Polynomial Skew Products

Manfred Denker[1] and Stefan-M. Heinemann[2]

[1] Institut für Mathematische Stochastik
Universität Göttingen
Lotzestraße 13
37083 Göttingen

[2] Institut für Theoretische Physik
TU Clausthal
Arnold-Sommerfeld-Straße 6
38678 Clausthal-Zellerfeld

Abstract. This is a brief overview on a new theory[1] for the dynamical behaviour of polynomial skew products in $\mathbb{C}^n$, including some non-published facts.

1 A New Direction in Fractal Analysis and Iteration Theory of Holomorphic Maps

With the development of modern high-performance computers the 'chaos theory' in the form of fractal geometry has become a popular field of research. Simple iterative algorithms, i.e. simple repetitive step-by-step calculations, permit the composition of complicated shapes, which sometimes even appear to possess artistic value. One feature of the design underlying all these methods is self-similarity: manifestations, which when enlarged are visible to the naked eye, are repeated again and again in miniature and, given a corresponding enlargement, each step, no matter how small, will be similar to the next object as a whole. This explains how simple equations or geometrical forms can lead to complex and multilayered structures.

In recent years, the conceptual approach inherent in this theory has been extended to every realm of knowledge. Chaos theory has been applied in the attempt to explain complicated processes which cannot always be predicted, or at least not with precision. The best-known of these are the so-called Lorenz equations in physics, which describe the vertical flow of a gas. In medicine, scientists have conducted experiments aimed at modelling the generation of creativity in the brain through chaos. Electrical discharges or the creation of polymer compounds display clearly fractal structures. A further example is the modulation of population dynamics in biology, where the size of the population is subject to irregular fluctuations. The list is endless. This whole field of research has developed from problems met with

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when solving equations using the so-called Newton method. When employing this algorithm, initial numbers cannot just be selected at random if the aim is to achieve a useful solution to an equation. Experiments are now in train to use this system for solving equations with multiple variables. Their mathematical investigation leads towards a new theory of higher-dimensional Julia sets, which is partly reviewed and extended here. Geometric, analytic and probabilistic structures which occur under simplest dynamics of four (and higher)-dimensional space, and which comprise just these points upon which the greatest chaos reigns, are discussed (for an extended exposition see [DHP95]).

The aim is best explained by the following example. For analytic endomorphisms of the Riemann sphere S^2 it is well known that the Julia sets $J(f_c)$ of mappings of the form

$$f_c : z \mapsto z^2 + c$$

with $|c|$ small are Jordan curves (in general non-rectifiable) and $f|_{J(f_c)}$ shows similar dynamical behaviour as $f_0 : z \mapsto z^2$ restricted to S^1. This holds at least if $|c| \leq 1/4 - \varepsilon$ for some $\varepsilon > 0$. Also, as a consequence of the choice of the parameter c, the mapping has all inverse branches everywhere defined on its Julia set, hence has bounded distortion and is hyperbolic. The term is used in the sense of expanding, i.e.

$$\liminf_{\substack{x \to y \\ x \neq y \in J(f_c)}} \frac{|f_c(x) - f_c(y)|}{|x - y|} > 1.$$

This property in turn enforces probability structures by which stationary time series based on Hölder functions are approximated by Brownian motion (like in the case of independent, identically distributed discrete time processes).

One of the main points of our discussion here is concerned with similar questions in higher dimensions, in particular the analytic and geometric structures and the long time behaviour of stationary processes generated by them.

2 Another look at the one-dimensional case

An analytic endomorphism (or a rational function) $f : S^2 \to S^2$ on the Riemann sphere S^2 has always a nonempty, fully invariant Julia set, defined as the set of non-normal points for the family of functions $f^n : S^2 \to S^2$ $(n \geq 0)$. It is quite clear that this set constitutes the regime of greatest chaos, whence the dynamics creates large diversity in the long time behaviour of time series $X_n = \phi \circ f^n$, $n \geq 0$, as well as some disordering structure which is visible in the local structure around periodic points. In fact, such different aspects of the same phenomenon have been the centre of investigation for a long time, starting with Julia and Fatou in the beginning of this century. In case of polynomials $P : \mathbb{C} \to \mathbb{C}$ one can summarise the essential features as follows (see [Hei93] for the first time when this was observed).

Theorem 1. *For a polynomial map $P : \mathbb{C} \to \mathbb{C}$ of degree at least 2, the Julia set $J(P)$ equals each of the following sets:*

1. $\{z \in \mathbb{C} : \{P^k : k \geq 0\}$ *is not normal at* $z\}$.
2. *The boundary $\partial(K(P))$ where $K(P) = \{z \in \mathbb{C} : \sup_k |P^k(z)| < \infty\}$.*
3. *The closure $\mathcal{R}$ of the set of repelling periodic points of P.*
4. *The limit of the pull backs by P^{-k} of the boundary $\partial(\{z \in \mathbb{C} : |z| \leq r\})$ for sufficiently large $r > 0$.*
5. *The support supp(μ_{max}) of the measure of maximal entropy for P.*

Before continuing it is necessary to clarify the meaning of the 5 statements in the theorem briefly. Clearly, (1.) is the description arising from geometric function theory and allows to introduce methods from complex function theory to study $J(P)$. In particular, bounded distortion properties play a fundamental role. (2.) tells us that the complement of $J(P)$ splits into connected components and hence the analysis on $J(P)$ can be studied using the theory of holomorphic functions on domains, in particular using harmonic analysis (Green's function). Certainly, this complements the description in (1.). However, as is known, this boundary is equal to the Shilov boundary $\partial_{SH}(K(P))$, which is defined to be the minimal compact set Q with the property that functions holomorphic in some neighbourhood of $K(P)$ attain their maximal modulus (over $K(P)$) in Q. Certainly, $\partial_{SH}(K(P))$ is contained in $\partial K(P)$ – it is not difficult to construct holomorphic functions which attain their maximal modulus in a given open set intersecting $K(P)$ which shows that even $\partial K(P) = \partial_{SH}(K(P))$. This is one possibility to introduce some abstract boundary theory in order to study Julia sets. Other types of boundaries are certainly well suited to approach specific problems (e.g. rigidity questions). (3.) is a dynamic description. It implies that the dynamics is essentially determined by some hyperbolic behaviour. The tracing property of repelling period points and the value of the topological entropy are an almost immediate consequence. (4.) means that we may use the well known boundary theory for large centred disks and obtain the Julia set as a pull back. Finally, we obtain the existence of a probability measure maximising entropy. Intuitively this means that in addition to (3.) we obtain a probabilistic structure which contains a maximum of randomness. Since the map P is not a homeomorphism, there is also a natural filtration given by the pull backs of the Borel σ-algebra. It is clear that this filtration can be used to introduce martingale and mixing structures which are geometrically meaningful.

Proof. (of theorem 1) First, let us remark that the set $K(P)$ is always well defined and compact. Namely, we find a radius r_P such that $|z| > r_P$ implies that $|P(z)| > |z|$ and the forward orbit of z converges to infinity. Clearly,

$$K(P) = \bigcap_{k=0}^{\infty} P^{-k}\left(\overline{B_{r_P}}\right) = \lim_{k\to\infty} P^{-k}\left(\overline{B_{r_P}}\right). \tag{1}$$

Since P is a polynomial, we get the dynamical trichotomy (with respect to complete invariance under P) of the invariant sets $\complement K(P)$, $\partial K(P)$, and $\mathring{K}(P)$ (the latter set may be empty). This also yields that $\partial K(P)$ is a perfect set. We should note here that the proof uses Montel's theorem, i.e. one makes use of the fact that $\{P^k\}$ is a normal family on some open set U once one knows that the union of the images $P^k(U)$ leaves out at least 3 points in $\mathbb{C}$. A similar result in $\mathbb{C}^n$ (i.e. providing statements about $\mathbb{C}^n$ minus a set of finitely many points) does not hold. Hence one has to develop new tools for the higher-dimensional setting.

1. $\Leftrightarrow$ 2. : On $\complement K(P)$, by definition, we obtain convergence to infinity. The open bounded set $\mathring{K}(P)$ is mapped to itself, hence $\{P^k\}$ restricted to this set is also normal. A point z in $\partial K(P)$ has a bounded forward orbit, but any neighbourhood contains an open set from $\complement K(P)$, hence we cannot get normal convergence there.

1. $\Leftrightarrow$ 3. : It is obvious that one does not get normal convergence on $\mathcal{R}$. Furthermore, let us note that the number of non-repelling periodic points is finite. Hence, it remains to show that we find a periodic point in any neighbourhood U of a point in the Julia set. We fix a point w in $U \cap J$ such that w is not a critical value of P^2. From $P^{-2}(w)$ (which has cardinality at least 4) we choose three points w_1, w_2, w_3 distinct from w. We find disjoint compact neighbourhoods Z, W_1, W_2, W_3 of w, w_1, w_2, w_3, resp., such that for each j, P^2 maps W_j homeomorphic to Z. With S_j we denote the inverses $S_j : Z \to W_j$. If, for all $n \geq 1$, all j, and all $z \in Z$, we have that $P^n(z) \neq S_j(z)$, then $\{P^n\}$ is normal in Z which contradicts $w \in Z \cap J$. But $P^n(z) = S_j(z)$ implies $P^{n+2}(z) = z$, thus we have found a periodic point in U.

2. $\Leftrightarrow$ 4. : From equation (1) we deduce that, for $r \geq r_P$, we also get that $\lim_{k\to\infty} \partial(P^{-k}\overline{B_r}) = \partial K(P)$. We should note that, for disks B_r, Shilov boundary and topological boundary coincide, whence also for $K(P)$.

1. $\Leftrightarrow$ 5. : This equivalence is non-elementary. It requires the application of Gromov's result ([Gro]) on topological entropy of holomorphic maps and of thermodynamic formalism. This makes a proof rather lengthy and technical. We refer to [Lyu83]. $\square$

3 The characterisation programme in $\mathbb{C}^n$

In order to extend the one-dimensional theory to higher-dimensional complex manifolds, consider polynomial maps $P : \mathbb{C}^n \to \mathbb{C}^n$. We are interested in determining the mappings for which there exists a non-empty set $J(P)$ which can be described in the following equivalent ways:

1. $J(P) := \{z \in \mathbb{C}^n : \{P^k : k \geq 0\}$ is not weakly normal at $z\}$.
2. $J(P)$ is equal to the Shilov boundary $\partial_{SH}(K(P))$ where $K(P) := \{z \in \mathbb{C}^n : \sup_k \|P^k(z)\| < \infty\}$.
3. $J(P)$ is equal to the closure $\mathcal{R}$ of the set of repelling periodic points of P.
4. $J(P)$ is equal to the limit of the pull backs by P^{-k} of the Shilov boundary $\partial_{SH}(\{z \in \mathbb{C}^n : \|z\| \leq r\})$, for $r > 0$ sufficiently large.
5. $J(P)$ is given by the support $\mathrm{supp}(\mu_{\max})$ of the measure of maximal entropy for P.

By consideration of the example $(z_1, \ldots, z_n) \mapsto (z_1^{d_1}, \ldots, z_n^{d_n})$ where $d_j \in \mathbb{N}$, one easily sees that normality alone is too weak if one wants to detect the set of maximal randomness. Therefore, we are using the term *weak normality.* The notion is defined as follows: A family of functions $\{f_k\}$ is called *weakly normal* in a point $z \in U$ if there is

– an open neighbourhood V of z;
– a family $\mathcal{C}_x$ of at least one-dimensional (complex) analytic sets indexed by the points $x \in V$,

such that

– each x lies in the corresponding analytic set C_x;
– for each $x \in V$ the family $\{f_k\}$ restricted to $\mathcal{C}_x \cap V$ is normal (including convergence to infinity).

The condition (5.) has been used in [Bri96] to define the Julia set. It is shown there that (3.) then follows if the map is hyperbolic. The same definition has been used in [Jon97]. In the projective setting (1.) has been used in [FS94] to define a Julia set, but no results were proven. Equivalence of these five conditions has been obtained for certain skew products (see definition in section 4) in [Hei94], [Hei96], [Hei98c], [Hei98b], and [Hei98a]. In addition, these papers contain results on hyperbolicity and parameter spaces (Mandelbrot set).

It is clear that the definition selects a set of maximal randomness. It is known from [DH98] that torus like maps of the form

$$P(x, y) = (x^2 + k(y), y^2 + l(x))$$

satisfy this characterisation as long as the norms of the polynomials $k(y)$ and $l(x)$ are sufficiently small in some neighbourhood of 0. In case $k = l = 0$ one obtains the product map

$$P(x, y) = (x^2, y^2).$$

The analysis of this map is easy: The Julia set equals the torus

$$\mathbb{T}^2 = \{(x, y) : |x| = |y| = 1\}$$

and P is expanding on $\mathbb{T}^2$. For small perturbations one expects stability: The Julia set is still homeomorphic to a torus and the map is still expanding. [Hei95] shows even more: For the class of mappings for which the characterisation theorem holds (which is considerably larger than just a perturbed family) we obtain a complete analogue of the one-dimensional result mentioned in section 2. The Julia set is homotopic to a torus and P acts as an expanding transformation on $J(P)$. Hence $(J(P), P)$ forms an expanding repeller and thus defines the complete scenery of thermodynamic formalism including its probabilistic structure for stationary sequences.

A polynomial map P from $\mathbb{C}^n$ to $\mathbb{C}^n$ of the form

$$P(z_1, \dots, z_n) = \left(\prod_{l=1}^{n} z_l^{a_{1l}}, \dots, \prod_{l=1}^{n} z_l^{a_{nl}} \right)$$

leaves the n-dimensional torus $\mathbb{T}^n$ invariant. The investigation of such maps can be carried out using the theory of torus automorphisms defined by the matrix (a_{ij}) (see [Lin78] and others). However, not all of these maps have a non-empty Julia set as defined in the charactarisation programme. $\mathbb{T}^n$ carries a measure of maximal entropy for P but it may be the case that no repelling periodic point exists. Therefore, an important preliminary question is to determine whether $J(P)$ is non-empty. There is a simple and easily verifiable criterion for this fact: A polynomial map $P : \mathbb{C}^2 \mapsto \mathbb{C}^2$ satisfies the regularity condition and is called a (p,q)*-regular map* (cf. [Hei94]) if there is

$$R > 0, p \in \mathbb{Q}_+^*, q \in \mathbb{N}, k_1, k_2 > 0$$

such that, for all z with $\|z\| > R$, we have that

$$k_1 \|z\|^p \leq \|P(z)\| \leq k_2 \|z\|^q. \tag{2}$$

In particular, the left hand side of (2) shows that the dynamics of P is compatible with the simple Alexandroff-compactification, i.e. it makes sense to work on $\mathbb{C}^N := \mathbb{C}^n \cup \{\infty\}$ (cf. [Hei96]). It is important to note that the approach here is different and more successful than that of using a compactification by complex manifolds, e.g. projective space or complex solenoids ([Hei98b]).

In general, a non-empty $J(P)$ is a highly irregular fractal set as computer simulations show. The most simple ones are those where the Julia set looks like a torus. More complicated objects are discussed in the next section.

4 J^*-continuity

Polynomial skew products of $\mathbb{C}^2$ have the form

$$P(x, y) = (p_y(x), q(y)). \tag{3}$$

As just explained, the only interesting Julia sets occur for the subclasses of polynomial maps satisfying the regularity condition. We had been mentioning that for the higher-dimensional case one has to develop new tools which can replace Montel's theorem.

For skew products, we have been developing the notion of J^*-continuity (cf. [Hei98c]) which on the one hand allows us to reduce the dimension of our dynamical systems and on the other hand has turned out to be essential in establishing the equivalence in the characterisation programme. First let us note that skew products of the form (3) contain an abundance of one-dimensional subsystems. For each $y \in K(q)$ we can fix the complex line $\mathbb{C} \times \{y\}$ and investigate the dynamical behaviour of the family

$$\mathcal{P}_y := \{p_y, p_{q(y)} \circ p_y, \dots\}.$$

This yields the pre-Julia sets $J_y^* := J(\mathcal{P}_y)$ which are obtained using one-dimensional techniques.

Definition 2. ***J*****-continuity [Hei98c] We call a skew product $P : \mathbb{C}^2 \to \mathbb{C}^2$ *J^*-continuous* if the map

$$\mathbb{J} : J(q) \to \mathbb{C}$$

which maps $y \in J(q)$ to its pre-Julia set J_y^* is continuous with respect to the Hausdorff distance d_H in $\mathbb{C}$.

If one knows that $\pi_2(J(P)) \subseteq J(q)$ then J^*-continuity yields as an immediate consequence that

$$J(P) = \bigcup_{y \in J(q)} J_y^* \times \{y\}.$$

Moreover, J^*-continuity also gives the equivalence of the characterisations (1.) to (4.) (for equivalence to (5.) cf. [Hei98a], [Hei99]).

Theorem 3. *For J^*-continuous P with $\pi_2(J(P)) \subseteq J(q)$ we have that*

$$J = \mathcal{R} = \partial_{SH} K(P) = \lim_{k\to\infty} P^{-k} \partial_{SH}(\overline{B_r}).$$

Proof. 1. $\Leftrightarrow$ 2. : For $(x^*, y^*) \in J$ and $\eta > 0$, we construct a peak function for $B_\eta(x^*, y^*)$

$$\Phi : K(P) \to \mathbb{C}.$$

There exists $\zeta > 0$, $\zeta \leq \eta/2$, such that $|y - y^*| < \zeta$ implies

$$d_H(J_y, J_{y^*}) \ < \ \eta/2.$$

We choose $y' \in \partial_{CH} J(q)$ (where ∂_{CH} denotes the Choquet boundary (which is a dense subset of the Shilov boundary) and $K_y := K(P) \cap (\mathbb{C} \times \{y\})$) with

$$|y' - y^*| \ < \ \zeta,$$

and find $x' \in \partial_{CH} K_{y'} \subseteq J_{y'}$ such that

$$|x' - x^*| \; < \; \eta/2.$$

For $0 < \delta < 1/2$, there is a peak function φ for $B_{\eta/2}(x')$ in $K_{y'}$ such that

$$\begin{aligned} \varphi(x') \quad &= \; 1 \; = \|\varphi\|_{\pi_1(K_{y'})}, \\ \|\varphi\|_{K_{y'} \setminus B_{\eta/2}(x')} &< \delta/4. \end{aligned}$$

φ can be approximated by a polynomial $\tilde{P}$ such that

$$\|\tilde{P} - \varphi\|_{K_{y'}} \; < \; \delta/4.$$

For every $0 < \tau < \infty$, the derivative $\tilde{P}'$ of $\tilde{P}$ is bounded on the disk B_τ, i.e. for some $\omega < \infty$, we have that

$$\|\tilde{P}'\|_{B_\tau} \; < \; \omega.$$

Let us fix τ such that $\pi_1(K(P)) \subseteq B_\tau$.
It is clear that $\tilde{P}$ still is a peak function for $B_{\eta/2}(x')$ in $K_{y'}$. This follows since

$$\begin{aligned} \|\tilde{P}\|_{B_{\eta/2}(x') \cap K_{y'}} &> 1 - \delta/4, \\ \|\tilde{P}\|_{K_{y'} \setminus B_{\eta/2}(x')} &< \delta/2. \end{aligned}$$

Now we choose $0 < \xi < \eta/2$, such that $|y - y'| < \xi$ implies

$$d_H(J_y, J_{y'}) \; < \; \min\{\delta/(4\omega), \eta/2\}.$$

Since $y' \in \partial_{CH} J(q)$ we find $Q \in \mathbb{A}(J(q)$ such that

$$\begin{aligned} Q(y') \qquad &= \qquad 1 \qquad = \|Q\|_{J(q)}, \\ \|Q\|_{J(q) \setminus B_\xi(y')} &\leq \delta/(1 + \delta/4 + 2\tau\omega). \end{aligned}$$

We obtain the desired peak function Φ for $B_\eta(x, y)$ by setting

$$\Phi(x, y) \; := \; \tilde{P}(x) \cdot Q(y),$$

namely

$$\|\Phi\|_{B_\eta(x^*, y^*)} \; \geq \; \|\tilde{P} \cdot Q\|_{B_{\eta/2}(x') \times B_\xi(y')} \; > \; (1 - \delta/4) \cdot 1 \; = \; 1 - \delta/4,$$

whereas

$$\|\Phi\|_{K(P) \setminus B_\eta(x^*, y^*)} \; \leq \; \max \begin{cases} \|\Phi\|_{K(P) \setminus (\mathbb{C} \times (J(q) \setminus B_\xi(y')))} \; < \; \delta, \\ \|\Phi\|_{(\mathbb{C} \setminus B_\eta(x, y)) \times B_\xi(y')} \; < \; 3\delta/4. \end{cases}$$

1. $\Leftrightarrow$ 3. : It is clear that we cannot have weakly normal convergence in a repelling periodic point, hence $\mathcal{R}$ is contained in J. In order to show

the inclusion of J in $\overline{\mathcal{R}}$ we make use of the fact that for one-dimensional Julia sets the assertion of the theorem holds. We find $\delta > 0$ such that, for arbitrary $(x_0, y_0) \in J$ and $\eta > 0$, $y' \in J(q)$ together with $|y' - y_0| < \delta$ implies $d_H(J_{y_0}, J_{y'}) < \eta/2$.
In $J(q)$ repelling periodic points of q are dense, we find y_1 in $J(q)$ which is k-periodic for some $k \in \mathbb{N}$ and repelling with

$$|y_1 - y_0| \;<\; \min\{\delta, \eta\}.$$

In J_{y_1} we find (x', y_1) such that

$$|x' - x_0| \;<\; \eta/2.$$

Finally, as in

$$J_{y_1} \;=\; J(\underbrace{p_{q^{k-1}(y_1)} \circ \ldots \circ p_{q(y_1)} \circ p_{y_1}}_{=:\, P_{y_1}}) \times \{y_1\}$$

repelling periodic points of P_{y_1} are dense, we find x_1 which is repelling periodic under P_{y_1} such that $|x_1 - x'| < \eta/2$. Evidently, (x_1, y_1) is a repelling periodic point of f and (using the maximum norm)

$$\begin{aligned} \|(x_1, y_1) - (x_0, y_0)\| &\leq \max\{|x_1 - x_0|, |y_1 - y_0|\} \\ &\leq \max\{|x_1 - x'| + |x' - x_0|, |y_1 - y_0|\} \\ &\leq \eta. \end{aligned}$$

1. $\Leftrightarrow$ 4. : This is an easy consequence of the one-dimensional statement as $\lim_{k\to\infty} \pi_2(P^{-k}(\overline{B_r})) = J(q)$. □

Examples of skew products where the complete characterisation programme can be verified include Cantor skews ([Hei96]), certain noodle type maps ([Hei98c]), homogeneous maps ([Hei98b]) and doughnut type maps ([Hei98a]).

5 Thermodynamical formalism for skew products

Consider a polynomial skew product P of $\mathbb{C} \times \mathbb{C}^n$ where $n \geq 1$, which factors over the base $\mathbb{C}$ and the polynomial $p : \mathbb{C} \to \mathbb{C}$. The projection $\pi : \mathbb{C}^{n+1} \to \mathbb{C}$ commutes with P and p. Assume that the Julia set defined by weak normality is nonempty and completely invariant.

In this situation one can establish a relativised thermodynamic formalism. Recall that $J_z = \pi^{-1}(\{z\}) \cap J(P)$ denotes the fibre over $z \in J(p)$, and the skew product P induces canonical maps between the fibres

$$P_z : J_z \to J_{p(z)}$$

whence a relativised thermodynamic formalism for a potential $\Psi : J(P) \to \mathbb{R}_+$ is well defined by the transfer operators

$$V_z \ : \ C(J_z) \to C(J_{p(z)})$$
$$V_z\varphi(y) = \sum_{\substack{P(y')=y;\\ \pi(y')=z}} \varphi(y')\Psi(y').$$

(In case y is a critical point, take preimages according to their multiplicity, so that the operators act on continuous functions.) A real valued function $\lambda : \mathbb{C} \to \mathbb{R}$ is called an *eigenvalue-function* if there exists a family μ_z $(z \in J(P))$ of probability measures μ_z on J_z so that

$$V_z^*\mu_{p(z)} = \lambda(z)\mu_z \qquad z \in J(p).$$

These families are called *Gibbs* and are a basic notion for the relativised thermodynamic formalism.

A weakly continuous Gibbs family (i.e. for every bounded continuous function $g : J(P) \to \mathbb{R}$ the map $z \mapsto \int g(y)\mu_z(dy)$ is continuous) can be characterised by the property that the Jacobian satisfies the equation $\frac{d\mu_{p(z)} \circ P_z}{d\mu_z} = \frac{\lambda(z)}{\Psi(y)}$. Let ν be a Gibbs measure for p and the potential $\phi : J(p) \to \mathbb{R}_+$, that is

$$V^*\nu = \nu,$$

where $Vf(z) = \sum_{p(y)=z} f(y)\phi(y)$ denotes the *Frobenius-Perron operator* for p and ϕ. Then the probability measure $\mu = \int_{\mathbb{C}} \mu_z\nu(dz)$ has two Radon-Nikodym derivatives

$$\frac{d\mu \circ P_z}{d\mu} = \frac{\lambda(z)}{\Psi(\cdot)} \qquad \mu_z \text{ a.e.}$$
$$\frac{d\mu \circ \pi^{-1} \circ p}{d\mu \circ \pi^{-1}} = \phi^{-1} \qquad \mu \circ \pi^{-1} \text{ a.e.}$$

which may be considered to be partial derivations of the measure μ with respect to the two actions on $J(P)$.

The expanding property of P depends on the metric, but equivalent metrics do not change this property. Here we choose the metric $\|x - y\| = \sum_{i=1}^{n+1} |x_i - y_i|$, and P is *expanding (hyperbolic)* if there exists $\Lambda > 1$ such that $\|P(z) - P(y)\| \geq \Lambda\|z - y\|$.

Theorem 4. *Let $J(P) \neq \emptyset$ be a bounded Julia set of the hyperbolic regular skew product $P : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ such that $\pi(J(P)) = J(p)$. For any Hölder continuous potential $\Psi : J(P) \to \mathbb{R}_+$ there exists a weakly Hölder continuous Gibbs family*

$$z \mapsto \mu_z,$$

and the eigenvalue-function $\lambda : J(p) \to \mathbb{R}$ is Hölder continuous.

Proof. Since $P|_{J(P)}$ is hyperbolic, there are no critical points in $J(P)$, hence the number of preimages of P_z is constant (the degree of P_z) by regularity. Also, as is easily seen after possibly passing to an equivalent metric, the fibre maps P_z are uniformly expanding and open (as maps $J_z \to J_{p(z)}$). The factor map $\pi : \mathbb{C}^{n+1} \to \mathbb{C}$ is a contraction, i.e. $|\pi(z_1) - \pi(z_2)| \le \|z_1 - z_2\|$ holds for all $z_1, z_2 \in \mathbb{C}^{n+1}$. Since P is hyperbolic (expanding) on its Julia set, the system $(J(P), J(p), \pi, P, p)$ is relatively Hölder continuous in the sense that for some $\alpha, b, c > 0$ the following holds:
If $y, y_0' \in \mathbb{C}^{n+1}$, $x' \in \mathbb{C}$ are such that $p(x') = \pi(y_0')$, $|\pi(y) - x'| < 2b$, $\|P(y) - y_0'\| < 2c$, then there exists a unique $y' \in \mathbb{C}^{n+1}$ such that $\pi(y') = x'$, $P(y') = y_0'$, $\|y - y'\| < 2c$, and

$$\|y - y'\| \cdot |\pi(P(y)) - \pi(P(y'))|^\alpha \le \|P(y) - P(y')\| \cdot |\pi(y) - \pi(y')|^\alpha. \quad (4)$$

As a hyperbolic map P is open and locally invertible, there is some $c' > 0$ (small enough) such that $P : B(z, c') \to \mathbb{C}^{n+1}$ is injective, whence there exists $c > 0$ such that each inverse branch P_z^{-1} of P sending $P(z)$ to z is well defined on balls of the form $B(P(z), 2c)$. Let $P(y)$, $x \in \mathbb{C}$ and $y_0' \in B(P(y), 2c)$ with $\pi(y_0') = p(x)$ be given. Then $y' = P_y^{-1}(y_0')$ is uniquely determined by the condition $\|y' - y\| < 2c$, which also implies $\pi(y') = x$ and $|\pi(y) - x| \le 2c$. Since p is continuous there exists there exists a constant K such that $|p(t) - p(s)| \le K|t - s|$ for $t, s \in J(p) = \pi J(P)$, and since P is expanding, there exists $\Lambda > 1$ such that $\|P(z) - P(z')\| \ge \Lambda \|z - z'\|$ whenever $z' \in B(z, 2c)$. Choose α so that $\Lambda^{-1} K^\alpha \le 1$. It follows that equation (4) holds.

The theorem follows from this by application of Theorem 2.10 in [DG99]. □

This is a typical structure theorem for probability measures which are nonsingular with respect to polynomial skew products and have given conditional Jacobians. In the non-hyperbolic case, say if the polynomial is only assumed to be expansive, a change of the metric makes the map expanding (by a result essentially due to Coven and Reddy). Therefore the above theorem applies proving the existence of continuous Gibbs families. Specific examples are given in [DGH]. The reader may also notice that the above theorem strengthens the result of Jonsson in [Jon97] where a fibred structure is put on the measure of maximal entropy using a decomposition of the current defined by dd^c. Naturally, one also gets such a decomposition in more general situations.

In case of a constant potential Ψ the Gibbs family is invariant. Integrating this family with respect to the measure of maximal entropy for the base map p defines an invariant measure for P, which is called a measure of maximal entropy. In the hyperbolic case, it is unique (as a measure on $J(P)$), and possibly also as a measure on $\mathbb{C}^{n+1}$. The natural generalisation of this is the question whether invariant measures exist which are fibrewise absolutely continuous with respect to the Gibbs family for a Hölder continuous potential Ψ. This problem leads to new aspects for the theory of equilibrium states in higher dimensions. If p would be invertible, classical techniques can be used to prove the existence of such measures. However, p is not invertible and one needs to introduce new concepts. Following [DGH] a potential Ψ is called *basic* if for each $x \in J(p)$ and $z \in \pi^{-1}(\{p(x)\} \cap J(P))$

$$V_x 1(z) = \lambda(x),$$

where λ denotes the eigenvalue-function as before (see also section 3 in [DG98]). For potentials of this type one can reduce the variational problem to the relativised problem (see [LW76]) and the variational problem for p and the potential λ.

For a measure preserving transformation T of a probability space $(\Omega, \mathcal{B}, m)$ denote the metric entropy by $h_m(T)$ and the relative entropy of T given the factor S by $h_m(T|S)$. For a compact metric space Ω with Borel σ-field $\mathcal{B}$ we let

$$\mathbf{P}(T, g) = \sup\{h_m(T) + \int g dm : \ m \circ T^{-1} = m\}$$

denote the pressure of the continuous function $g \in C(\Omega)$.

In the situation of the theorem one can show that for a basic potential Ψ with eigenvalue-function λ a fibrewise absolutely continuous invariant measure m exists satisfying the relative variational principle:

$$h_m(P|p) + \int \Psi(y) m(dy) = \int \lambda dm \circ \pi^{-1},$$

and the variational problem for P reduces to that of p:

$$\sup\{h_\nu(P) + \int \Psi d\nu : \nu \circ P^{-1} = \nu\} = \sup\{h_\rho(p) + \int \lambda d\rho : \rho \circ p^{-1} = \rho\}.$$

It follows that the pressure for P and Ψ can be derived from p and λ:

$$\mathbf{P}(P, \Psi) = \mathbf{P}(p, \lambda).$$

A large class of example falls into this category: Maps of doughnut type (see [Hei99]), or Cannelloni or Spaghetti type (cf. [Hei98c]) as explained above. Details can be found in [DGH].

6 Questions and open problems

In dimension one the ergodic theory is completely described if the rational map $f : S^2 \to S^2$ is hyperbolic (expanding) or expansive on its Julia set $J(f)$. It is not hard to show that f is hyperbolic if and only if $J(f)$ does not contain any critical point nor any parabolic point. (Recall that a periodic point z with period p is parabolic if $(f^p)'(z)$ is a root of unity.) Also, it is known that f is expansive if and only if $J(f)$ does not contain any critical point.

In dimension ≥ 2 hyperbolicity used in a more general meaning is not the same as the expanding property. Hénon maps are invertible, and they are never expanding (their Julia sets in our sense here is always empty) but can be hyperbolic. Thus we have the first problem:

Problem 5. Which skew products $P : \mathbb{C}^n \to \mathbb{C}^n$ admit a completely invariant set on which P is expanding?

Examples when this is true were described in previous sections. Consider a polynomial skew product

$$P\binom{x}{y} = \binom{p_y(x)}{q(y)}.$$

A point $\binom{x}{y}$ is called parabolic if y is parabolic for q or if y is periodic (with period m say) and if x is parabolic for P_y (cf. section 4). Denote by $\mathcal{C}$ the set of critical points for P. It is not difficult to see that P is expanding if and only if $\overline{O^+(\mathcal{C})} \cap J(P) = \emptyset$, where $O^+(\mathcal{C})$ denotes the forward orbit of $\mathcal{C}$. Thus an expanding P cannot have a critical or a parabolic point.

Problem 6. Is a regular skew product $P(\binom{x}{y}) = \binom{p_y(x)}{q(y)}$ (together with its Julia set) expanding if and only if the omega-limit set of the critical locus is disjoint from $J(P)$?

As a consequence of this classification on obtains the existence of unique equilibrium states: Let $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ be a Hölder continuous function. Then there exists a probability measure μ with

$$\mathbf{P}(P, \varphi) := h_\mu(P) + \int \varphi d\mu.$$

Again, in one dimension such equilibrium measures (Gibbs measures) always exist for rational functions f and are unique for Hölder continuous φ satisfying $\mathbf{P}(f, \varphi) > \sup_{z \in J(f)} \varphi(z)$.

Problem 7. If P is a regular skew product in $\mathbb{C}^n$ and φ Hölder continuous with $\mathbf{P}(P, \varphi) > \sup_{z \in J(P)} \varphi(z)$, does there exists a unique invariant Gibbs measure?

Recall that P is expansive on $J(P)$ (more precisely positively expansive) if there exists $\varepsilon > 0$ such that

$$\sup_{n \geq 0} |P^n(x) - P^n(y)| > \varepsilon$$

for all $x \neq y \in J(P)$. Again, in one dimension, a rational map is expansive if and only if its Julia set does not contain any critical point. For these maps the ergodic theory is well developed, hence one needs to characterise this class in higher dimensions.

Problem 8. Which regular polynomial $P : \mathbb{C}^n \to \mathbb{C}^n$ admit a completely invariant set in which P is open and expansive?

If one considers a regular skew product

$$P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p_y(x) \\ q(y) \end{pmatrix},$$

then P is not expansive on $J(P)$ if $J(P)$ contains a critical point. Conversely, there are reasons to believe that P is expansive if $J(P)$ does not contain a critical point and

$$\overline{O^+(\mathcal{C}_x)} \cap J(P)$$

is finite. Here

$$\mathcal{C}_x = \{(x,y) : P_y'(x) = 0\}$$

denotes the critical locus over the base.

Problem 9. Is a regular skew product $P\binom{x}{y} = \binom{p_y(x)}{q(y)}$ expansive (on $J(P)$) if and only if $\overline{O^+(\mathcal{C}_x)} \cap J(P)$ is finite and $J(P)$ does not contain any critical point?

If this is true, new dynamics occur for higher dimensional maps. Then there are non–expansive maps with no critical point in its Julia set. On the other hand, an open and expansive map is – after changing the metric – expanding in Ruelle's sense. Hence there exist Markov partitions for P. Since we also have an $\exp[\mathbf{P}(P,\varphi)-\varphi]$ conformal measure μ, the dynamical system $(J(P), P, \mu)$ forms a Markov fibred system (see [ADU93]).

It is not known whether skew products of $\mathbb{C}^n$ admit new probabilistic structures different from those well known arising in the one-dimensional case or in the case of group extensions. Certainly, the thermodynamic formalism as part of the dynamic behaviour is different. Thus one can expect some refined probabilistic theorems respecting canonical filtrations given by the dynamics. On the other hand, the existence of an infinite number of parabolic points may cause different spectral behaviour of the transfer operators and hence different ergodic theorems.

The fractal geometry of skew products on $\mathbb{C}^n$ for $n \geq 2$ does not fall into category of self–similarity. It is known that even simple affine maps in higher dimension need not to have equal Hausdorff– and box–dimensions. Also for skew products

$$P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p_y(x) \\ q(y) \end{pmatrix}$$

one certainly obtains new phenomena when the Hausdorff dimension on fibres $J(P)\cap(\mathbb{C}\times\{y\})$ is not continuous. In fact, for the skew products as discussed previously, this can only happen for product maps. On the other hand, it is not clear whether the usual dynamic representations for the Hausdorff–dimension are valid (as the smallest zero of the pressure function). Thus we obtain the following question.

Problem 10. Characterise those regular skew products $P\binom{x}{y} = \binom{p_y(x)}{q(y)}$ for which the fibres $J_x = J(P) \cap \{x\} \times \mathbb{C}$ are continuous in the Hausdorff metric (J^*-continuity). Compute the Hausdorff dimension in this case.

References

[ADU93] J. Aaronson, M. Denker, and M. Urbański. Ergodic theory of Markov fibred systems and parabolic rational maps. *Transactions American Mathematical Society*, 337:495–548, 1993.

[Bri96] J.-Y. Briend. Exposants de Liapounoff et de points périodiques d'endomorphismes holomorphes de $\mathbb{C}P^k$. *Comptes Rendus d'Academie des Seances Paris*, 323:805–808, 1996.

[DG98] M. Denker and M. Gordin. Problems on complex dynamical systems. In S. Morosawa, editor, *RIMS Workshop*, volume 1042, pages 1–10, 1998.

[DG99] M. Denker and M. Gordin. Gibbs measures for fibred systems. *to appear: Advances in Mathematics*, 1999.

[DGH] M. Denker, M. Gordin, and St. M. Heinemann. Conditional equilibrium measures and Gibbs families. preprint.

[DH98] M. Denker and St. M. Heinemann. Jordan Tori as Julia sets in $\mathbb{C}^n$. *Fundamenta Mathematicae*, 157:139–159, 1998.

[DHP95] M. Denker, St. M. Heinemann, and S. J. Patterson. Erinnert stark an abstrakte Malerei. *forschung*, 4:18–21, 95.

[FS94] J. E. Fornæss and N. Sibony. Complex dynamics in higher dimension, I. *Astérisque*, 222:201–231, 1994.

[Gro] M. Gromov. On the entropy of holomorphic mappings. *Preprint d'Institute des Hautes Etudes scientifiques.*

[Hei93] St. M. Heinemann. Iteration holomorpher Abbildungen in $\mathbb{C}^n$. *Diplomarbeit Universität Göttingen*, 1993.

[Hei94] St. M. Heinemann. Dynamische Aspekte holomorpher Abbildungen in $\mathbb{C}^n$. *Dissertation Universität Göttingen*, 1994.

[Hei95] St. M. Heinemann. Jordan tori as Julia sets in $\mathbb{C}^n$. *Mathematica Gottingensis*, 44, 1995.

[Hei96] St. M. Heinemann. Julia sets for endomorphisms of $\mathbb{C}^n$. *Ergodic Theory and Dynamical Systems*, 16:1275–1296, 1996.

[Hei98a] St. M. Heinemann. Iteration of Holomorphic Endomorphisms in $\mathbb{C}^n$ – A Case Study, 1998. Habilitationsschrift, Universität Göttingen.

[Hei98b] St. M. Heinemann. Julia sets in $\mathbb{C}^n$. *Progress in holomorphic dynamics*, 387:159–185, 1998.

[Hei98c] St. M. Heinemann. Julia-sets of skew products in $\mathbb{C}^2$. *Kyushu Journal of Mathematics*, 52(2):299–329, 1998.

[Hei99] St. M. Heinemann. Shilov boundary, dynamics and entropy in $\mathbb{C}^2$. *Mathematica Gottingensis*, 6, March 1999.

[Jon97] M. Jonsson. *Dynamical Studies in Several Complex Variables.* PhD thesis, Royal Institute of Technology, Stockholm, 1997.

[Lin78] D. Lind. Ergodic group automorphisms and specification. In M. Denker and K. Jacobs, editors, *Ergodic Theory*, volume 729 of *Lecture Notes in Mathematics*, pages 93–104, Oberwolfach, 1978.

[LW76] F. Ledrappier and P. Walters. A relativised variational principle for continuous transformations. *Journal London Mathematical Society*, 10:568–576, 1976.
[Lyu83] M. Yu. Lyubich. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergodic Theory and Dynamical Systems*, 3:351–385, 1983.

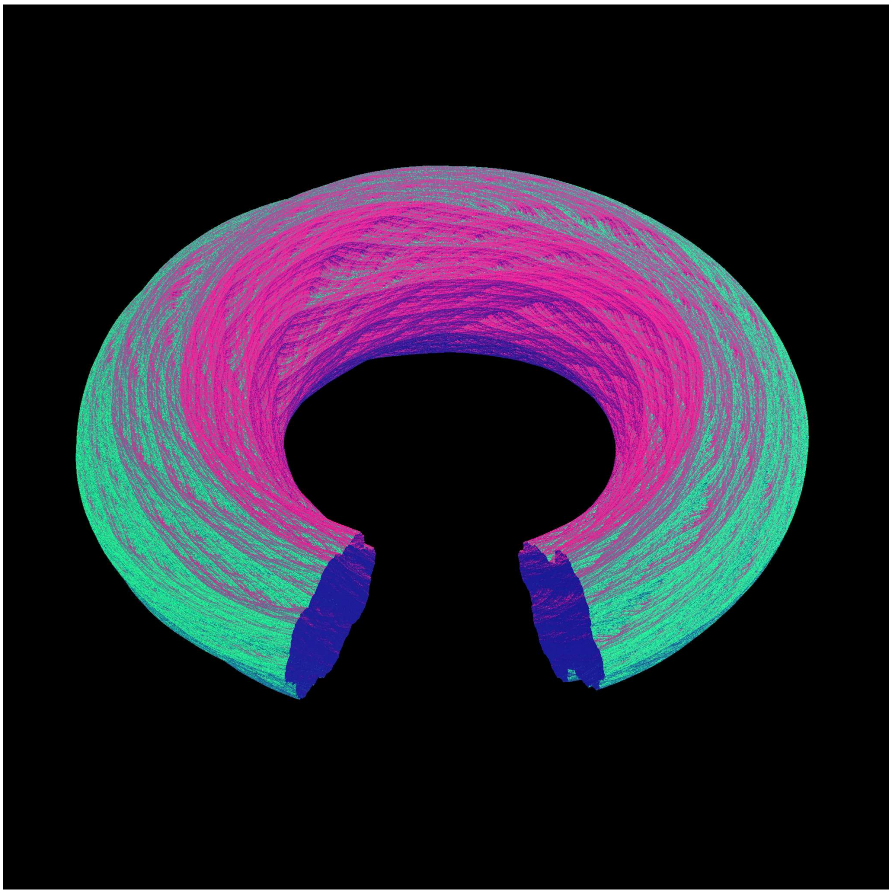

Fig. 1. The Julia Set of the Doughnut Map $(x, y) \mapsto (x^2+(1+i)\cdot(y^2-y)/10-1/5, y^2)$ (cut open at $\Im(y) > 9/10$).

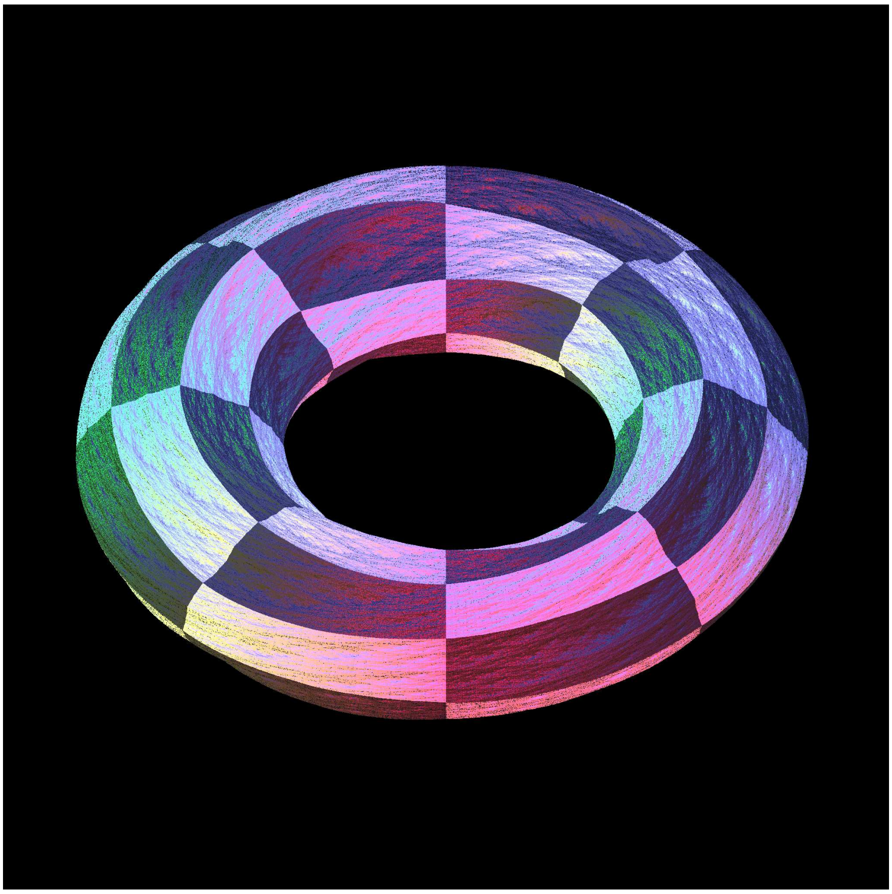

Fig. 2. The Julia Set of the Doughnut Map $(x, y) \mapsto (x^2+(1+i)\cdot(y^2-y)/10-1/5, y^2)$ with the canonical Markov partition (here, of iteration order 3) induced by the dynamics.

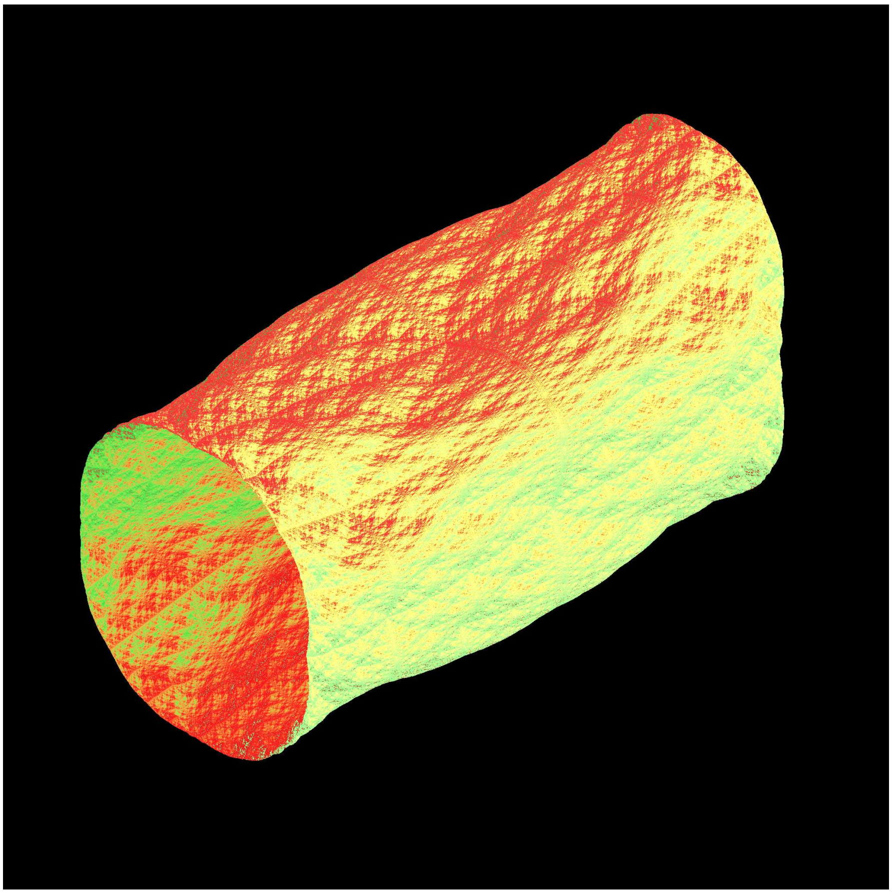

Fig. 3. The Julia Set of the Skew Product of Cannellono Type $(x, y) \mapsto (x^2 + (y^2 - y - 1)/10, y^2 - 2)$.

Fig. 4. The Julia Set of the Skew Product $(x, y) \mapsto (x^2 - (y^2 - y - 1)/10, y^2 - 21/10)$.