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SINGULAR MEASURES OF PIECEWISE SMOOTH CIRCLE HOMEOMORPHISMS WITH TWO BREAK POINTS

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Abstract. Let $T_f : S^1 \to S^1$ be a circle homeomorphism with two break points a_b, c_b, that means the derivative Df of its lift $f : \mathbb{R} \to \mathbb{R}$ has discontinuities at the points $\tilde{a}_b, \tilde{c}_b$ which are the representative points of a_b, c_b in the interval $[0, 1)$, and irrational rotation number ρ_f. Suppose that Df is absolutely continuous on every connected interval of the set $[0,1]\backslash\{\tilde{a}_b, \tilde{c}_b\}$, that $DlogDf \in L^1([0,1])$ and the product of the jump ratios of Df at the break points is nontrivial, i.e. $\frac{Df_-(\tilde{a}_b)}{Df_+(\tilde{a}_b)}\frac{Df_-(\tilde{c}_b)}{Df_+(\tilde{c}_b)} \neq 1$. We prove that the unique T_f-invariant probability measure μ_f is then singular with respect to Lebesgue measure on S^1.

1. **Introduction.** This paper continues, and in some sense completes our study of circle maps with break points in [5]. Let T_f be an orientation preserving homeomorphism of the circle $S^1 \equiv \mathbb{R}/\mathbb{Z}$ with lift $f : \mathbb{R} \to \mathbb{R}$, f continuous, strictly increasing and $f(\hat{x} + 1) = f(\hat{x}) + 1$, $\hat{x} \in \mathbb{R}$. The circle homeomorphism T_f is then defined by $T_f x = f(\tilde{x})\ (mod\,1)$, $x \in S^1$, $x \equiv \tilde{x} + \mathbb{Z}$ with $\tilde{x} \in [0, 1)$. If T_f is a circle diffeomorphism with irrational rotation number $\rho = \rho(f)$ and $logDf$ is of bounded variation, then T_f is conjugate to the pure rotation T_ρ, that is, there exists an essentially unique homeomorphism T_φ of the circle with $T_f = T_\varphi^{-1} \circ T_\rho \circ T_\varphi$. This classical result of Denjoy [3] can be extended to circle homeomorphisms with break points. The exact statement of the corresponding theorem will be given later.

It is well known, that circle homeomorphisms T_f with irrational rotation number ρ admit a unique T_f- invariant probability measure μ_f. Since the conjugating map T_φ and the invariant measure μ_f are related by $T_\varphi x = \mu_f([0, x])$ (see [2]), regularity properties of the conjugating map T_φ imply corresponding properties of the density of the absolutely continuous invariant measure μ_f. This problem of smoothness

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of the conjugacy of smooth diffeomorphisms is by now very well understood (see for instance [1, 14, 6, 8, 9, 18]). An important class of circle homeomorphisms are homeomorphisms with break points or shortly, class P-homeomorphisms. In general their ergodic properties like the invariant measures, their renormalizations and also their rigidity properties are rather different from the properties of diffeomorphisms (see [13] chapter I and IV, [6] chapter VI, [10]).

The class of P-homeomorphisms consists of orientation preserving circle homeomorphisms T_f whose lifts f are differentiable away from countable many points, the so called break points of T_f , at which left and right derivatives, denoted respectively by Df_- and Df_+, exist, and such that

i) there exist constants $0 < c_1 < c_2 < \infty$ with $c_1 < Df(\tilde{x}) < c_2$ for all $\tilde{x} \in [0,1)\backslash BP(f)$, $c_1 < Df_-(\tilde{x}_b) < c_2$ and $c_1 < Df_+(\tilde{x}_b) < c_2$ for all $\tilde{x}_b \in BP(f)$, the set of break points of f in $[0,1)$;
ii) $logDf$ has bounded variation.

The ratio $\sigma_f(x_b) := \frac{Df_-(\tilde{x}_b)}{Df_+(\tilde{x}_b)}$ is called the **jump ratio** of T_f in x_b. Here and later $\tilde{x}$ denotes the representative point of $x \in S^1$ in the unit interval $[0,1)$. The total variation of $logDf$ we denote by v i.e. $v = Var(logDf)$.

Piecewise linear (PL) orientation preserving circle homeomorphisms with two break points are the simplest examples of class P-homeomorphisms. They show up in many other areas of mathematics as for instance in group theory, homotopy theory and logic via the Thompson group and its generalizations (see [16]). The invariant measures of PL homeomorphisms were first studied by Herman in [6]:

Theorem 1.1. *(Herman) A PL circle homeomorphism with two break points and irrational rotation number ρ has an invariant measure absolutely continuous with respect to Lebesgue measure if and only if its break points lie on the same orbit.*

Invariant measures of general class P-homeomorphisms with one break point have been studied by Dzhalilov and Khanin in [4]. Their properties are quite different from the ones for $C^{2+\varepsilon}$ diffeomorphisms. The main result in [4] is the following:

Theorem 1.2. *Let T_f be a class P-homeomorphism with one break point c_b. If the rotation number ρ_f is irrational and $T_f \in C^{2+\varepsilon}(S^1\backslash\{c_b\})$ for some $\varepsilon > 0$, then the T_f-invariant probability measure μ_f is singular with respect to Lebesgue measure l on S^1, i.e. there exists a measurable subset $A \subset S^1$ such that $\mu_f(A) = 1$ and $l(A) = 0$.*

I. Liousse got in [12] the same result for "generic" PL circle homeomorphisms with several break points and with irrational rotation number of bounded type. In a next step Dzhalilov and I. Liousse studied in [5] a class of circle homeomorphisms with two break points. Their result is the following:

Theorem 1.3. *Let T_f be a class P-homeomorphism satisfying the following conditions:*

i) T_f has irrational rotation number ρ_f of bounded type;
ii) T_f has two break points a_b, c_b not on the same orbit of T_f;
iii) there exist constants $k_i > 0$ such that $|Df(\tilde{x}) - Df(\tilde{y})| \leq k_i|\tilde{x} - \tilde{y}|$ on every continuity interval of Df.

Then the T_f- invariant probability measure μ_f is singular with respect to Lebesgue measure.

In the present paper we study circle homeomorphisms T_f with two break points whose rotation number ρ_f is not necessary of bounded type. The main purpose of the paper is to prove the following:

Theorem 1.4. *Let T_f be a class P-homeomorphism satisfying the following conditions:*

(a) the rotation number $\rho = \rho_f$ of T_f is irrational;
(b) T_f has two break points a_b, c_b such that $\sigma_f(a_b) \cdot \sigma_f(c_b) \neq 1$;
(c) Df is absolutely continuous on every connected interval of $[0,1]\backslash\{\tilde{a}_b, \tilde{c}_b\}$ and $D^2 f \in L^1([0,1], dl)$.

Then the T_f- invariant probability measure μ_f is singular with respect to Lebesgue measure.

Remark 1. Obviously condition (c) is weaker than a Lipschiz condition on Df. If T_f has two break points on the same orbit then our Theorem 1.4 sharpens Theorem 1.2 since T_f is not necessarily in $C^{2+\varepsilon}(S^1\backslash\{c_b\})$.

The smoothness condition (c) on T_f in Theorem1.4 we call the Katznelson-Ornstein (KO) condition.

Consider then a circle homeomorphism T_f with two break points a_b, c_b which satisfies the (KO) condition and whose jump ratios fulfill $\sigma_f(a_b) \cdot \sigma_f(c_b) = 1$. If the break points of T_f lie on the same orbit and the rotation number ρ_f is irrational of bounded type, then the T_f-invariant measure is absolutely continuous with respect to Lebesgue measure (see [5]).

The second main result of this paper concerns circle homeomorphisms with two break points not lying on the same orbit.

Theorem 1.5. *Let T_f be a class P-homeomorphism satisfying condition (c) of Theorem 1.4 and the conditions*

(a') the rotation number ρ_f of T_f is irrational of bounded type;
(b') T_f has two break points a_b, c_b on disjoint orbits and $\sigma_f(a_b) \cdot \sigma_f(c_b) = 1$.

Then the T_f-invariant measure μ_f is singular with respect to Lebesgue measure.

The main analytic tool for proving Theorems 1.4 and 1.5 are cross-ratios. This technique has been introduced in [18] for discussing real analytic circle homeomorphism with critical points. Recently it has been used by Teplinsky and Khanin (see [11], [17]) for establishing a sharp version of Hermann's Theorem and by Navas in the context of group actions on the circle ([15]). One can expect that this technique will play an important role in the future for handling circle homeomorphisms with singularities. The properties of invariant measures of circle homeomorphisms with two break points satisfying the condition $\sigma_f(a_b) \cdot \sigma_f(c_b) = 1$ and with rotation number of unbounded type are not yet known.

2. **Preliminaries and Notations.** Let T_f be an orientation preserving homeomorphism of the circle with lift f and irrational rotation number $\rho = \rho_f$. We denote by $\{k_n, n \in \mathbb{N}\}$ the sequence of entries in the continued fraction expansion $\rho = [k_1, k_2, ..., k_n, ...] = 1/(k_1 + 1/(k_2 + ... + 1/(k_n + ...)))$. For $n \in \mathbb{N}$ denote by $p_n/q_n = [k_1, k_2, ..., k_n]$ the convergents of ρ. Their denominators q_n satisfy the well known recursion relation $q_{n+1} = k_{n+1}q_n + q_{n-1},\ n \geq 1,\ q_0 = 1,\ q_1 = k_1$.

For an arbitrary point $x_0 \in S^1$ define $\Delta_0^{(n)}(x_0)$ as the closed interval on S^1 with endpoints x_0 and $x_{q_n} = T_f^{q_n} x_0$, such that for n odd x_{q_n} is to the left of x_0 and for

n even it is to its right. Denote by $\Delta_i^{(n)}(x_0) := T_f^i \Delta_0^{(n)}(x_0), i \geq 1$, the iterates of the interval $\Delta_0^{(n)}(x_0)$ under T_f.

It is well known from the work of Denjoy, that the set $\xi_n(x_0)$ of intervals with mutually disjoint interiors defined as

$$\xi_n(x_0) = \Big\{\Delta_i^{(n-1)}(x_0),\ 0 \leq i < q_n;\ \Delta_j^{(n)}(x_0),\ 0 \leq j < q_{n-1}\Big\} \tag{1}$$

determines a partition of the circle for any n. The partition $\xi_n(x_0)$ is called the n-th **dynamical partition** of the point x_0 with **generators** $\Delta_0^{(n-1)}(x_0)$ and $\Delta_0^{(n)}(x_0)$. Obviously the partition $\xi_{n+1}(x_0)$ is a refinement of the partition $\xi_n(x_0)$: indeed the intervals of order n are members of $\xi_{n+1}(x_0)$ and each interval $\Delta_i^{(n-1)}(x_0) \in \xi_n(x_0)$ $0 \leq i < q_n$, is partitioned into $k_{n+1}+1$ intervals belonging to $\xi_{n+1}(x_0)$ such that

$$\Delta_i^{(n-1)}(x_0) = \Delta_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{k_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^{(n)}(x_0).$$

Recall the following definition introduced in [8]:

Definition 2.1. An interval $I = (\tau, t) \subset S^1$ is q_n-small and its endpoints τ, t are q_n-close if the intervals $T_f^i(I),\ 0 \leq i < q_n$ are disjoint.

It is known that the interval (τ, t) is q_n-small if, depending on the parity of n, either $t \prec \tau \preceq T_f^{q_{n-1}}(t) \prec t$ or $T_f^{q_{n-1}}(\tau) \preceq t \prec \tau \prec T_f^{q_{n-1}}\tau$ in the counter clockwise order on the circle S^1. Then we can show

Lemma 2.2. *Let T_f be a P-homeomorphism with a finite number of break points $z^{(i)}$, $i = 1, 2, ..., m$, and irrational rotation number ρ. Assume $x, y \in S^1$ are q_n-close and $z^{(i)} \notin \Big\{T_f^j x,\ T_f^j y, 0 \leq j < q_n, \Big\}, i = 1, 2, ..., m$. Then for any $0 \leq k < q_n$ the following inequality holds:*

$$e^{-v} \leq \frac{Df^k(\tilde{x})}{Df^k(\tilde{y})} \leq e^v. \tag{2}$$

where $\tilde{x}$, $\tilde{y}$ are the representative points of x, y in the interval $[0, 1)$.

Proof. Take any two q_n-close points $x, y \in S^1$ and $0 \leq k \leq\ q_n - 1$. Denote by I the open interval with endpoints x and y. Because the intervals $T_f^i(I),\ 0 \leq i < q_n$ are disjoint, we obtain

$$|logDf^k(\tilde{x}) - logDf^k(\tilde{y})| \leq \sum_{s=0}^{q_n-1} |logDf(f^s(\tilde{x})) - logDf(f^s(\tilde{y}))| \leq v,$$

from which inequality (2) follows immediately. □

The following Lemma, which can be proven easily using the assertion of Lemma 2.2, plays a key role for studying metrical properties of the homeomorphism T_f :

Lemma 2.3. *Let T_f be a P-homeomorphism with a finite number of break points $z^{(i)}$, $i = 1, 2, ..., m$, and irrational rotation number ρ. If $x_0 \in S^1$, $n \geq 1$ and $z^{(i)} \notin \Big\{T_f^j x_0, 0 \leq j < q_n\Big\}$, $i = 1, 2, ..., m$ then*

$$e^{-v} \leq \prod_{i=0}^{q_n-1} Df^i(\tilde{x}_0) \leq e^v. \tag{3}$$

Inequality (3) is called the **Denjoy inequality**. The proof of Lemma 2.3 is as for circle diffeomorphisms (see for instance [9]). Using Lemma 2.3 it can be shown that the intervals of the dynamical partition $\xi_n(x_0)$ in (1) have exponentially small length . Indeed one finds

Corollary 1. *Let $\Delta^{(n)}$ be an arbitrary element of the dynamical partition $\xi_n(x_0)$. Then*

$$l(\Delta^{(n)}) \leq const\ \lambda^n, \tag{4}$$

where $\lambda = (1 + e^{-v})^{-1/2} < 1$.

From Corollary 1 it follows that the trajectory of every point $x \in S^1$ is dense in S^1. This together with monotonicity of the homeomorphism T_f implies the following

Theorem 2.4. *Suppose that a homeomorphism T_f satisfies the conditions of Lemma 2.3.Then T_f is topologically conjugate to the linear rotation T_ρ.*

Remember that homeomorphisms T_f of the circle satisfying the conditions of Lemma 2.3 are ergodic with respect to the Lebesgue measure , i.e. every T_f-invariant set has full or vanishing measure (see [7]).
In the following discussion we have to compare different intervals. For this we use

Definition 2.5. Let $C > 1$. We call two intervals of S^1 **C-comparable** if the ratio of their lengths is in $[C^{-1}, C]$.

Lemma 2.3 then implies (see [8])

Corollary 2. *Suppose, that a homeomorphism T_f satisfies the conditions of Lemma 2.3. Then for any interval $I \subset S^1$ the intervals I and $T_f^{q_n} I$ are e^v-comparable. If the interval I is q_n-small then $l(T_f^i I) < const\, \lambda^n$ for all $i = 1, 2, ..., q_n - 1$.*

3. **The Cross-ratio Tools.**

Let us first recall two definitions:

Definition 3.1. The **cross-ratio** $Cr(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ of four points $\hat{z}_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, $\hat{z}_1 < \hat{z}_2 < \hat{z}_3 < \hat{z}_4$ is defined as

$$Cr(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4) = \frac{(\hat{z}_2 - \hat{z}_1)(\hat{z}_4 - \hat{z}_3)}{(\hat{z}_3 - \hat{z}_1)(\hat{z}_4 - \hat{z}_2)}.$$

Definition 3.2. The **cross-ratio distortion** $Dist(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4; f)$ of four points $\hat{z}_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, $\hat{z}_1 < \hat{z}_2 < \hat{z}_3 < \hat{z}_4$ with respect to a strictly increasing function f on $\mathbb{R}$ is defined as

$$Dist(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4; f) = \frac{Cr(f(\hat{z}_1), f(\hat{z}_2), f(\hat{z}_3), f(\hat{z}_4))}{Cr(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)}.$$

For $k \geq 3$ let be $z_i \in S^1, i = 1, \cdots, k$, with $z_1 \prec z_2 \prec ... \prec z_k \prec z_1$ in the anti-clockwise order on the circle and $\tilde{z}_i = z_i\ (mod\, 1)$, $i = 1, \cdots, k$. Define $\hat{z}_1 = \tilde{z}_1$ and

$$\hat{z}_i := \begin{cases} \tilde{z}_i, & \text{if } \tilde{z}_1 < \tilde{z}_i < 1 \\ 1 + \tilde{z}_i, & \text{if } 0 \leq \tilde{z}_i < \tilde{z}_1 \end{cases}$$

for $i = 2, 3, ...k$. It is obvious that $\hat{z}_1 < \hat{z}_2 < ... < \hat{z}_k$. The vector $(\hat{z}_1, \hat{z}_2, ..., \hat{z}_k) \in \mathbb{R}^k$ is called the **lifted vector** of $(z_1, z_2, ..., z_k) \in (S^1)^k$.

Consider a circle homeomorphism T_f with lift f. We define the cross- ratio distortion of (z_1, z_2, z_3, z_4) with respect to T_f by

$$Dist(z_1, z_2, z_3, z_4; T_f) := Dist(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4; f)$$

where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of (z_1, z_2, z_3, z_4).

It is well known that for $T_f \in C^{2+\varepsilon}([z_1, z_4])$, $\varepsilon > 0$, with $[z_1, z_4] \subset (0, 1)$

$$Dist(z_1, z_2, z_3, z_4; T_f) = 1 + O(|\hat{z}_4 - \hat{z}_1|^{1+\varepsilon}).$$

Next we will estimate $Dist(z_1, z_2, z_3, z_4; T_f)$ for circle homeomorphisms T_f satisfying the conditions of Theorem 1.4. Fix an arbitrary $\varepsilon > 0$. Since $D^2 f$ is a periodic function on $\mathbb{R}$ with period 1 and hence $D^2 f \in L^1([0, 1], dl)$, it can be written in the form

$$D^2 f(\hat{x}) = g_\varepsilon(\hat{x}) + \theta_\varepsilon(\hat{x}),\ \hat{x} \in \mathbb{R}, \tag{5}$$

with $g_\varepsilon(\hat{x})$ and $\theta_\varepsilon(\hat{x})$ periodic functions on $\mathbb{R}$ with period 1, and $g_\varepsilon(\hat{x})$ a continuous function on $[0, 1]$ and $\int_0^1 |\theta_\varepsilon| dl < \varepsilon$. Then we can prove the following

Theorem 3.3. *Suppose, the circle homeomorphism T_f with lift f satisfies the conditions of Theorem 1.4. Choose $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ such that the interval $[z_1, z_4]$ does not contain any break point of T_f. Then*

$$\begin{aligned} |Dist(z_1, z_2, z_3, z_4; T_f) - 1| \quad &\leq \quad C_1 |\hat{z}_4 - \hat{z}_1| \max_{\hat{x}, \hat{t} \in [\hat{z}_1, \hat{z}_4]} |g_\varepsilon(\hat{x}) - g_\varepsilon(\hat{t})| + \\ &+ \quad C_1 \int_{\hat{z}_1}^{\hat{z}_2} |\theta_\varepsilon(y)| dy + C_1 \Big(\int_{\hat{z}_1}^{\hat{z}_4} |D^2 f(y)| dy \Big)^2, \end{aligned}$$

where the constant $C_1 > 0$ depends only on the function f and $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of (z_1, z_2, z_3, z_4).

Proof. Take $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ and consider its lifted vector $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$. The following identities are easy to check:

$$f(\hat{z}_k) - f(\hat{z}_1) = Df(\hat{z}_1)(\hat{z}_k - \hat{z}_1) + \int_{\hat{z}_1}^{\hat{z}_k} D^2 f(y)(\hat{z}_k - y) dy,\ k = 2, 3;$$

$$f(\hat{z}_4) - f(\hat{z}_l) = Df(\hat{z}_4)(\hat{z}_4 - \hat{z}_l) - \int_{\hat{z}_l}^{\hat{z}_4} D^2 f(y)(y - \hat{z}_l) dy,\ l = 2, 3.$$

Using these identities we obtain:

$$\begin{aligned} Cr(f(\hat{z}_1), f(\hat{z}_2), f(\hat{z}_3), f(\hat{z}_4)) &= \frac{f(\hat{z}_2) - f(\hat{z}_1)}{f(\hat{z}_3) - f(\hat{z}_1)} \cdot \frac{f(\hat{z}_4) - f(\hat{z}_3)}{f(\hat{z}_4) - f(\hat{z}_2)} \\ = \quad Cr(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4) &\cdot \frac{1 + \frac{1}{Df(\hat{z}_1)(\hat{z}_2 - \hat{z}_1)} \int_{\hat{z}_1}^{\hat{z}_2} D^2 f(y)(\hat{z}_2 - y) dy}{1 + \frac{1}{Df(\hat{z}_1)(\hat{z}_3 - \hat{z}_1)} \int_{\hat{z}_1}^{\hat{z}_3} D^2 f(y)(\hat{z}_3 - y) dy} \end{aligned} \tag{6}$$

$$\cdot \quad \frac{1 - \frac{1}{Df(\hat{z}_4)(\hat{z}_4 - \hat{z}_3)} \int_{\hat{z}_3}^{\hat{z}_4} D^2 f(y)(y - \hat{z}_3) dy}{1 - \frac{1}{Df(\hat{z}_4)(\hat{z}_4 - \hat{z}_2)} \int_{\hat{z}_2}^{\hat{z}_4} D^2 f(y)(y - \hat{z}_2) dy}.$$

Setting

$$A(a,b) := \frac{1}{Df(a)(b-a)}\int_a^b D^2f(y)(b-y)dy,$$

and

$$B(a,b) := \frac{1}{Df(b)(b-a)}\int_a^b D^2f(y)(y-a)dy$$

the distortion $Dist(\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{z}_4;f)$ can be written as

$$\begin{aligned} Dist(\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{z}_4;f) &= \frac{1+A(\hat{z}_1,\hat{z}_2)}{1+A(\hat{z}_1,\hat{z}_3)}\cdot\frac{1-B(\hat{z}_3,\hat{z}_4)}{1-B(\hat{z}_2,\hat{z}_4)} = \\ &= (1+A(\hat{z}_1,\hat{z}_2))\cdot(1-A(\hat{z}_1,\hat{z}_3)+O(A(\hat{z}_1,\hat{z}_3)) \\ &\cdot\ (1-B(\hat{z}_3,\hat{z}_4))\cdot(1+B(\hat{z}_2,\hat{z}_4)+O(B(\hat{z}_2,\hat{z}_4))). \end{aligned}$$

Therefore

$$\begin{aligned} Dist(\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{z}_4;f) &= 1+A(\hat{z}_1,\hat{z}_2)-A(\hat{z}_1,\hat{z}_3)+B(\hat{z}_2,\hat{z}_4) \\ &- B(\hat{z}_3,\hat{z}_4)+O\left(\left(\int_{\hat{z}_1}^{\hat{z}_4}|D^2f(y)|dy\right)^2\right). \end{aligned} \tag{7}$$

Define next $M_1 = 0.5\Big(\inf_{\hat{x}\in(\hat{z}_1,\hat{z}_4)} Df(\hat{x})\Big)^{-1}$.

To continue the proof of Theorem 3.3 we need the following

Lemma 3.4. *Assume, that a circle homeomorphism T_f with lift f satisfies the conditions of Theorem 1.4 and the interval $[a,b]\subset\mathbb{R}$ does not contain the break points of f. Then the following identities hold:*

$$A(a,b) = \int_a^b \frac{D^2f(y)}{2Df(y)}dy + G_1(a,b),\quad B(a,b) = \int_a^b \frac{D^2f(y)}{2Df(y)}dy + G_2(a,b),$$

where

$$\begin{aligned} |G_i(a,b)| \le M_1(b-a)\max_{x,t\in[a,b]}|g_\varepsilon(x)-g_\varepsilon(t)| + \\ +M_1\int_a^b|\theta_\varepsilon(y)|dy + 2M_1^2\left(\int_a^b|D^2f(y)|dy\right)^2,\ i=1,2. \end{aligned} \tag{8}$$

Proof. We prove only the identity for $A(a,b)$, the one for $B(a,b)$ is quite similar. Set $G_1(a,b) := A(a,b) - \int_a^b \frac{D^2f(y)}{2Df(y)}dy$. It is clear, that

$$\begin{aligned} |G_1(a,b)| = &\ \Bigg|\left(A(a,b)-\int_a^b\frac{D^2f(y)}{2Df(a)}dy\right)+\left(\int_a^b\frac{D^2f(y)}{2Df(a)}dy-\int_a^b\frac{D^2f(y)}{2Df(y)}dy\right)\Bigg| \le \\ &\le \left|A(a,b)-\int_a^b\frac{D^2f(y)}{2Df(a)}dy\right| + \frac{1}{2}\left|\int_a^b\frac{D^2f(y)}{Df(y)Df(a)}dy\int_a^y D^2f(t)dt\right|. \end{aligned}$$

Using this and the bound $(Df(x))^{-1} \leq 2\,M_1$ we conclude

$$|G_1(a,b)| \leq \Big|A(a,b) - \int_a^b \frac{D^2f(y)}{2Df(a)}dy\Big| + 2M_1^2\Big(\int_a^b |D^2f(y)|dy\Big)^2. \tag{9}$$

To get finally the estimate (8) for $G_1(a,b)$, it is sufficient to estimate the difference $A(a,b) - \int_a^b \frac{D^2f(y)}{2Df(a)}dy$. Using the definition of $A(a,b)$ and the representation (5) we obtain:

$$\Big|A(a,b) - \int_a^b \frac{D^2f(y)}{2Df(a)}dy\Big| = \Big|\frac{1}{Df(a)}\int_a^b D^2f(y)\left(\frac{b-y}{b-a} - \frac{1}{2}\right)dy\Big| =$$

$$= \frac{1}{Df(a)}\Big|\int_a^b \big(g_\varepsilon(y) + \theta_\varepsilon(y)\big)\left(\frac{b-y}{b-a} - \frac{1}{2}\right)dy\Big|$$

$$\leq \frac{1}{Df(a)}\Big|g_\varepsilon(a)\int_a^b \left(\frac{b-y}{b-a} - \frac{1}{2}\right)dy\Big| + \frac{1}{Df(a)}\Big|\int_a^b |g_\varepsilon(y) - g_\varepsilon(a)|\Big|\frac{b-y}{b-a} - \frac{1}{2}\Big|dy\Big|$$

$$+ \frac{1}{Df(a)}\int_a^b |\theta_\varepsilon(y)|\Big|\frac{b-y}{b-a} - \frac{1}{2}\Big|dy$$

$$\leq M_1(b-a)\max_{x,t\in[a,b]}|g_\varepsilon(x) - g_\varepsilon(t)| + M_1\int_a^b |\theta_\varepsilon(y)|dy.$$

This together with estimate (9) proves estimate (8) for $G_1(a,b)$ in Lemma 3.4. □

We can now finish the proof of Theorem 3.3. Equation (7) and the formulas for $A(a,b)$ and $B(a,b)$ in Lemma 3.4 imply:

$$Dist(\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{z}_4;f) = 1 + \int_{\hat{z}_1}^{\hat{z}_2} \frac{D^2f(y)}{2Df(y)}dy + G_1(\hat{z}_1,\hat{z}_2) - \int_{\hat{z}_1}^{\hat{z}_3} \frac{D^2f(y)}{2Df(y)}dy - G_1(\hat{z}_1,\hat{z}_3)$$

$$+\int_{\hat{z}_2}^{\hat{z}_4} \frac{D^2f(y)}{2Df(y)}dy + G_2(\hat{z}_2,\hat{z}_4) - \int_{\hat{z}_3}^{\hat{z}_4} \frac{D^2f(y)}{2Df(y)}dy - G_2(\hat{z}_3,\hat{z}_4) + O\left(\left(\int_{\hat{z}_1}^{\hat{z}_4} |D^2f(y)|dy\right)^2\right)$$

$$= 1 + G_1(\hat{z}_1,\hat{z}_2) - G_1(\hat{z}_1,\hat{z}_3) + G_2(\hat{z}_2,\hat{z}_4) - G_2(\hat{z}_3,\hat{z}_4) + O\left(\left(\int_{\hat{z}_1}^{\hat{z}_4} |D^2f(y)|dy\right)^2\right).$$

Using next the bound (8) for the intervals $[\hat{z}_s,\hat{z}_{s+1}] \subset [\hat{z}_1,\hat{z}_4]$, $s = 1,2,3$, we obtain

$$|G_1(\hat{z}_s,\hat{z}_{s+1})| \leq M_1|\hat{z}_4 - \hat{z}_1|\max_{[\hat{z}_1,\hat{z}_4]}|g_\varepsilon(\hat{x}) - g_\varepsilon(\hat{t})| \quad + \quad M_1\int_{\hat{z}_1}^{\hat{z}_4} |\theta_\varepsilon(y)|dy$$

$$+ \quad 2M_1^2\Big(\int_{\hat{z}_1}^{\hat{z}_4} |D^2f(y)|dy\Big)^2.$$

The same bound holds for $|G_2(\hat{z}_s,\hat{z}_{s+1})|$. Theorem 3.3 then follows immediately from these bounds. □

Let us next discuss the case when the interval $[z_1, z_4]$ contains just one break point $x = x_b$. We will estimate the distortion of the cross ratio when the break point lies outside the middle interval $[z_2, z_3]$ i.e. $x_b \in [z_1, z_2] \cup [z_3, z_4]$.

For $z_i \in S^1$, $i = 1, 2, 3, 4$ with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ and $x_b \in [z_1, z_2]$ define the numbers α, β, γ, τ, ξ and z as follows

$$\alpha := \hat{z}_2 - \hat{z}_1,\ \beta := \hat{z}_3 - \hat{z}_2,\ \gamma := \hat{z}_4 - \hat{z}_3,\ \tau := \hat{z}_2 - \hat{x}_b,\ \xi := \frac{\beta}{\alpha},\ z := \frac{\tau}{\alpha}. \tag{10}$$

where $(\hat{z}_1, \hat{x}_b, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of $(z_1, x_b, z_2, z_3, z_4)$.

Lemma 3.5. *Assume, that the circle homeomorphism T_f with lift f satisfies the conditions of Theorem 1.4. Let $z_i \in S^1$, $i = 2, 3$, with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$. Suppose that $x_b \in [z_1, z_2]$, and the other break point of T_f is not contained in $[z_1, z_4]$. Then*

$$|Dist(z_1, z_2, z_3, z_4; T_f) - \tfrac{[\sigma(x_b)+(1-\sigma(x_b))z](1+\xi)}{\sigma(x_b)+(1-\sigma(x_b))z+\xi}| \le K_1 \int_{\hat{z}_1}^{\hat{z}_4} |D^2 f(y)| dy, \tag{11}$$

where the constant $K_1 > 0$ depends only on the function f.

Proof. . By assumption $x_b \in [z_1, z_2]$. Let $(\hat{z}_1, \hat{x}_b, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ be the lifted vector of $(z_1, x_b, z_2, z_3, z_4)$. Rewriting $Dist(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4; f)$ in the form

$$Dist(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4; f) = \frac{Cr(f(\hat{z}_1), f(\hat{z}_2), f(\hat{z}_3), f(\hat{z}_4))}{Cr(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)}$$
$$= \Big(\frac{f(\hat{z}_2) - f(\hat{z}_1)}{\hat{z}_2 - \hat{z}_1} : \frac{f(\hat{z}_3) - f(\hat{z}_1)}{\hat{z}_3 - \hat{z}_1}\Big)\Big(\frac{f(\hat{z}_4) - f(\hat{z}_3)}{\hat{z}_4 - \hat{z}_3} : \frac{f(\hat{z}_4) - f(\hat{z}_2)}{\hat{z}_4 - \hat{z}_2}\Big),$$

it is easy to check, that

$$\begin{aligned} & f(\hat{z}_2) - f(\hat{z}_1) = (f(\hat{x}_b) - f(\hat{z}_1)) + (f(\hat{z}_2) - f(\hat{x}_b)) \\ = \ & Df_-(\hat{x}_b)(\hat{x}_b - \hat{z}_1) - \int_{\hat{z}_1}^{\hat{x}_b} D^2 f(y)(y - \hat{z}_1) dy \\ + \ & Df_+(\hat{x}_b)(\hat{z}_2 - \hat{x}_b) + \int_{\hat{x}_b}^{\hat{z}_2} D^2 f(y)(\hat{z}_2 - y) dy \\ = \ & Df_+(\hat{x}_b)(\hat{z}_2 - \hat{z}_1)\left[\sigma(x_b) + (1 - \sigma(x_b))\frac{\tau}{\alpha})\right] \\ + \ & Df_+(\hat{x}_b)(\hat{z}_2 - \hat{z}_1)\frac{1}{Df_+(\hat{x}_b)}\int_{\hat{x}_b}^{\hat{z}_2} D^2 f(y)\frac{\hat{z}_2 - y}{\hat{z}_2 - \hat{z}_1} dy \\ - \ & Df_+(\hat{x}_b)(\hat{z}_2 - \hat{z}_1)\frac{1}{Df_+(\hat{x}_b)}\int_{\hat{z}_1}^{\hat{x}_b} D^2 f(y)\frac{y - \hat{z}_1}{\hat{z}_2 - \hat{z}_1} dy \end{aligned}$$

with $\sigma(x_b) = \frac{Df_-(\hat{x}_b)}{Df_+(\hat{x}_b)}$ the jump ratio of T_f at the point x_b. Hence we get

$$f(\hat{z}_2) - f(\hat{z}_1) = Df_+(\hat{x}_b)(\hat{z}_2 - \hat{z}_1)\left[\sigma(x_b) + (1 - \sigma(x_b))\frac{\tau}{\alpha} + r_1(\hat{x}_b, \hat{z}_1, \hat{z}_2)\right]; \tag{12}$$

in the same way we find

$$
\begin{aligned}
& f(\hat{z}_3) - f(\hat{z}_1) = (f(\hat{x}_b) - f(\hat{z}_1)) + (f(\hat{z}_3) - f(\hat{x}_b)) = \\
= \;& Df_-(\hat{x}_b)(\hat{x}_b - \hat{z}_1) - \int_{\hat{z}_1}^{\hat{x}_b} D^2 f(y)(y - \hat{z}_1) dy \\
+ \;& Df_+(\hat{x}_b)(\hat{z}_3 - \hat{x}_b) + \int_{\hat{x}_b}^{\hat{z}_3} D^2 f(y)(\hat{z}_3 - y) dy \\
= \;& Df_+(\hat{x}_b)(\hat{z}_3 - \hat{z}_1) \left[\frac{\hat{z}_3 - \hat{x}_b}{\hat{z}_3 - \hat{z}_1} + \sigma(x_b) \frac{\hat{x}_b - \hat{z}_1}{\hat{z}_3 - \hat{z}_1}) \right] + \\
+ \;& Df_+(\hat{x}_b)(\hat{z}_3 - \hat{z}_1) \frac{1}{Df_+(\hat{x}_b)} \int_{\hat{x}_b}^{\hat{z}_3} D^2 f(y) \frac{\hat{z}_3 - y}{\hat{z}_3 - \hat{z}_1} dy \\
- \;& Df_+(\hat{x}_b)(\hat{z}_3 - \hat{z}_1) \frac{1}{Df_+(\hat{x}_b)} \int_{\hat{z}_1}^{\hat{x}_b} D^2 f(y) \frac{y - \hat{z}_1}{\hat{z}_3 - \hat{z}_1} dy.
\end{aligned}
$$

respectively

$$
f(\hat{z}_3) - f(\hat{z}_1) = Df_+(\hat{x}_b)(\hat{z}_3 - \hat{z}_1) \left[\frac{\tau + \beta}{\alpha + \beta} + \sigma(x_b) \frac{\alpha - \tau}{\alpha + \beta} + r_2(\hat{x}_b, \hat{z}_1, \hat{z}_3) \right]. \tag{13}
$$

For the functions $r_1(\hat{x}_b, \hat{z}_1, \hat{z}_2)$ and $r_2(\hat{x}_b, \hat{z}_1, \hat{z}_3)$ the following estimates hold

$$
|r_1(\hat{x}_b, \hat{z}_1, \hat{z}_2)|, |r_2(\hat{x}_b, \hat{z}_1, \hat{z}_3)| \leq \frac{2}{Df_+(\hat{x}_b)} \int_{\hat{z}_1}^{\hat{z}_3} |D^2 f(y)| dy. \tag{14}
$$

This together with equations (12) and (13) shows

$$
\begin{aligned}
\Big| \frac{f(\hat{z}_2) - f(\hat{z}_1)}{\hat{z}_2 - \hat{z}_1} : \frac{f(\hat{z}_3) - f(\hat{z}_1)}{\hat{z}_3 - \hat{z}_1} - \frac{[\sigma(x_b) + (1 - \sigma(x_b))z](1 + \xi)}{\sigma(x_b) + \xi + (1 - \sigma(x_b))z} \Big| \\
\leq K_2 \int_{\hat{z}_1}^{\hat{z}_3} |D^2 f(y)| dy,
\end{aligned} \tag{15}
$$

with ξ and z as defined in (10) and where the constant $K_2 > 0$ is depending only on the function f.

Since the interval $[\hat{z}_2, \hat{z}_4]$ does not contain the break point $\hat{x}_b$, it can easily be shown that

$$
\Big| \frac{f(\hat{z}_4) - f(\hat{z}_3)}{\hat{z}_4 - \hat{z}_3} : \frac{f(\hat{z}_4) - f(\hat{z}_2)}{\hat{z}_4 - \hat{z}_2} - 1 \Big| \leq K_3 \int_{\hat{z}_2}^{\hat{z}_4} |D^2 f(y)| dy,
$$

where also the constant $K_3 > 0$ depends only on f. This inequality and the bounds (14) and (15) imply Lemma 3.5. □

Remark 2. If the break point $x = x_b$ belongs to the right interval $[z_3, z_4]$, then one can prove the analogous estimate:

$$|Dist(z_1, z_2, z_3, z_4; T_f) - \frac{[\sigma(x_b) + (1-\sigma(x_b))\vartheta](1+\eta)}{\sigma(x_b) + (1-\sigma(x_b))\vartheta + \eta}| \le K_4 \int_{\hat{z}_1}^{\hat{z}_4} |D^2 f(y)| dy,$$

where $\eta = \frac{\hat{z}_3 - \hat{z}_2}{\hat{z}_4 - \hat{z}_3}$, $\vartheta = \frac{\hat{x}_b - \hat{z}_3}{\hat{z}_4 - \hat{z}_3}$ and the constant $K_4 > 0$ depends only on the function f.

4. **The Proofs of Theorem 1.4 and Theorem 1.5.** For the proofs of Theorems 1.4 and 1.5 we need several Lemmas which we formulate next. Their proofs will be given later. Recall that the length of an interval $[a, b] \subset S^1$ is defined by

$$l([a, b]) := \begin{cases} \tilde{b} - \tilde{a}, & \text{if } \tilde{a} < \tilde{b} < 1, \\ 1 + \tilde{b} - \tilde{a}, & \text{if } 0 \le \tilde{b} < \tilde{a} < 1. \end{cases}$$

For $x_0 \in S^1$ with representative point $\tilde{x}_0$ in $(0, 1)$ define $d(x_0) := \min\{\tilde{x}_0, (1 - \tilde{x}_0)\}$.

Lemma 4.1. *Assume, that the lift φ of the conjugating homeomorphism T_φ has a positive derivative $D\varphi(\tilde{x}_0) = \omega$ at the point $\tilde{x}_0 \in (0, 1)$, and that the following conditions hold for $z_i \in S^1$, $i = 1, .., 4$, with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ and some constant $R_1 > 1$:*

(a) $R_1^{-1} l([z_2, z_3]) \le l([z_1, z_2]) \le R_1 l([z_2, z_3])$
$R_1^{-1} l([z_2, z_3]) \le l([z_3, z_4]) \le R_1 l([z_2, z_3])$;
(b) $\max_{1 \le i \le 4} l([x_0, z_i]) \le R_1 l([z_1, z_2])$.

Then for any $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) \in (0, d(x_0))$ such that

$$|Dist(z_1, z_2, z_3, z_4; T_\varphi) - 1| \le C_2 \varepsilon, \tag{16}$$

if $z_i \in U_\delta(x_0)$ a δ-neighbourhood of x_0 for all $i = 1, 2, 3, 4$, where the constant $C_2 > 0$ depends only on $R_1$and ω, but not on ε.
If furthermore the points $T_f^{q_n} z_i$, $i = 1, 2, 3, 4$, fulfill conditions (a) and (b) and $T_f^{q_n} z_i \in U_\delta(x_0)$ for $i = 1, 2, 3, 4$, then also

$$|Dist(z_1, z_2, z_3, z_4; T_f^{q_n}) - 1| < C_2 \epsilon. \tag{17}$$

The strategy for proving that the invariant measure of T_f is singular with respect to Lebesgue measure, is to construct a quadruple of points z_i, $i = 1, 2, 3, 4$, for which the above distortion control is violated.

Suppose now $D\varphi(\tilde{x}_0) = \omega > 0$ for some point $\tilde{x}_0 \in (0, 1)$. Consider the $n-$ th dynamical partition $\xi_n(x_0)$ for the corresponding point $x_0 \in S^1$ under the homeomorphism T_f. Assume w.l.o.g. n to be odd. Then $\Delta_0^{(n)}(x_0) = [T_f^{q_n} x_0, x_0]$ and $\Delta_0^{(n-1)}(x_0) = [x_0, T_f^{q_{n-1}} x_0]$ are its two generators. Denote by $\overline{a}_b$ and $\overline{c}_b$ the unique points in the interval $\left[T_f^{q_n} x_0, T_f^{q_{n-1}} x_0\right]$ such that $\overline{a}_b = T_f^{-l} a_b$ and $\overline{c}_b = T_f^{-p} c_b$ for some $0 \le l, p \le q_n - 1$.

We need to introduce the notion of a regular cover of the break points. For this consider $z_i \in S^1, i = 1, 2, 3, 4$ with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ and define for

$0 \le m \le q_n - 1$ the length ratios

$$\xi(m) := \frac{l([T_f^m z_2, T_f^m z_3])}{l([T_f^m z_1, T_f^m z_2])}, \quad z(m) := \frac{l([T_f^m \bar{c}_b, T_f^m z_2])}{l([T_f^m z_1, T_f^m z_2])},$$
$$\eta(m) := \frac{l([T_f^m z_2, T_f^m z_3])}{l([T_f^m z_3, T_f^m z_4])}, \quad \vartheta(m) := \frac{l([T_f^m z_3, T_f^m \bar{c}_b])}{l([T_f^m z_3, T_f^m z_4])} \tag{18}$$

The numbers $z(m)$ for $\bar{c}_b \in [z_1, z_2]$ (respectively $\vartheta(m)$ for $\bar{c}_b \in [z_3, z_4]$) are called **normalized coordinates** of the point $\bar{c}_b$. It is clear that the normalized coordinates $z(m)$ respectively $\vartheta(m)$ change from 0 to 1, when the break point c_b is moving from $T_f^p z_2$ to $T_f^p z_1$ respectively from $T_f^p z_3$ to $T_f^p z_4$.

Definition 4.2. The intervals $\left\{T_f^j[z_1, z_2], T_f^j[z_2, z_3], T_f^j[z_3, z_4] : 0 \le j \le q_n - 1\right\}$ cover the break points a_b, c_b regularly with constants $C \ge 1$ and $\zeta \in [0, 1]$, if

1) the intervals $\left\{T_f^j[z_1, z_4], 0 \le j \le q_n - 1\right\}$ cover each break point exactly once;
2) either $z_2 = T_f^{-l} a_b$ and $T_f^{-p} c_b \in [z_1, z_2]$ or $z_3 = T_f^{-l} a_b$ and $T_f^{-p} c_b \in [z_3, z_4]$ for some $0 \le l, p \le q_n - 1$;
3) $\xi(0) \ge C$ and $z(0) \in [0, \zeta]$ if $\bar{c}_b = T_f^{-p} c_b \in [z_1, z_2]$,
   $\eta(0) \ge C$ and $\vartheta(0) \in [0, \zeta]$ if $\bar{c}_b = T_f^{-p} c_b \in [z_3, z_4]$,

or if conditions 1),2) and 3) hold by interchanging the roles of a_b and c_b.

Define next for $x > 0$ and $0 \le t \le 1$ the functions $G(x)$ and $F(x,t)$ as

$$G(x) := \frac{\sigma(a_b)(1 + x)}{\sigma(a_b) + x}, \ F(x,t) := \frac{[\sigma(c_b) + (1 - \sigma(c_b))t](1 + x)}{\sigma(c_b) + (1 - \sigma(c_b))t + x}, \tag{19}$$

The next result shows via a Denjoy-like argument that the distortion outside the breakpoints is controlled by the jumps at these points:

Lemma 4.3. *Suppose that the homeomorphism T_f satisfies the conditions of Theorem 1.4. If the intervals $\left\{T_f^j[z_1, z_2], T_f^j[z_2, z_3], T_f^j[z_3, z_4] : 0 \le j \le q_n - 1\right\}$ cover the break points a_b and c_b, then*

I) $Dist(z_1, z_2, z_3, z_4; T_f^{q_n}) = [G(\xi(l)) + \chi_1][F(\xi(p), z(p)) + \chi_2] \times$
$\times \prod_{\substack{0 \le i < q_n \\ i \ne l,p}} Dist(T_f^i z_1, T_f^i z_2, T_f^i z_3, T_f^i z_4; T_f)$ *if* $z_2 = \bar{a}_b$, $\bar{c}_b \in [z_1, z_2]$;

II) $Dist(z_1, z_2, z_3, z_4; T_f^{q_n}) = [G(\eta(l)) + \chi_3][F(\eta(p), \vartheta(p)) + \chi_4] \times$
$\times \prod_{\substack{0 \le i < q_n \\ i \ne l,p}} Dist(T_f^i z_1, T_f^i z_2, T_f^i z_3, T_f^i z_4; T_f)$ *if* $z_3 = \bar{a}_b$, $\bar{c}_b \in [z_3, z_4]$,

where

$$|\chi_j| = |\chi_j(z_1, z_2, z_3, z_4)| \le K_1 \int_{f^r(\hat{z}_1)}^{f^r(\hat{z}_4)} |D^2 f(y)| dy, \ 1 \le j \le 4, \tag{20}$$

for either $r = l$ or $r = p$.

Lemma 2.2 implies the following inequalities

$$e^{-v}\xi(0) \le \xi(m) \le e^v \xi(0), \quad e^{-v} z(0) \le z(m) \le e^v z(0),$$
$$e^{-v}\eta(0) \le \eta(m) \le e^v \eta(0), \quad e^{-v}\vartheta(0) \le \vartheta(m) \le e^v \vartheta(0) \tag{21}$$

for all $1 \le m \le q_n - 1$ From this it follows, that the normalized coordinates $\xi(m)$, $\eta(m)$, $z(m)$ and $\vartheta(m)$ are uniformly (in x_0 and m) comparable with the normalized coordinates $\xi(0)$, $\eta(0)$, $z(0)$ and $\vartheta(0)$ respectively. Formulas (19) lead for z small and ξ large to the following

Lemma 4.4. *If a circle homeomorphism T_f satisfies the conditions of Theorem 1.4, then for any $x_0 \in S^1$ with $\tilde{x}_0 \in (0,1)$ and any $\delta \in (0, d(x_0))$ there exist constants $C_0 = C_0(f, \sigma(a_b), \sigma(c_b)) > 1$ and $\zeta_0 = \zeta_0(f, \sigma(a_b), \sigma(c_b)) \in (0,1)$, such that for all triple of intervals $[z_s, z_{s+1}] \subset U_\delta(x_0)$, $s = 1,2,3$, covering the break points a_b, c_b regularly with constants C_0 and ζ_0 the following inequalities hold:*

$$|G(\xi(l))F(\xi(p), z(p)) - 1| \geq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{4} \;\; if \;\; z_2 = \overline{a}_b, \; and \; \overline{c}_b \in [z_1, z_2]$$

respectively

$$|G(\eta(l))F(\eta(p), \vartheta(p)) - 1| \geq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{4} \;\; if \;\; z_3 = \overline{a}_b, \; and \; \overline{c}_b \in [z_3, z_4].$$

The next Lemma will show the existence of a quadruple of points z_1, z_2, z_3, z_4 for which the distortion control in (17) is indeed violated as we will see.

Lemma 4.5. *If the circle homeomorphism T_f with two break points a_b, c_b satisfies the conditions of Lemma 2.3, then for any $x_0 \in S^1$ with $\tilde{x}_0 \in (0,1)$ and any $\delta \in (0, d(x_0))$ there exists a number $N = N(\delta, x_0) > 1$, such that for all $n > N(\delta, x_0)$ there is a triple of intervals $[z_s, z_{s+1}] \subset U_\delta(x_0)$, $s = 1,2,3$ with the following properties:*

1) *the interval $[z_1, z_4]$ is q_n-small;*
2) *the intervals $[z_s, z_{s+1}]$ and $[T_f^{q_n} z_s, T_f^{q_n} z_{s+1}]$ $s = 1,2,3$ satisfy conditions a) and b) of Lemma 4.1 with some constant $R_1 > 1$ depending on C_0, ζ_0 and the total variation v of $logDf$;*
3) *the intervals $\left\{T_f^i[z_1, z_2],\ T_f^i[z_2, z_3]\ T_f^i[z_3, z_4],\ 0 \leq i \leq q_n - 1\right\}$ either cover both break points a_b, c_b regularly with constants C_0 and ζ_0, or cover only the break point a_b where $\overline{a}_b = T_f^{-l} a_b$ for some $0 \leq l < q_n$ coincides with either z_2 or z_3.*

Indeed with the points z_1, z_2, z_3, z_4 of Lemma 4.5 we can now formulate Lemma 4.6 which in some sense is the criterion for the singularity of the invariant measure of T_f.

Lemma 4.6. *Suppose, the circle homeomorphism T_f satisfies the conditions of Theorem 1.4 and the intervals $[z_s, z_{s+1}]$, $s = 1,2,3$ satisfy the conditions 1)-3) of Lemma 4.5. Then for sufficiently large n one finds*

$$|Dist(z_1, z_2, z_3, z_4; T_f^{q_n}) - 1| > const > 0,$$

where the constant depends only on the function f.

After these preparations we can now proceed to the proof of Theorem 1.4.

Proof of Theorem 1.4. Let T_f be a circle homeomorphism satisfying the conditions of Theorem 1.4. Since ρ_f is irrational, the T_f-invariant measure μ_f is nonatomic i.e. every one point subset of the circle has zero μ_f-measure. The conjugating map T_φ, related to μ_f by $T_\varphi x = \mu_f([0,x])$ for $x \in S^1$, is given by a continuous and monotone increasing function φ on $\mathbb{R}$. Hence T_φ has a finite derivative almost everywhere (w.r.t. Lebesgue measure) on the circle. We show that $D\varphi(\hat{x}) = 0$ at all points $\hat{x}$ at which the derivative is defined.

Assume, $D\varphi(\tilde{x}_0)=\omega>0$ for some point $\tilde{x}_0\in(0,1)$ corresponding to the point $x_0\in S^1$. Choose an $\varepsilon>0$ and points $z_i, i=1,2,3,4$ such that $[z_s,z_{s+1}]$ and $[T_f^{q_n}z_s,T_f^{q_n}z_{s+1}]\subset U_\delta(x_0),\ s=1,2,3,$ satisfy the conditions of Lemma 4.1. It then follows from this Lemma that

$$|Dist(z_1,z_2,z_3,z_4;T_f^{q_n})-1|\leq C_2\varepsilon, \tag{22}$$

where the constant $C_2>0$ does not depend on ε and n. But this contradicts Lemma 4.6 according to which

$$|Dist(z_1,z_2,z_3,z_4;T_f^{q_n}-1|>const>0$$

for sufficiently large n. This contradiction proves Theorem 1.4. □

5. The proofs of Lemmas 4.1-4.6.

We start with the proof of Lemma 4.1.

Proof of Lemma 4.1. Suppose, that the derivative $D\varphi(\tilde{x}_0)$ exists and $D\varphi(\tilde{x}_0)=\omega>0$ for some $\tilde{x}_0\in(0,1)$ corresponding to the point x_0 in S^1. By the definition of the derivative there exists for any $\varepsilon>0$ a number $\delta=\delta(x_0,\varepsilon)\in(0,d(x_0))$ such that, for all $\hat{x}\in(\tilde{x}_0-\delta,\ \tilde{x}_0+\delta)$,

$$\omega-\varepsilon<\frac{\varphi(\hat{x})-\varphi(\tilde{x}_0)}{\hat{x}-\tilde{x}_0}<\omega+\varepsilon. \tag{23}$$

Now take four points $z_i\in U_\delta(x_0)\subset S^1$ satisfying conditions (a) and (b) of Lemma 4.1. W.l.o.g. we can assume that $[z_1,z_4]\subset U_\delta(x_0)$ with $z_4\prec x_0$. Relation (23) then implies for $\hat{x}=\hat{z}_i,\ i=1,2,3,4$

$$(\omega-\varepsilon)(\tilde{x}_0-\hat{z}_i)<\varphi(\tilde{x}_0)-\varphi(\hat{z}_i)<(\omega+\varepsilon)(\tilde{x}_0-\hat{z}_i).$$

This yields the following inequalities for $s=1,2,3$:

$$\begin{aligned}\omega-\varepsilon\frac{(\tilde{x}_0-\hat{z}_{s+1})+(\tilde{x}_0-\hat{z}_s)}{\hat{z}_{s+1}-\hat{z}_s} &\leq \frac{\varphi(\hat{z}_{s+1})-\varphi(\hat{z}_s)}{\hat{z}_{s+1}-\hat{z}_s}\\ &\leq \omega+\varepsilon\frac{(\tilde{x}_0-\hat{z}_{s+1})+(\tilde{x}_0-\hat{z}_s)}{\hat{z}_{s+1}-\hat{z}_s}\end{aligned} \tag{24}$$

respectively for $s=1,2$.

$$\begin{aligned}\omega-\varepsilon\frac{(\tilde{x}_0-\hat{z}_{s+2})+(\tilde{x}_0-\hat{z}_s)}{\hat{z}_{s+2}-\hat{z}_s} &\leq \frac{\varphi(\hat{z}_{s+2})-\varphi(\hat{z}_s)}{\hat{z}_{s+2}-\hat{z}_s}\\ &\leq \omega+\varepsilon\frac{(\tilde{x}_0-\hat{z}_{s+2})+(\tilde{x}_0-\hat{z}_s)}{\hat{z}_{s+2}-\hat{z}_s}.\end{aligned} \tag{25}$$

Conditions (a) and (b) of Lemma 4.1 on the other hand imply

$$\max_{1\leq i\leq 4}\left\{\frac{\tilde{x}_0-\hat{z}_i}{\hat{z}_2-\hat{z}_1},\frac{\tilde{x}_0-\hat{z}_i}{\hat{z}_3-\hat{z}_1},\frac{\tilde{x}_0-\hat{z}_i}{\hat{z}_4-\hat{z}_2},\frac{\tilde{x}_0-\hat{z}_i}{\hat{z}_4-\hat{z}_3}\right\}\leq K_1, \tag{26}$$

where the constant $K_1>0$ depends on R_1 but not on ε.
Since

$$Dist(z_1,z_2,z_3,z_4;T_\varphi)=\frac{\varphi(\hat{z}_2)-\varphi(\hat{z}_1)}{\hat{z}_2-\hat{z}_1}\cdot\frac{\varphi(\hat{z}_4)-\varphi(\hat{z}_3)}{\hat{z}_4-\hat{z}_3}\cdot\frac{\hat{z}_3-\hat{z}_1}{\varphi(\hat{z}_3)-\varphi(\hat{z}_1)}\cdot\frac{\hat{z}_4-\hat{z}_2}{\varphi(\hat{z}_4)-\varphi(\hat{z}_2)}$$

inequalities (24)-(26) then prove the first part of Lemma 4.1.
It follows from this and our assumptions on the points $T_f^{q_n}z_i,\ i=1,2,3,4,$ that also

$$|Dist(T_f^{q_n}z_1, T_f^{q_n}z_2, T_f^{q_n}z_3, T_f^{q_n}z_4; T_\varphi) - 1| \leq C_1\varepsilon. \tag{27}$$

By definition

$$\begin{aligned} & Dist(T_f^{q_n}z_1, T_f^{q_n}z_2, T_f^{q_n}z_3, T_f^{q_n}z_4; T_\varphi) = \\ = \quad & \frac{Cr(T_\varphi(T_f^{q_n}z_1), T_\varphi(T_f^{q_n}z_2), T_\varphi(T_f^{q_n}z_3), T_\varphi(T_f^{q_n}z_4))}{Cr(T_f^{q_n}z_1, T_f^{q_n}z_2, T_f^{q_n}z_3, T_f^{q_n}z_4)}. \end{aligned} \tag{28}$$

Since T_φ conjugates T_f with the linear rotation T_ρ, we can readily see that $Cr(T_\varphi(T_f^{q_n}z_1), (T_\varphi(T_f^{q_n}z_2), (T_\varphi(T_f^{q_n}z_3), (T_\varphi(T_f^{q_n}z_4))) = Cr(T_\varphi z_1, T_\varphi z_2, T_\varphi z_3, T_\varphi z_4)$ and hence

$$Dist(T_f^{q_n}z_1, T_f^{q_n}z_2, T_f^{q_n}z_3, T_f^{q_n}z_4; T_\varphi) = \frac{Cr(T_\varphi z_1, T_\varphi z_2, T_\varphi z_3, T_\varphi z_4)}{Cr(T_f^{q_n}z_1, T_f^{q_n}z_2, T_f^{q_n}z_3, T_f^{q_n}z_4)}$$

This together with relations (16), (27) and (28) implies

$$|Dist(z_1, z_2, z_3, z_4; T_f^{q_n}) - 1| \leq C_2\varepsilon, \tag{29}$$

where the constant $C_2 > 0$ does not depend on ε and n.
This finishes the proof of Lemma 4.1. □

Next we will prove Lemma 4.3.

Proof of Lemma 4.3. We restrict ourselves to the case $z_2 = T_f^{-l}a_b,\ T_f^{-p}c_b \in [z_1, z_2]$ for some $0 \leq l, p \leq q_n - 1$; the case $z_3 = T_f^{-l}a_b,\ T_f^{-p}c_b \in [z_3, z_4]$ for some $0 \leq l, p \leq q_n - 1$ can be treated similarly. Rewrite $Dist(z_1, z_2, z_3, z_4; T_f^{q_n})$ in the following form

$$\begin{aligned} & Dist(z_1, z_2, z_3, z_4; T_f^{q_n}) = Dist(T_f^l z_1, T_f^l z_2, T_f^l z_3, T_f^l z_4; T_f) \\ \cdot \quad & Dist(T_f^p z_1, T_f^p z_2, T_f^p z_3, T_f^p z_4; T_f) \prod_{\substack{0 \leq i < q_n \\ i \neq l, p}} Dist(T_f^i z_1, T_f^i z_2, T_f^i z_3, T_f^i z_4; T_f) \end{aligned} \tag{30}$$

By assumption only the two intervals $[T_f^l z_1, T_f^l z_2]$ and $[T_f^p z_1, T_f^p z_2]$ contain the break points: namely $a_b = T_f^l z_2$, and $c_b \in [T_f^p z_1, T_f^p z_2]$ for some $0 \leq l, p \leq q_n - 1$.

Using Lemma 3.5 and the definition of the functions $G(x)$, $F(x, t)$ in (19) we get

$$\begin{aligned} Dist(T_f^l z_1, T_f^l z_2, T_f^l z_3, T_f^l z_4; T_f) &= \frac{\sigma(a_b)(1 + \xi(l))}{\sigma(a_b) + \xi(l)} + \chi_1 = G(\xi(l)) + \chi_1, \\ Dist(T_f^p z_1, T_f^p z_2, T_f^p z_3, T_f^p z_4; T_f) &= \frac{[\sigma(c_b) + (1 - \sigma(c_b))z(p)](1 + \xi(p))}{\sigma(c_b) + (1 - \sigma(c_b))z(p) + \xi(p)} + \chi_2 = \\ &= F(\xi(p), z(p)) + \chi_2, \end{aligned}$$

where the functions $\chi_j = \chi_j(z_1, z_2, z_3, z_4)$ can be bounded in absolute value either for $r = l$ or for $r = p$ by
$K_1 \int\limits_{f^r(\hat{z}_1)}^{f^r(\hat{z}_4)} |D^2 f(y)| dy,\ 1 \leq j \leq 2$. This together with (30) implies Lemma 4.3. □

Next we will prove Lemma 4.4.

Proof of Lemma 4.4. We prove only the bound for $G(\xi(l))F(\xi(p), z(p))$ since the one for
$G(\eta(l))F(\eta(p), \vartheta(p))$ can be proved similarly. We start by rewriting the expression $G(\xi(l))F(\xi(p), z(p))$ as follows:

$$G(\xi(l))F(\xi(p), z(p)) = \frac{\sigma(a_b)(1+\xi(l))}{\sigma(a_b)+\xi(l)} \cdot \frac{[\sigma(c_b)+(1-\sigma(c_b))z(p)](1+\xi(p)]}{\sigma(c_b)+(1-\sigma(c_b))z(p)+\xi(p)} =$$
$$= [\sigma(a_b)\sigma(c_b)+(1-\sigma(c_b))\sigma(a_b)z(p)] \times [\frac{(1+\xi(l))}{\sigma(a_b)+\xi(l)}$$
$$\cdot \frac{(1+\xi(p))}{\sigma(c_b)+(1-\sigma(c_b))z(p)+\xi(p)}] \equiv \Phi_1(z(p)) \times \Phi_2(\xi(l), \xi(p), z(p)) \quad (31)$$

where $z(p) \in [0, 1]$ and $\xi(l)$, $\xi(p) > 0$.

It is clear, that $\Phi_1(0) = \sigma(a_b)\sigma(c_b)$ and $\Phi_2(\xi(l), \xi(p), z(p))$ tends to 1 as $\xi(l)$ and $\xi(p)$ tend to ∞. This we can achieve by choosing $\xi(0)$ sufficiently large. Recall, that we assumed $\sigma(a_b)\sigma(c_b) \neq 1$. Next we want to see under which conditions $\Phi_1(z(p))\Phi_2(\xi(l), \xi(p), z(p))$ stays away from 1. Obviously

$$|\Phi_1\Phi_2 - 1| = |(\Phi_1 - 1) + \Phi_1(\Phi_2 - 1)| \geq ||\Phi_1 - 1| - \Phi_1|\Phi_2 - 1|. \quad (32)$$

Using the bounds for $z(m)$ in (21) we get

$$\begin{aligned} |\Phi_1 - 1| &= |\sigma(a_b)\sigma(c_b) + (1-\sigma(c_b))\sigma(a_b)z(p) - 1| \\ &\geq |\sigma(a_b)\sigma(c_b) - 1| - |1-\sigma(c_b)|\sigma(a_b)z(p) \\ &\geq |\sigma(a_b)\sigma(c_b) - 1| - |1-\sigma(c_b)|\sigma(a_b)e^v z(0). \end{aligned}$$

If $z(0)$ then fulfills the inequality

$$|\sigma(a_b)\sigma(c_b) - 1| - |1-\sigma(c_b)|\sigma(a_b)e^v z(0) \geq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{2},$$

and hence

$$z(0) \leq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{2e^v|\sigma(a_b)\sigma(c_b) - \sigma(a_b)|},$$

then obviously

$$|\Phi_1 - 1| \geq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{2} \quad if \;\; 0 \leq z(0) \leq \zeta_0, \quad (33)$$

where

$$\zeta_0 := min\left\{\frac{|\sigma(a_b)\sigma(c_b) - 1|}{2e^v|\sigma(a_b)\sigma(c_b) - \sigma(a_b)|}, 1\right\}. \quad (34)$$

Next we determine conditions on $\xi(0)$ which imply

$$\Phi_1|\Phi_2 - 1| \leq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{4}. \quad (35)$$

Obviously, for $z(p) \in [0, 1]$ one has $\Phi_1(z(p)) \leq max\,\{\sigma(a_b)\sigma(c_b), \sigma(a_b)\} := m_\sigma$. Inequality (35) then certainly holds if

$$|\Phi_2 - 1| \leq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{4m_\sigma}. \quad (36)$$

Since

$$\Phi_2 - 1 = \frac{(1+\xi(l))}{\sigma(a_b)+\xi(l)} \cdot \frac{(1+\xi(p))}{\sigma(c_b)+(1-\sigma(c_b))z(p)+\xi(p)} - 1, \quad (37)$$

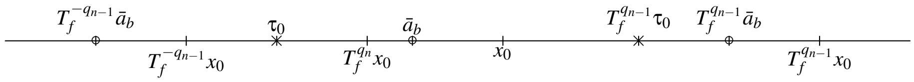

FIGURE 1. Structure of the intervals $[x_0, T_f^{q_{n-1}}x_0]$ and $[\bar{a}_b, T_f^{q_{n-1}}\bar{a}_b]$

the right hand side of (37) behaves for $\xi(l)$ and $\xi(p)$ sufficiently large as $(1+O(\frac{1}{\xi(l)}))(1+O(\frac{1}{\xi(p)}))-1$, which can be bounded by $R_6\left(\frac{1}{\xi(l)}+\frac{1}{\xi(p)}\right)$ for some constant $R_6>1$ not depending on $\xi(l)$ and $\xi(p)$. On the other hand, according to relations (21), for $0<m\le q_n$ the value of $\xi(m)$ is comparable to $\xi(0)$, and hence

$$|\Phi_2-1|\le R_6\left(\frac{1}{\xi(l)}+\frac{1}{\xi(p)}\right)\le 2R_6e^v\frac{1}{\xi(0)}. \tag{38}$$

Therefore, if

$$2R_6e^v\frac{1}{\xi(0)}\le\frac{|\sigma(a_b)\sigma(c_b)-1|}{4m_\sigma}$$

or equivalently

$$\xi(0)\ge\frac{4R_6e^v m_\sigma}{|\sigma(a_b)\sigma(c_b)-1|}, \tag{39}$$

then inequality (36) is certainly fulfilled. With C_0 defined as

$$C_0:=\max\left\{\frac{4R_6e^v m_\sigma}{|\sigma(a_b)\sigma(c_b)-1|},1\right\}. \tag{40}$$

Lemma 4.4 then follows immediately from relations (33)-(40). □

Next we prove Lemma 4.5.

Proof of Lemma 4.5. W.l.o.g. we assume n odd, the case n even can be deduced from the odd one just by reversing the orientation of the circle. From the definition of the dynamical partition $\xi_n(x_0)$ it follows that for some $0\le l,p<q_n$ the points $\bar{a}_b=T_f^{-l}a_b$ and $\bar{c}_b=T_f^{-p}c_b$ belong to the interval $[T_f^{q_n}x_0,T_f^{q_{n-1}}x_0]$. Consider then the neighborhood $[T_f^{-q_{n-1}}\bar{a}_b,T_f^{q_{n-1}}\bar{a}_b]$ of the point $\bar{a}_b$. By Corollary 2 the intervals $[a,b]$, $T_f^{q_n}[a,b]$, $T_f^{-q_n}[a,b]$ are e^v- comparable for any $a,b\in S^1$. Since $\bar{a}_b\in[T_f^{q_n}x_0,T_f^{q_{n-1}}x_0]$, it can easily be shown, that the pairs of intervals $([T_f^{-q_{n-1}}x_0,x_0],[T_f^{-q_{n-1}}\bar{a}_b,\bar{a}_b])$, $([x_0,T_f^{q_{n-1}}x_0],[\bar{a}_b,T_f^{q_{n-1}}\bar{a}_b])$ as well as $([T_f^{-q_{n-1}}x_0,T_f^{q_{n-1}}x_0],[T_f^{-q_{n-1}}\bar{a}_b,T_f^{q_{n-1}}\bar{a}_b])$ are e^v- comparable.

Let τ_0 be the middle point of the interval $[T_f^{-q_{n-1}}\bar{a}_b,\bar{a}_b]$. Since $[\bar{a}_b,T_f^{q_{n-1}}\tau_0]=T_f^{q_{n-1}}[T_f^{-q_{n-1}}\bar{a}_b,\tau_0]$ and $l([T_f^{-q_{n-1}}\bar{a}_b,\tau_0])=l([\tau_0,\bar{a}_b])$ we conclude, that the intervals $[\tau_0,\bar{a}_b]$ and $[\bar{a}_b,T_f^{q_{n-1}}\tau_0]$ are e^v- comparable (see figure 1).

Set

$$d_n:=\frac{1}{2}min\left\{l([\bar{a}_b,T_f^{q_{n-1}}\bar{a}_b]),l([T_f^{-q_{n-1}}\bar{a}_b,\bar{a}_b])\right\}. \tag{41}$$

Then Corollary 2 implies

$$e^{-v}\frac{1}{2}l([\bar{a}_b,T_f^{q_{n-1}}\bar{a}_b])\le d_n\le e^v\frac{1}{2}\,l([\bar{a}_b,T_f^{q_{n-1}}\bar{a}_b]). \tag{42}$$

Since the interval $[\tau_0,T_f^{q_{n-1}}\tau_0]$ is one of the two generators of the partition $\xi_n(\tau_0)$, the intervals $T_f^i[\tau_0,T_f^{q_{n-1}}\tau_0]$, $0\le i<q_n$ cover the break point a_b only once. Using

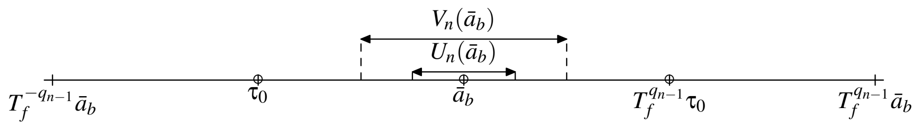

FIGURE 2. The neighborhoods $U_n(\bar{a}_b)$ and $V_n(\bar{a}_b)$ are comparable with $[T_f^{-q_{n-1}}\bar{a}_b, T_f^{q_{n-1}}\bar{a}_b]$. The point τ_0 is the middle point of the interval $[T_f^{-q_{n-1}}\bar{a}_b, \bar{a}_b]$

the constants C_0 and ζ_0 in Lemma 4.4 we define two neighbourhoods (see figure 2) of the point $\overline{a}_b$:

$$V_n(\overline{a}_b) = V_{\delta_n}(\overline{a}_b)\, with\, \delta_n = \frac{1}{2}e^{-v}C_0^{-1}d_n,$$

$$U_n(\overline{a}_b) = U_{\gamma_n}(\overline{a}_b)\, with\, \gamma_n = \frac{1}{2}\zeta_0\, l(V_n(\overline{a}_b)).$$

It is clear that $U_n(\overline{a}_b) \subset V_n(\overline{a}_b) \subset [\tau_0, T_f^{q_{n-1}}\tau_0]$. The construction of the intervals $[z_s, z_{s+1}]$ depends now on the location of $\overline{c}_b$ in the interval $V_n(\overline{a}_b)$. There are two cases to consider:

$$\textbf{either}\ \ \overline{c}_b \notin U_n(\overline{a}_b), i.e.\ \overline{c}_b \in [T_f^{q_n}\tau_0, T_f^{q_{n-1}}\tau_0] \setminus U_n(\overline{a}_b)\ \ \textbf{or}\ \overline{c}_b \in U_n(\overline{a}_b)). \tag{43}$$

If $\overline{c}_b \in V_n(\overline{a}_b) \setminus U_n(\overline{a}_b))$ we set

$$\tilde{z}_2 := \tilde{\overline{a}}_b, \tilde{z}_3 := \tilde{\overline{a}}_b + \frac{1}{4}l(U_n(\overline{a}_b)), \tilde{z}_4 := \tilde{\overline{a}}_b + \frac{1}{2}l(U_n(\overline{a}_b))\ and\ \tilde{z}_1 := \tilde{\overline{a}}_b - \frac{1}{4}l(U_n(\overline{a}_b))$$

corresponding to the points $z_i \in S^1$ for $i = 1, 2, 3, 4$. Obviously $[z_1, z_4] \subset [\tau_0, T_f^{q_{n-1}}\tau_0]$ and $\overline{c}_b \notin [z_1, z_4]$. We have to check, that the intervals $[z_s, z_{s+1}]$, $s = 1, 2, 3$ satisfy properties 1)-3) in Lemma 4.5. The interval $[z_1, z_4]$ is q_n-small, because $[z_1, z_4] \subset [\tau_0, T_f^{q_{n-1}}\tau_0]$, which is one of the generators of the dynamical partition $\xi_n(\tau_0)$. By construction $l([z_1, z_4]) = \frac{3}{4}e^{-v}C_0^{-1}\zeta_0 d_n$, but $d_n = \frac{1}{2}l([T_f^{-q_{n-1}}\overline{a}_b, \overline{a}_b]) = \frac{1}{2}l([\overline{a}_b, T_f^{q_{n-1}}\overline{a}_b])$ and hence $[z_1, z_4]$ is $4e^vC_0\zeta_0^{-1}$- comparable with $[x_0, T_f^{q_{n-1}}x_0]$. Next we show the assumptions a) and b) in Lemma 4.1 are fulfilled by the two intervals $[z_s, z_{s+1}]$ and $[T_f^{q_n}z_s, T_f^{q_n}z_{s+1}]$. For a) notice that $l([z_s, z_{s+1}]) = \frac{1}{4}e^{-v}C_0^{-1}\zeta_0 d_n$ for $s = 1, 2, 3$ and the intervals $[z_s, z_{s+1}]$ and $T_f^{q_{n-1}}[z_s, z_{s+1}]$ are e^v-comparable, so that they fulfill the assumption a) of Lemma 4.1 with constant e^v.

Next consider assumption b) of Lemma 4.1. It is easy to see, that for $i = 1, 2, 3, 4$ one has

$$\begin{gathered} l([z_i, x_0]) \leq l([z_2, x_0]) + l([z_1, z_4]) = l([z_2, x_0]) + d_n, \\ l([T_f^{q_n}z_i, x_0]) \leq l([z_2, x_0]) + l([T_f^{q_n}z_2, z_2]) + l([T_f^{q_n}z_1, T_f^{q_n}z_4]). \end{gathered} \tag{44}$$

The point z_2 belongs to $[T_f^{q_n}x_0, T_f^{q_{n-1}}x_0] \subset [T_f^{-q_{n-1}}x_0, T_f^{q_{n-1}}x_0]$, which is e^v- comparable with $[T_f^{-q_{n-1}}\overline{a}_b, T_f^{q_{n-1}}\overline{a}_b]$. But the length of this last interval is $4e^v$-comparable with d_n. In complete analogy we can estimate the second expression in (44) by $12e^vC_0\zeta_0^{-1}d_n$.

Furthermore the intervals $T_f^i[z_s, z_{s+1}]$ do not cover the break point c_b and cover the point a_b exactly once since $z_2 = T_f^{-l}a_b$.

Next consider the case $\overline{c}_b \in U_n(\overline{a}_b)$. Then again two cases can happen: if $\tilde{\overline{c}}_b \in [\tilde{\overline{a}}_b - \frac{1}{2}\zeta_0 l(V_n(\overline{a}_b)),\, \tilde{\overline{a}}_b]$, we define

$$\tilde{z}_1 := \tilde{\overline{a}}_b - \frac{1}{2} l(V_n(\overline{a}_b)), \tilde{z}_2 := \tilde{\overline{a}}_b, \tilde{z}_3 := \tilde{\overline{a}}_b + \frac{1}{2} C_0 l(V_n(\overline{a}_b)), \tilde{z}_4 := \tilde{\overline{a}}_b + C_0 l(V_n(\overline{a}_b))$$

corresponding to the points $z_i \in S^1$ for $i = 1,2,3,4$. Then $l([z_1, z_2] = \frac{1}{2} e^{-v} C_0^{-1} d_n$, where as all other intervals have a length equal to $\frac{1}{2} e^{-v} d_n$. The lengths of these intervals are hence $2e^v C_0$-comparable with d_n. The first two statements of Lemma 4.5 for these intervals can be checked in complete analogy to the first case in (43).

Next we show, that in the present case the intervals $\Big\{T_f^i[z_1, z_2],\ T_f^i[z_2, z_3],\ T_f^i[z_3, z_4],\ 0 \le i \le q_n\Big\}$ cover both break points a_b, c_b regularly with constants C_0 and ζ_0. By construction these intervals cover both break points exactly once. Moreover we have $z_2 = \overline{a}_b$ and $\overline{c}_b \in [z_1, z_2]$. It is easy to see, that $\xi(0) = \frac{l([z_2,z_3])}{l([z_1,z_2])} = C_0$. Since $\tilde{\overline{c}}_b \in [\tilde{\overline{a}}_b - \frac{1}{2}\zeta_0 l(V_n(\tau_0)),\, \tilde{\overline{a}}_b]$, we find that $z(0) = \frac{l([\overline{c}_b,z_2])}{l([z_1,z_2])} \le \zeta_0$ and hence the intervals $[z_s, z_{s+1}],\ \ s = 1,2,3$ satisfy Lemma 4.5.

It only remains to consider the case where the point $\tilde{\overline{c}}_b$ belongs to the interval $[\tilde{\overline{a}}_b, \tilde{\overline{a}}_b + \frac{1}{2}\zeta_0 l(V_n(\tau_0))]$. In this case we define

$$\tilde{z}_1 := \tilde{\overline{a}}_b - C_0 l(V_n(\overline{a}_b)), \tilde{z}_2 := \tilde{\overline{a}}_b - \frac{1}{2} C_0 l(V_n(\overline{a}_b)), \tilde{z}_3 := \tilde{\overline{a}}_b, \tilde{z}_4 := \tilde{\overline{a}}_b + \frac{1}{2} l(V_n(\overline{a}_b))$$

corresponding to the points $z_i \in S^1$ for $i = 1,2,3,4$. The proof of Lemma 4.5 for these intervals $[z_s, z_{s+1}],\ \ s = 1,2,3$ proceeds now exactly as in the previous case. This concludes the proof of Lemma 4.5. $\square$

Finally we prove Lemma 4.6.

Proof of Lemma 4.6. Assume, that the circle homeomorphism T_f satisfies the assumptions of Theorem 1.4 and the intervals $[z_s, z_{s+1}]$, $s = 1,2,3$ satisfy Lemma 4.5. Consider first the case when the intervals $\Big\{T_f^i[z_1, z_2],\ T_f^i[z_2, z_3]\, T_f^i[z_3, z_4],\ \ 0 \le i \le q_n - 1\Big\}$ cover both break points a_b, c_b regularly with constants C_0 and ζ_0. Suppose that $z_2 = \overline{a}_b = T_f^{-l} a_b$ and $\overline{c}_b = T_f^{-p} c_b$, for some $0 \le l, p \le q_n$ and

$$\frac{l([z_2, z_3])}{l([z_1, z_2])} \le C_0, \quad 0 \le \frac{l([\overline{c}_b, z_3])}{l([z_1, z_2])} \le \zeta_0. \tag{45}$$

Lemma 4.3 shows that for $z_2 = \overline{a}_b$, and $\overline{c}_b \in [z_1, z_2]$

$$\begin{aligned} Dist(z_1, z_2, z_3, z_4; T_f^{q_n}) = [G(\xi(l) + \chi_1][F(\xi(p), z(p)) + \chi_2] \times \\ \times \prod_{\substack{0 \le i < q_n \\ i \ne l,p}} Dist(T_f^i z_1, T_f^i z_2, T_f^i z_3, T_f^i z_4; T_f) \end{aligned} \tag{46}$$

with

$$|\chi_j| = |\chi_j(z_1, z_2, z_3, z_4)| \le K_1 \int_{f^r(\hat{z}_1)}^{f^r(\hat{z}_4)} |D^2 f(y)| dy, \ 1 \le j \le 2, \ r = l, p. \tag{47}$$

for some constant K_1 not depending on neither n nor ε. Next we estimate the right hand side of equation (46). Fix some $\varepsilon > 0$. Since $D^2 f$ is a periodic function on $\mathbb{R}$

with period 1 and hence $D^2 f \in L^1([0,1], dl)$, it can be written in the form

$$D^2 f(\hat{x}) = g_\varepsilon(\hat{x}) + \theta_\varepsilon(\hat{x}), \hat{x} \in \mathbb{R}, \tag{48}$$

with $g_\varepsilon(\hat{x})$ and $\theta_\varepsilon(\hat{x})$ periodic functions on $\mathbb{R}$ with period 1, and $g_\varepsilon(\hat{x})$ a continuous function on $[0,1]$ and $\int_0^1 |\theta_\varepsilon| dl < \varepsilon$. By assumption, among the intervals $T_f^i[z_s, z_{s+1}]$, $0 \le i \le q_n$, only the intervals $T_f^l[z_1, z_4]$ and $T_f^p[z_1, z_4]$ contain the break points a_b respectively c_b .

Obviously

$$|\prod_{\substack{0 \le i < q_n \\ i \ne l,p}} Dist(T_f^i z_1, T_f^i z_2, T_f^i z_3, T_f^i z_4; T_f) - 1| =$$

$$= |exp\{\sum_{i=0,\ i \ne l,b}^{q_n - 1} \log(1 + (Dist(T_f^i z_1, T_f^i z_2, T_f^i z_3, T_f^i z_4; T_f) - 1))\} - 1|. \tag{49}$$

Applying Theorem 3.3 we obtain

$$|Dist(T_f^i z_1, T_f^i z_2, T_f^i z_3, T_f^i z_4; T_f) - 1| \le C_1 l([T_f^i z_1, T_f^i z_4]) \times$$

$$\max_{x,t \in [T_f^i z_1, T_f^i z_4]} |g_\varepsilon(x) - g_\varepsilon(t)| + C_1 \int\limits_{f^i(\hat{z}_1)}^{f^i(\hat{z}_4)} |\theta_\varepsilon(y)| dy + C_1 \Big(\int\limits_{f^i(\hat{z}_1)}^{f^i(\hat{z}_4)} |D^2 f(y)| dy \Big)^2 \tag{50}$$

where the constant $C_1 > 0$ depends only on the function f.

But for $D^2 f \in L^1([0,1])$ the function $\Psi(\tilde{x}) = \int\limits_0^{\tilde{x}} |D^2 f(y)| dy$ defines an absolutely continuous function on $[0,1]$ and the functions $\Psi(\tilde{x})$ and $g_\varepsilon(\tilde{x})$ are then uniformly continuous on the closed interval $[0,1]$. Hence there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that for all $\tilde{x}, \tilde{t} \in [0,1]$ with $l([\tilde{t}, \tilde{x}]) < \delta_0$, the inequalities

$$|\Psi(\tilde{x}) - \Psi(\tilde{t})| < \varepsilon,\ |g_\varepsilon(\tilde{x}) - g_\varepsilon(\tilde{t})| < \varepsilon, \tag{51}$$

hold.

Since by assumption the interval $[z_1, z_4]$ is q_n-small Corollary 2 shows, that $l(T_f^i[z_1, z_4]) \le const\, \lambda^n$ for $0 \le i \le q_n - 1$, with $\lambda = (1 + e^{-v})^{-\frac{1}{2}} < 1$. Hence there exists a number $N_0 = N_0(\delta) > 0$ such that for all $n > N_0$ and all $0 \le i \le q_n - 1$ one finds $l(T_f^i[z_1, z_4]) \le \delta_0$. This together with (51) implies that for all $n > N_0$ the inequalities

$$|\Psi(\tilde{x}) - \Psi(\tilde{y})| < \varepsilon, \quad |g_\varepsilon(\tilde{x}) - g_\varepsilon(\tilde{y})| < \varepsilon, \tag{52}$$

hold for all $\tilde{x}, \tilde{y} \in [f^i \tilde{z}_1, f^i \tilde{z}_4]\,(mod\,1)$ and all $0 \le i \le q_n - 1$.

Since for $[z_1, z_4]$ q_n- small the intervals $T_f^i[z_1, z_4]$ are non intersecting for $0 \le i \le q_n - 1$ we have

$$\sum_{i=0}^{q_n} l(T_f^i[z_1, z_4]) \le 1.$$

Because $\| \theta_\varepsilon \|_{L^1} < \varepsilon$ we find, using relations (49)-(52),

$$|\prod_{\substack{0 \le i < q_n \\ i \ne l,p}} Dist(T_f^i z_1, T_f^i z_2, T_f^i z_3, T_f^i z_4; T_f) - 1| \le C_2 \varepsilon \sum_{i=0}^{q_n - 1} l([T_f^i z_1, T_f^i z_4]) +$$

$$+C_2 \sum_{i=0}^{q_n-1} \int\limits_{f^i(\hat{z}_1)}^{f^i(\hat{z}_4)} |\theta_\varepsilon(y)|dy + C_2 \sum_{i=0}^{q_n-1} |\Psi(f^i\hat{z}_4) - \Psi(f^i\hat{z}_1)| \int\limits_{f^i(\hat{z}_1)}^{f^i(\hat{z}_4)} |D^2 f(y)|dy \leq$$

$$\leq C_2 \left\{ 2\varepsilon + \int\limits_0^1 |\theta_\varepsilon(y)|dy + \varepsilon \int\limits_0^1 |D^2 f_1(y)|dy \right\} \leq C_2(3 + ||D^2 f||_{L^1})\varepsilon, \qquad (53)$$

where the constant C_2 depends only on f.

Next we estimate the expression $(G(\xi(l)) + \chi_1)(F(\xi(p), z(p)) + \chi_2)$, where $|\chi_i|$ is bounded from above by $K_1 \int\limits_{f^r(\hat{z}_1)}^{f^r(\hat{z}_4)} |D^2 f(y)|dy$ for $j = 1, 2$ and $r = l, p$ with some constant K_1 not depending on n and ε. For $n > N_0$ and $[z_1, z_4]$ q_n-small we have $\int\limits_{\hat{z}_1}^{\hat{z}_4} |D^2 f(y)|dy = \Psi(\tilde{z}_4) - \Psi(\tilde{z}_1) < \varepsilon$. Since $G(x)$ is bounded for $x > 0$ and $F(x, t)$ is bounded for $x > 0$ and $1 \leq t \leq 1$, it is sufficient to estimate the term $G(\xi(l))F(\xi(p), z(p))$ in the above product. Since the intervals $\left\{T_f^i[z_1, z_2],\ T_f^i[z_2, z_3]\, T_f^i[z_3, z_4],\ \ 0 \leq i \leq q_n - 1\right\}$ cover both break points a_b, c_b regularly with constants C_0 and ζ_0 Lemma 4.4 implies

$$|G(\xi(l))F(\xi(p), z(p)) - 1| \geq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{4} > 0,$$

because by assumption $\sigma(a_b)\sigma(c_b) \neq 1$. Therefore for sufficiently large n and small ε the inequality

$$|[G(\xi(l)) + \chi_1][F(\xi(p), z(p)) + \chi_2] - 1| \geq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{8}$$

holds. Together with relations (50) and (53) this implies the assertion of Lemma 4.6 in the case of a regular covering of the two break points.

Next consider the case where the intervals $\left\{T_f^i[z_1, z_2],\ T_f^i[z_2, z_3]\, T_f^i[z_3, z_4],\ \ 0 \leq i \leq q_n - 1\right\}$ cover w.l.o.g. only the break point a_b with $z_2 = \overline{a}_b = T_f^{-l} a_b$ for some $0 \leq l \leq q_n$ and satisfy properties 1), 2) of Lemma 4.5.

Then we write $Dist(z_1, z_2, z_3, z_4; T_f^{q_n})$ in the following form

$$Dist(z_1, z_2, z_3, z_4; T_f^{q_n}) = Dist(T_f^l z_1, T_f^l z_2, T_f^l z_3, T_f^l z_4; T_f) \times$$
$$\times \prod_{\substack{0 \leq i < q_n \\ i \neq l}} Dist(T_f^i z_1, T_f^i z_2, T_f^i z_3, T_f^i z_4; T_f). \qquad (54)$$

For sufficiently large n and any $\varepsilon > 0$, the value of the product over $i \neq l, p$ in (54) lies in an $\varepsilon-$neighbourhood of 1. By assumption, only the interval $T_f^l[z_1, z_4]$ contains the break point a_b with $a_b = T_f^l z_2$. Using Lemma 3.5 we find

$$|Dist(T_f^l z_1, T_f^l z_2, T_f^l z_3, T_f^l z_4; T_f) - \frac{\sigma(a_b)(1 + \xi(l))}{\sigma(a_b) + \xi(l)}|$$
$$\leq K_1 \int\limits_{f^l(\hat{z}_1)}^{f^l(\hat{z}_4)} |D^2 f(y)|dy, \qquad (55)$$

where the constant $K_1 > 0$ depends only on the function f and where $\xi(l) = \frac{l([T_f^l z_2, T_f^l z_3])}{l([T_f^l z_1, T_f^l z_2])}$. Obviously

$$\frac{\sigma(a_b)(1+\xi(l))}{\sigma(a_b)+\xi(l)} - 1 = \frac{(\sigma(a_b)-1)\xi(l)}{\sigma(a_b)+\xi(l)}.$$

This, together with the inequality $R_2^{-1} \leq \xi(l) \leq R_2$ following from (21) and the comparability of the intervals $[z_s, z_{s+1}]$ for $s = 1, 2, 3$, shows

$$R_3^{-1} \leq \frac{(\sigma(a_b)-1)\xi(l)}{\sigma(a_b)+\xi(l)} \leq R_3,$$

where the constants $R_i > 0,\ i = 1, 2$ depend only on f. Finally we obtain

$$|Dist(T_f^l z_1, T_f^l z_2, T_f^l z_3, T_f^l z_4; T_f) - 1| \geq const > 0, \tag{56}$$

where the constant again depends only on f. Since the inequality

$$|\prod_{\substack{0 \leq i < q_n \\ i \neq l}} Dist(T_f^i z_1, T_f^i z_2, T_f^i z_3, T_f^i z_4; T_f) - 1| \leq const\ \varepsilon$$

holds also in the present case this, together with relations (54) and (56) proves Lemma 4.6. □

Let us finally prove Theorem 1.5:

Proof of Theorem 1.5. Consider the $n-$th dynamical partition $\xi_n(\overline{a}_b)$ of the unique point $\overline{a}_b$ of the break point a_b in the interval $[T_f^{q_n} x_0, T_f^{q_n - 1} x_0]$ around the point x_0, at whose representative point $\tilde{x}_0$ there exists a positive derivative $D\varphi(\tilde{x}_0)$ of the lift φ of the conjugating homeomorphism T_φ. Notice that in the case of an irrational rotation number of bounded type the following important fact was proved in [5](see [5], Proposition 3): there exists a subsequence $\{n_k, k = 1, 2, ...\} \in \mathbb{N}$ such that for every n_k the interval $[\overline{a}_b, T_f^{q_{n_k}-1} \overline{a}_b]$ (respectively the interval $[T_f^{q_{n_k}} \overline{a}_b, \overline{a}_b]$) contains the point $\overline{c}_b = T_f^{-p} c_b$ for some $0 \leq p \leq q_{n_k} - 1$, and furthermore $K_3^{-1} \leq \frac{l([\overline{a}_b, \overline{c}_b])}{l([\overline{c}_b, T_f^{q_{n_k}-1} \overline{a}_b])} \leq K_3$ (respectively $K_3^{-1} \leq \frac{l([\overline{c}_b, \overline{a}_b])}{l([T_f^{q_{n_k}} \overline{a}_b, \overline{c}_b])} \leq K_3$) , where the constant K_3 depends only on f.

Set $d_{n_k} = min\left\{ l([\overline{a}_b, \overline{c}_b]), l([\overline{c}_b, T_f^{q_{n_k}-1} \overline{a}_b]) \right\}$. Then the closed $d_{n_k}-$ neighbourhood $U_{d_{n_k}}(\overline{c}_b)$ of $\overline{c}_b$ is $e^{3v}-$ comparable with $[T_f^{-q_{n_k}-1}(x_0), T_f^{q_{n_k}-1}(x_0)]$. Furthermore, the intervals $T_f^i U_{d_{n_k}}(\overline{c}_b)$, $0 \leq i < q_{n_k}$, cover the break point c_b exactly once. It is hence enough to give the construction of the intervals $[z_s, z_{s+1}], i = 1, 2, 3$, in the neighbourhood $U_{d_{n_k}}(\overline{c}_b)$ as in the case of a homeomorphism T_f with a single break point c_b. We define

$$\tilde{z}_2 := \tilde{\overline{c}}_b, \tilde{z}_1 := \tilde{\overline{c}}_b - \frac{1}{2} d_{n_k}, \tilde{z}_3 := \tilde{\overline{c}}_b + \frac{1}{2} d_{n_k}, \tilde{z}_4 := \tilde{\overline{c}}_b + d_{n_k}.$$

and the corresponding points $z_i \in S^1,\ i = 1, 2, 3, 4$. As in the proof of Lemma 4.6 it can be checked, that the intervals $[z_s, z_{s+1}],\ s = 1, 2, 3$ satisfy the conditions of the Lemma 4.5. Applying Lemma 4.1 we get that $Dist(z_1, z_2, z_3, z_4; T_f^{q_n})$ is close to 1. But by Lemma 4.6 it should stay away from 1. This is a contradiction and hence also Theorem 1.5 is proved. □

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