(\vec{v}u)_x - u_{xx} = 0.$$

Case-IV: Modified multicomponent dispersionless long wave equation

In this case, we just set $c_1 = c_2 = c_3 = 0$, i.e. non-centrally extended part.

Lemma 5

$$ad^*_{\hat{f}}\hat{u} =$$
$$\begin{pmatrix} (2f'(x)u(x) + f(x)u'(x) + \vec{a}'(x)\vec{v}(x) \\ f'\vec{v}(x) + f(x)\vec{v}'(x) \\ 0 \end{pmatrix}.$$

Thus, we obtain

$$u_t + 6uu_x + 2\vec{v}^t\vec{v}_x = 0$$
$$\vec{v}_t + 2(\vec{v}u)_x = 0.$$

5 Second Category of multiple component integrable systems

In this section we consider the Euler-Arnold equation on the dual space of $Vect(S^1) \bigodot C^\infty(S^1)^{k+1}$. We split the space $C^\infty(S^1)^{k+1}$ into $C^\infty(S^1) \times C^\infty(S^1)^k$. A typical element of $Vect(S^1) \bigodot C^\infty(S^1)^{k+1}$ is $\begin{pmatrix} f(x)\frac{d}{dx} \\ a(x) \\ \vec{p} \end{pmatrix}$. This type of construction has been discussed by Kupershmidt [20].

We redefine the cocycles in the following form:

$$\omega_1((f\frac{d}{dx}, a, \vec{p}), (g\frac{d}{dx}, b, \vec{q})) = \int_{S^1} f'(x)g''(x)dx \qquad (18)$$

$$\omega_2((f\frac{d}{dx},a,\vec{p}),(g\frac{d}{dx},b,\vec{q})) = \int_{S^1} f''(x)b(x) - g''a(x))dx \qquad (19)$$

$$\omega_3((f\frac{d}{dx},a,\vec{p}),(g\frac{d}{dx},b,\vec{q})) = 2\int_{S^1} a(x)b'(x)dx. \qquad (20)$$

This means

$$H^2(Vect(S^1) \bigodot C^\infty(S^1)^{k+1}) = \mathbf{R}^3.$$

Definition 7 *The commutation relation in $\hat{\mathcal{G}}$ is given by*

$$[\begin{pmatrix} f(x)\frac{d}{dx} \\ a(x) \\ \vec{p} \\ \alpha \end{pmatrix}, \begin{pmatrix} g(x)\frac{d}{dx} \\ b(x) \\ \vec{q} \\ \beta \end{pmatrix}] := \begin{pmatrix} (fg' - f'g)\frac{d}{dx} \\ fb' - ga' \\ f\vec{q}' - g\vec{p}' \\ \omega \end{pmatrix}. \qquad (21)$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbf{R}^3$, $\omega = (\omega_1, \omega_2, \omega_3)$ are the two cocycles.

Definition 8 *The regular part of the dual space $\hat{\mathcal{G}}^*$ to the Lie algebra $\hat{\mathcal{G}}$ as follows: Consider*

$$\hat{\mathcal{G}}^*_{reg} = C^\infty(S^1) \oplus \underbrace{C^\infty(S^1) \oplus \cdots C^\infty(S^1)}_{k} \oplus \mathbf{R}^3.$$

and fix the pairing between this space and $\hat{\mathcal{G}}$, $< \cdot, \cdot >: \hat{\mathcal{G}}^*_{reg} \otimes \hat{\mathcal{G}} \to \mathbf{R}$:

$$< \hat{u}, \hat{f} >= \int_{S^1} f(x)u(x)dx + \int_{S^1} \vec{a}(x)\vec{v}(x)dx + \alpha \cdot \gamma, \qquad (22)$$

where $\hat{u} = (u(x), v, \vec{w}, \gamma)$, $\hat{f} = (f\frac{d}{dx}, a, \vec{p}, \alpha)$.

Again from the coadjoint action, we obtain the following set of integrable Hamiltonian system

$$u_t + 6uu_x + 2vv_x + v_{xx} + u_{xxx} = 0$$

$$v_t + 2(vu)_x - u_{xx} = 0$$

$$\vec{w}_t + 2(\vec{w}u)_x = 0.$$

This is an *avatar* of the Ito equation. Similarly, we can derive the other sets of equations also.

This equation admits a bi-Hamiltonian structure

$$\mathbf{D}_2\delta H_n = \mathbf{D}_1\delta H_{n+1},$$

where

$$\mathbf{D}_2 = \begin{pmatrix} D^3 + 4uD + 2u_x & 2vD & 0 \\ 2v_x + 2vD & D^2 & 0 \\ 2\vec{w}_x + 2\vec{w}D & 0 & 0 \end{pmatrix}$$

$$\mathbf{D}_1 = \begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix}$$

with the Hamiltonian functionals

$$H_1[u, v, \vec{w}] = \frac{1}{2}\int (u^2 + v^2 + \mathbf{w}^2)dx \tag{23}$$

$$H_2[u, v, \vec{w}] = \frac{1}{2}\int (u^3 + uv_x - \frac{1}{2}u_x^2\vec{w} + uv^2 + u\mathbf{w}^2)dx. \tag{24}$$

The recursion operator arising from a Hamiltonian pair

$$\mathbf{R} = \mathbf{D}_2{\mathbf{D}_1}^{-1} = \begin{pmatrix} D^2 + 4u + 2u_xD^{-1} & 2v & 0 \\ 2v_xD^{-1} + 2v & D & 0 \\ 2\vec{w}_xD^{-1} + 2\vec{w} & 0 & 0 \end{pmatrix}$$

is a hereditory operator which yields infinitely many conserved quantities.

6 Conclusion and Outlook

In this paper we have continued to study a question posed by Marcel-Ovsienko-Roger that what are the integrable systems associated to the coadjoint orbit of the extended Bott-Virasoro group. In particular, in our earlier paper, we have identified a large class of two component integrable systems are associated to $\widehat{Diff(S^1) \bigodot C^\infty}(S^1)$. These are mostly trihamiltonian systems.

In this paper we further extend the earlier group to $Diff(S^1)\widehat{\bigodot C^\infty}(S^1)^k$. We have shown that a large class of multicomponent integrable systems can be derived as a geodesic flow on this space. Some of the integrable systems have been already discussed by Mareck Antonowicz and Allan Fordy and Kupershmidt. Thus we study the multicomponent integrable systems from this geometric point of view.

In this paper we have left out two more generalizations of this construction. The first one is the supersymmetric generalization to obtain the super multicomponent integrable systems, and the second one is the H^1 counterpart of the multicomponent integrable systems considered here.

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