On the relative variational principle for fibre expanding maps

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Abstract

A fibred dynamical system whose fibre maps are uniformly expanding and exact, possesses, for every Hölder continuous potential, a Gibbs family of conditional measures on its fibres [DG2]. Such a family is constructed by means of the relative transfer operator. It is investigated whether the relative variational principle may be reduced to the study of this operator (which happens to be the case if non-fibred expanding systems are considered).

On the one hand it turns out that the maximal value for the free energy in the relative variational problem can be represented in terms of the transfer operator. On the other hand, for a general potential, the possibility to reduce the construction of an equilibrium measure to the search for an appropriate family of conditional measures on the fibres critically depends on the invertibility of the base transformation.

A certain class of potentials (called *basic)* which allow the above-mentioned reduction is introduced, and the properties of the corresponding equilibrium measures are studied. Any measure of this kind gives rise to

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a *regular factor*; under a natural assumption the latter property is shown to be equivalent to the validity of the relative version of Rokhlin's formula for the entropy of a measure preserving transformation. Several examples are presented, among them families of polynomial skew products in $\mathbb{C}^n$ (cf. [H3]).

1 Introduction

Let T be a continuous map of a compact space Y onto itself, φ a continuous real valued function on Y and μ a T-invariant probability measure on Y. Let $P_T(\varphi)$ and $h_\mu(T)$, respectively, denote the pressure of the function φ with respect to T and the measure theoretical entropy of T with respect to μ (see [Bo], [DGS] or [Ro1]). Then the following variational principle holds.

$$\sup_{\mu\in M(T)}\left\{h_\mu(T)+\int_Y\varphi(y)\mu(dy)\right\}=P_T(\varphi), \tag{1}$$

where $M(T)$ denotes the set of all T-invariant probability measures on Y. If, for $\mu\in M(T)$, it holds that

$$P_T(\varphi)=h_\mu(T)+\int_Y\varphi(y)\mu(dy), \tag{2}$$

then μ is called an *equilibrium measure* for the *potential* φ with respect to T. Recall that in this setting $h_\mu(T)+\int_Y\varphi(y)\mu(dy)$ is the *free energy* of the probability measure (or *state*) μ (with respect to the potential φ). One can characterise the equilibrium measures as those probability measures which maximise free energy. Under suitable additional assumptions on T like expansiveness (as defined below) the existence of an equilibrium measure μ is well known. Uniqueness and the Gibbs property of equilibrium measures require much stronger assumptions on T and φ. These topics constitute a part of the theory of thermodynamic formalism.

For a wide class of non-invertible locally expanding maps the main technical tool is the *transfer operator* (also called *Frobenius-Perron* or *Ruelle operator*). This operator is defined on a certain Banach space C of functions g by

$$(V_\varphi g)(x)=\sum_{y:T(y)=x}g(y)\exp(\varphi(y)) \tag{3}$$

(We shall omit the index φ and simply write Vg in the sequel). In order that this definition makes sense, we shall restrict ourself to maps T which are finite-to-one. The dual operator V^* then acts on the space $M=C^*$. The pattern for the results is given by Ruelle's theorem for transfer operators (see [Bo]). In this case C and M are the spaces of continuous real-valued functions on Y, of bounded signed measures on Y, respectively. If φ is Hölder continuous, then the theorem asserts that, under certain natural assumptions on T, V and V^* have a positive eigenfunction $h \in C$ and a probability eigenmeasure $\mu \in M$, respectively, with a common eigenvalue $\lambda > 0$. Furthermore, for any $f \in C$, it holds that

$$\lim_{n\to\infty} \lambda^{-n}(V^n f) = h \int_Y f d\mu,$$

provided that h fulfils the norming condition $\int_Y h d\mu = 1$.

In addition, it holds that λ is a simple eigenvalue for both operators and that $P_T(\varphi) = \log\lambda$. Moreover, the subspace of Hölder continuous functions which satisfy $\int_Y f\, d\mu = 0$ is V-invariant, and the spectral radius of the restriction of V to this space is strictly less than λ. Multiplication of μ by h gives an invariant probability measure $h\mu$ for T which is equivalent to μ.

The part of the theorem dealing with the operator V^* and its eigenmeasure μ can be viewed as a result on the existence and uniqueness of Gibbs measures (not necessarily T-invariant). The crucial assumption is that T is an *expanding* map. Generalisations of this result appear in many papers.

One of important consequences of these results is that they allow, when applicable, to include the pressure into the framework of operator theoretic and analytic considerations.

In the present note the above questions are studied in a relativised setting. We consider a fibred situation where the map T is foliated over a continuous map $S : X \to X$ on some compact space X, with a continuous *factorisation map* $\pi : Y \to X$ which semi-conjugates T to S

$$\begin{tikzcd} Y \arrow[r, "T"] \arrow[d, "\pi"'] & Y \arrow[d, "\pi"] \\ X \arrow[r, "S"] & X. \end{tikzcd}$$

X is called the *base space* and S the *base transformation* (we always suppose that T, S and π are surjective). The non-relativised case is obtained by consideration of a one point space X.

Such relativised situations are considered in the literature. Ledrappier and Walters ([LW]) extended the variational principle (1) to the fibred case. They proved that it holds that

$$\sup_{\mu\in M(T,\nu)} \left\{ h_\mu(T|S) + \int_Y \varphi\ d\mu \right\} = \int_X P_T(\varphi, Y_x)\nu(dx), \tag{4}$$

where $\nu \in M(S)$, $M(T,\nu) = \{\mu : \mu \in M(S), \mu\circ\pi^{-1} = \nu\}$, $Y_x = \pi^{-1}(\{x\})$ for $x \in X$, $h_\mu(T|S)$ denotes the mean relative entropy of T with respect to S, and $P_T(\psi, Z)$ denotes the relative pressure of a function $\psi : Z \to \mathbb{R}$ with respect to T and a closed set $Z \subset Y$. In analogy to the non-fibred case, the quantity $h_\mu(T|S) + \int_Y \varphi \, d\mu$ is called the *relative free energy* (of μ). Furthermore, the probability measures μ which maximise this expression are called *relative equilibrium measures*. The problems of determining the maximum (or supremum) of free energy, existence, uniqueness and other properties of equilibrium measures are of obvious interest. In particular, it is a natural question to ask to what extent the relative equilibrium measure may be constructed 'fibrewise', i.e. without consideration of the base transformation. More precisely, any probability measure μ on Y projects to the measure $\mu \circ \pi^{-1}$ on X and gives rise to a family $\{\mu_x : x \in X\}$ of conditional probability measures on the fibres. This correspondence is one-to-one and onto if the family $\{\mu_x : x \in X\}$ is considered modulo $\mu \circ \pi^{-1}$. It is clear from the definition that $\mu \in M(T, \nu)$ can only vary in the family $\{\mu_x : x \in X\}$. This suggests the question whether it is possible to reduce the relativised variational problem to a problem merely concerning the system $\{\mu_x : x \in X\}$ (which gives rise to a T-invariant measure on Y).

The main goal of this paper is to give, under appropriate assumptions, a description of the relative maximum of the free energy in terms which are closely related with the fibrewise transfer operator (see [DG2]). Our assumptions are mostly concerned with properties of T restricted to the fibres. We present a formula for the maximal value of the relative free energy which seems to be more appropriate for our setting than the general result in [LW]. More precisely, the maximal value is equal to the integral of the logarithm of the *gauge function* (a function defined on the base space which plays in the relativised setting the role of a substitute for the eigenvalue λ) with respect to a prescribed invariant measure on the base space.

It is worthwhile to compare this result with (4). According to the latter formula, the maximal value of the relative free energy is equal to the integral of the relative pressure. Thus, two different functions, the logarithm of the gauge function and the relative pressure generated by the same potential, have the same integrals with respect to each measure in $M(S)$. This gives rise to the question about the precise relation between these two functions. Whether they are additively cohomologous (which property implies the coincidence of the integrals mentioned above) is an open question for a general S (for hyperbolic S Livshits' theorem suggests an affirmative answer).

Concerning the description of the equilibrium measures, it turns out (rather unexpectedly), that the situation differs considerably depending on whether S is invertible or not. In case of an invertible S, the relative variational problem (4) admits a complete reduction to a variational problem for the conditional measures μ_x. Moreover, the equilibrium system $\{\mu_x : x \in X\}$ of measures on fibres does not depend on the prescribed S-invariant measure ν and is completely determined by the function φ. This is not the case if S is non-invertible. For general S, the reduction is possible only for a special class of potentials which we call *basic potentials*.

The reason for this difficulty is purely 'functorial'. The conjugated relative transfer operator acting on the measures on the fibres can also be considered as a global operator acting on the measures on the space Y. However, the relative transfer operator itself acting on the functions on the fibres may be considered as globally defined on some function space on Y only under the rather restrictive assumption of invertibility of the base transformation S. More precisely, the relative transfer operator is given by a family of operators between spaces of all bounded Borel measurable functions B_x on the fibres Y_x. For $x \in X$ and $k \geq 1$ the operators

$$V_x^{(k)} : B_x \to B_{S^k(x)}$$

are defined by

$$(V_x^{(k)} g)(y) = \sum_{\substack{T^k(y')=y \\ \pi(y')=x}} g(y') \exp\left(\sum_{j=0}^{k-1} \varphi(T^j(y'))\right), \tag{5}$$

where $y \in Y_{S^k(x)}$. Under suitable continuity assumptions $V_x^{(1)}$ preserves continuous functions and $V_x^{(1)*}$ maps measures of bounded total variation on $Y_{S(x)}$ to measures of bounded total variation on Y_x. Unlike for $V_x^{(1)*}$, there is no is natural way to derive a global operator on B_Y from the $V_x^{(1)}$, except for the case when S is invertible. Here, B_Y denotes the space of all bounded Borel measurable functions on Y. The definition in (5) depends on the choice of the preimage of $\pi(y)$ under S.

Let us note that for invertible base maps S the relative transfer operator coincides with the usual one. In this special case (for a fixed invariant probability measure on the base) some partial generalisation of Ruelle's theory was developed in [BG]. On the other hand, in the non-invertible case we showed in [DG2] existence, uniqueness and continuity of Gibbs families of probability measures on the fibres which partially generalises Ruelle's result in a different direction.

We now give a brief survey of the contents of the paper. In paragraphs **2** and **3** we present necessary tools for the proof of our main theorem **4.5** which describes the equilibrium states for a fibred system with respect to invariant measures on the base. More precisely, in section **2.1** we recall some notation and facts concerning the thermodynamics of fibred systems, in particular, we derive a representation for the gauge function. We establish in paragraph **2.2** that for an *invertible base map* S and *basic potentials* φ the conditional equilibrium measures on the sets of preimages of points over fixed fibres of π (i.e. on sets of the form $T^{-1}(y) \cap \pi^{-1}(x)$ with $S(x) = \pi(y)$) are uniquely determined by the potential. Moreover, it is shown, without the assumption of invertibility of S, that adding a coboundary to an arbitrary potential may change the corresponding gauge function only within its cohomology class. For an invertible S and a given potential φ, this allows to reduce the calculation of the free energy in terms of the gauge function to the case of a basic potential cohomologous to φ.

In paragraph **2.3** we make some step to reduce, by means of the construction of a *natural base extension* from [DG2], the proof of theorem **4.5** to the case of invertible base transformations.

In paragraph **3** we contribute (in a purely measure theoretical setup) some results to the relativised entropy theory. In particular, the relative Abramov-Rokhlin theorem **3.13** generalises the classical Abramov-Rokhlin formula found in [AbRo] and [LW]. More results concerning this topic are to be found in [Sch]. The main result of this paragraph is tightly related with [Ro]. V.A. Rokhlin proved in [Ro2] a remarkably simple integral representation for the entropy of a measure preserving transformation having a one-sided generator of finite entropy. We investigate in this paragraph under which assumptions a relative analogue of this formula holds. We give various criteria for this property. It turns out that the literal fibred analogue of Rokhlin's requires very special assumptions about the fibred transformation (called in ergodic theory *extension*) under consideration. Equivalently, we arrive at a special class of factors of a measure-preserving transformation which we call *regular* factors. They have very simple probabilistic interpretation in terms of conditional independence. On the other hand, from a general viewpoint they may be described as a class of Cartesian squares. Without any connection with measure preserving transformations an analogous concept was investigated in the theory of classification of monotonic sequences of σ-fields under the name of *immersed* filtrations [BSÉ]. Besides the application of this in the next section, it seems to be of some independent interest.
Paragraph **4** contains our main result:

$$\int_X P_T(\varphi, Y_x)\nu(dx) = \int_X \log A_\varphi d\nu,$$

where A_φ denotes the gauge function of the potential φ (see (6)). Also, a class of basic potentials is introduced here. It is shown that, together with an arbitrary measure $\nu \in M(S)$, each potential of this type leads to an S-invariant measure μ on the space Y such that (X, S, ν) is a regular factor of (Y, T, μ).

Finally, in section **5** we present a large class of polynomial skew products in $\mathbb{C}^n$ for which our assumptions are satisfied, and where one can establish a relativised thermodynamic formalism. Likewise this applies to rational semigroups provided they are finitely generated and studied in he context of skew products whose base is given by a sequence space.

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2 Fibred systems and conditional measures

2.1 Gibbs families for fibre expanding maps, relative transfer operators and gauge functions

We begin by recalling the notion of a fibred system as introduced in [DG2] (see also [DG1]). Let X and Y be compact metric spaces with metrics d_X and d, respectively. Let $S : X \to X$, $\pi : Y \to X$ and $T : Y \to Y$ be continuous surjective maps such that $\pi \circ T = S \circ \pi$. The system $\mathcal{Y} = (Y, T, X, S, \pi)$ is called a (compact) *fibred system*. It is called *fibre expanding*, if the fibre maps $T_x : Y_x = \pi^{-1}(\{x\}) \to Y_{S(x)}$ are uniformly expanding in Ruelle's sense (see [Ru]). This means that there exists $a > 0$ and $\lambda \in (0, 1)$ such that the following holds.

If $u, v' \in Y, \pi(T(u)) = \pi(v')$ and $d(T(u), v') < 2a$, then there exists a unique $v \in Y$ such that $\pi(v) = \pi(u), \quad T(v) = v'$ and $d(u, v) < 2a$. Furthermore, we have that

$$d(u, v) \leq \lambda d(T(u), T(v)).$$

Let us now fix a fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ where the map is bounded-to-one and where the fibres contain at least two points. Throughout the paper we make the assumption that T is bounded-to-one on fibres. This means there exists a constant $M > 0$ such that, for all $y \in Y$, we have that

$$\mathrm{card}\{y' \in Y : \; T(y) = T(y'); \; \pi(y) = \pi(y')\} \leq M,$$

in other words, the number of preimages for each fibre map is bounded by M.

The fibred system $\mathcal{Y}$ is called *topologically exact along fibres* if, for every $\varepsilon > 0$, there is an N such that, for any $y \in Y$ and $n \geq N$, we have that

$$T^n(B(y, \varepsilon) \cap Y_{\pi(y)}) \supseteq Y_{S^n(\pi(y))},$$

where $B(y, \varepsilon) \subset Y$ denotes the ball of radius ε centred at the point y.

For a Polish space Z, let $\mathcal{B}_Z$, B_Z and C_Z denote the Borel σ-field of Z, the space of bounded Borel measurable functions, and the space of bounded continuous functions $f : Z \to \mathbb{R}$, respectively. For a fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ and $x \in X$, we shall use the notations $\mathcal{B}_x$, B_x and C_x instead of $\mathcal{B}_{Y_x}$, B_{Y_x}, and C_{Y_x}. We recall the definition of a Gibbs family from [DG2].

Definition 2.1
Let $\mathcal{Y} = (Y, X, T, S, \pi)$ be a fibred system and let $\varphi : Y \to \mathbb{R}$ be a Borel measurable function. A system $\{\mu_x : x \in X\}$ of conditional probabilities for $\mathcal{Y}$ is called a *Gibbs family of conditional measures* (or simply a *Gibbs family*) for φ, if there exists a Borel measurable function $A_\varphi : X \to \mathbb{R}$ such that for $x \in X$ and $f \in L_1(\mu_x)$ we have that

$$\int V_x f(y) \mu_{S(x)}(dy) = A_\varphi(x) \int f(y) \mu_x(dy), \tag{6}$$

where $V_x := V_x^{(1)}$ denotes the relative transfer operator as defined in (5). The function A_φ is called a *gauge function* for $\mathcal{Y}$ and φ. In particular, if X reduces to a point, a Gibbs family consists of a single measure which is called the Gibbs measure.

Remark 2.2
It is possible to reformulate the above definition in terms of the Jacobian of the map T_x of the measure space (Y_x, μ_x) to the measure space $(Y_{S(x)}, \mu_S(x))$. It follows from the fibre expanding property that, for any $x \in Y_x$, the map T_x is a local homeomorphism in some neighbourhood of y in Y_x. We assume that this local homeomorphism sends (locally) μ_x to a measure on $Y_{S(x)}$ which is absolutely continuous with respect to $\mu_{S(x)}$, and that the following assertion holds. There exists a positive Borel measurable function $A_\varphi : X \to \mathbb{R}$ with the property that, for all $x \in X$ and all $y \in Y_x$, the Jacobian of μ_x with respect to the map T_x is given (in some Y_x-neighbourhood of y) by the expression

$$\frac{d\mu_{S(x)} \circ T_x}{d\mu_x} = A_\varphi(x) \exp(-\varphi) \qquad\qquad \mu_x\text{-a.e.}$$

We now prepare the description of the gauge function as a limit (see proposition **2.5**). We recall some definitions, notation and results from [DG2]. The following result asserts under appropriate assumptions existence and uniqueness of a Gibbs family.

Theorem 2.3
(cf. Th. 2.6 in [DG2]) Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibred system with metric d on Y. Assume that $\mathcal{Y}$ is fibrewise expanding and topologically exact along fibres. Then, for every Hölder continuous function $\varphi : Y \to \mathbb{R}$, there exists a unique Gibbs family $\{\mu_x : x \in X\}$ for φ. The function A_φ is also unique and one has that $\operatorname{supp}\{\mu_x\} = Y_x$. $\square$

The (weak) continuity of the Gibbs family depends on properties of the maps

$$i_n : Y \to Y_n := Y_n := \{(x, y) \in X \times Y : \ S^n(x) = \pi(y)\}$$

defined by $i_n(\cdot) = (\pi(\cdot), T^n(\cdot))$ where $n \geq 0$. We shall also need the two natural projections $\pi_n^X : Y_n \to X$ and $\pi_n^Y : Y_n \to Y$ defined by $\pi_n^X(x, y) = x$ and $\pi_n^Y(x, y) = y$.

Theorem 2.4
(Th. 2.7 in [DG2]) Let $\{\mu_x : x \in X\}$ and (φ, A) be as in theorem **2.3**. Assume that S and π are open maps and that i_1 is a local homeomorphism. Then the Gibbs family $\{\mu_x : x \in X\}$ and the function $A_\varphi : X \to \mathbb{R}$ are continuous. $\square$

In [DG2] also certain systems of conditional probabilities were constructed, which depend continuously on x under the assumptions of theorem **2.4**.

For every $n \geq 1$ denote by R_n the equivalence relation on Y defined by the relation: $y_1 R_n y_2$ if and only if $i_n(y_1) = i_n(y_2)$ (or, equivalently, $\pi(y_1) = \pi(y_2)$ and $T^n(y_1) = T^n(y_2)$). Let $R_n(y)$ be the equivalence class of R_n containing $y \in Y$. For $f \in C_Y$ we set, following [DG2],

$$(G^{(n)} f)(y) := \sum_{y' \in R_n(y)} \exp\left(\sum_{k=0}^{n-1} \varphi(T^k(y'))\right) \cdot f(y').$$

A similar expression

$$(G_x^{(n)} f)(y) := \sum_{y' \in R_n(y)} \exp\left(\sum_{k=0}^{n-1} \varphi(T^k(y'))\right) \cdot f(y')$$

defines an operator $G_x^{(n)} : C_x \to C_x$, for any $x \in X$ (and $n \geq 0$). Next, we introduce operators $P^{(n)}$, for $n \geq 1$, and $P_x^{(n)}$, for $n \geq 1, x \in X$, defined by

$$(P^{(n)} f)(y) := \frac{(G^{(n)} f)(y)}{(G^{(n)} \mathbf{1})(y)}, \qquad y \in Y, f \in C_Y,$$

and

$$(P_x^{(n)} f)(y) = \frac{(G_x^{(n)} f)(y)}{(G_x^{(n)} \mathbf{1})(y)}, \qquad y \in Y_x, f \in C_x.$$

All these operators are conditional expectations. It was shown in [DG2] that they converge in the strong operator topology. More precisely, under the assumptions of theorem **2.3**, for any $x \in X$, we have that $\{P_x^{(n)}, n \geq 0\}$ converges to a conditional expectation operator P_x, which gives the measure μ_x. Under the assumptions of theorem **2.4**, we even have that $\{P^{(n)},\ n \geq 0\}$ converges to the conditional expectation operator P defined by the family μ_x, $x \in X$. For $k \geq 0$ and $x \in X$, the operators $U_x^{(k)} : C_{S^k(x)} \to C_x$ are defined by

$$(U_x^{(k)} f)(y) = f(T^k y), y \in Y_x,$$

their global version $U^k : C_Y \to C_Y$ is defined by

$$(U^k f)(y) = f(T^k y), y \in Y.$$

Let us note that the relative transfer operators $V_x^{(k)}$ defined in (5) have a global version $V^k : C_Y \to C_Y$ only under the assumption that the base transformation S is invertible.

Proposition 2.5
Under the assumptions of theorem **2.3** we have that

$$A_\varphi(x) = \lim_{n\to\infty} (G^{(n+1)}\mathbf{1})(y))/(G^{(n)}\mathbf{1})(Ty))$$

uniformly in (x, y), where $x \in X$, $y \in Y_x$.
Proof: By proposition 5.5 in [DG2] we have that, for $x \in X$, $y \in Y_x$,

$$A_\varphi(x) = (U_x^{(1)} P_{S(x)} V_x^{(1)}\mathbf{1})(y).$$

Together with the above convergence results, this implies that

$$A_\varphi(x) = \lim_{n\to\infty} (U_x^{(1)} P_{S(x)}^{(n)} V_x^{(1)}\mathbf{1})(y)$$

uniformly in $x \in X$, $y \in Y_x$. In view of the identity

$$P_x^{(n)} f = \frac{G_x^{(n)} f}{G_x^{(n)}\mathbf{1}}$$

and the relation

$$G_x^{(n)} f = U_x^{(n)} V_x^{(n)} f$$

from section 5 in [DG2] and using obvious composition properties of the operators $U_x^{(k)}, V_x^{(k)}$ together with the multiplicativity of $U_x^{(k)}$ (also observed in [DG2], sec. 5) we get that, uniformly in $x \in X$ and $y \in Y_x$,

$$\begin{aligned}
A_\varphi(x) &= \lim_{n\to\infty} \left(U_x^{(1)} \left(\frac{G_{S(x)}^{(n)}(V_x^{(1)}\mathbf{1})}{G_{S(x)}^{(n)}\mathbf{1}} \right) \right)(y) \\
&= \lim_{n\to\infty} \frac{(U_x^{(1)} G_{S(x)}^{(n)}(V_x^{(1)}\mathbf{1}))(y)}{(U_x^{(1)} G_{S(x)}^{(n)}\mathbf{1})(y)} \\
&= \lim_{n\to\infty} \frac{(G_x^{(n+1)}\mathbf{1})(y)}{(G_{S(x)}^{(n)}\mathbf{1})(Ty)},
\end{aligned}$$

which completes the proof. □

The following proposition asserts that (additively T-)cohomologous potentials give rise to (multiplicatively S-)cohomologous gauge functions. Moreover, the corresponding coboundary is given explicitely.

Proposition 2.6
Assume the setting of proposition **2.5**. Let φ be a Hölder continuous potential and let

$$\varphi' = \varphi + \gamma - \gamma \circ T,$$

where $\gamma \in C_Y$. Then, for some $g \in C_X$, we have that

$$\log A_{\varphi'} = \log A_\varphi + g - g \circ S.$$

Furthermore, for $x \in X$ the function g can be defined by the formula

$$g(x) = \log \int \exp[\gamma(u)]\mu_x(du),$$

where $(\mu_x)_{x \in X}$ is the system of conditional Gibbs measures on fibres associated with the potential φ.

Proof: By proposition **2.5**, we have that

$$\begin{aligned}
A_{\varphi'}(x) &= \lim_{n\to\infty} \frac{\sum_{y'\in R_{n+1}(y)} \exp[\sum_{k=0}^{n} \varphi'(T^k(y'))]}{\sum_{y''\in R_n(T(y))} \exp[\sum_{k=0}^{n-1} \varphi'(T^k(y''))]} \\
&= \lim_{n\to\infty} \frac{\sum_{y'\in R_{n+1}(y)} \exp[\gamma(y') - \gamma(T^{n+1}(y')) + \sum_{k=0}^{n} \varphi(T^k(y'))]}{\sum_{y''\in R_n(T(y))} \exp[\gamma(y'') - \gamma(T^n(y'')) + \sum_{k=0}^{n-1} \varphi(T^k(y''))]} \\
&= \lim_{n\to\infty} \frac{\sum_{y'\in R_{n+1}(y)} \exp[\gamma(y') + \sum_{k=0}^{n} \varphi(T^k(y'))]}{\sum_{y''\in R_n(T(y))} \exp[\gamma(y'') + \sum_{k=0}^{n-1} \varphi(T^k(y''))]} \\
&= \lim_{n\to\infty} \frac{(G_x^{(n+1)} \exp[\gamma(\cdot)])(y)}{(G_{S(x)}^{(n)} \exp[\gamma(\cdot)])(T(y))} \\
&= \lim_{n\to\infty} \frac{(G_x^{(n+1)} \exp[\gamma(\cdot)])(y)}{(G_x^{(n+1)}\mathbf{1})(y)} \cdot \lim_{n\to\infty} \frac{(G_{S(x)}^{(n)}\mathbf{1})(T(y))}{(G_{S(x)}^{(n)} \exp[\gamma(\cdot)])(T(y))} \\
&\quad \cdot \lim_{n\to\infty} \frac{(G_x^{(n+1)}\mathbf{1})(y)}{(G_{S(x)}^{(n)}\mathbf{1})(T(y))} \\
&= \frac{\int \exp[\gamma(u)]\mu_x(du)}{\int \exp[\gamma(t)]\mu_{S(x)}(dt)} \, A_\varphi(x),
\end{aligned}$$

and the proposition follows with

$$g(x) = \log \int \exp[\gamma(u)]\mu_x(du)$$

for $x \in X$. □

2.2 Relative Ruelle's theorem in the case of invertible base transformation

In this paragraph S is supposed to be invertible. This implies that the general expression $(V_x^{(k)} g)(y)$ as defined in (5) reduces to the globally defined operator

$$(V^k g)(y) = \sum_{T^k(y')=y} g(y') \exp\left[\sum_{j=0}^{k-1} \varphi(T^j(y'))\right]. \tag{7}$$

Theorem 2.7
Assume the setting of theorem **2.4**. Then, uniformly in $y \in Y$, there exists the limit

$$h(y) := \lim_{n\to\infty} \frac{(V^n\mathbf{1})(y)}{A_\varphi(\pi(y))A_\varphi(S^{-1}(\pi(y)))\cdots A_\varphi(S^{-(n-1)}(\pi(y)))}.$$

$h \in C(Y)$ is continuous, strictly positive, and satisfies the identities

$$Vh(\cdot) = A_\varphi(S^{-1}(\pi(\cdot)))h(\cdot)$$

and for $x \in X$

$$\int_Y h(y)\mu_x(dy) = 1.$$

A strictly positive function $h \in C_Y$ with the latter two properties is unique.
Proof: It follows from ([DG2], remark 4.16) that, for $m, l \geq 1$, the functions $(\log V_x^{(l)}\mathbf{1} - \log V_x^{(m)}\mathbf{1})$ are uniformly Hölder continuous. Thus, there exist constants $K > 0$ and $0 < \varrho < 1$ such that, for all $m, l \geq 1$, we have that

$$\left| \frac{V_x^{(n+l)}\mathbf{1}(y)}{V_x^{(n+l)}\mathbf{1}(z)} \cdot \frac{V_x^{(n+m)}\mathbf{1}(z)}{V_x^{(n+m)}\mathbf{1}(y)} - 1 \right| \leq K\varrho^n, \quad \left| \frac{V_x^{(n+l)}\mathbf{1}(y)}{V_x^{(n+l)}\mathbf{1}(z)} \cdot \frac{V_x^{(n+m)}\mathbf{1}(z)}{V_x^{(n+m)}\mathbf{1}(y)} - 1 \right| \leq K\varrho^n,$$

for $y, z \in Y_x$, $x \in X$. By (6), the Gibbs family $\{\mu_x, x \in X\}$, which corresponds to the potential φ (cf. theorem **2.4**), satisfies, for any $x \in X$, the relation

$$\int_Y (V^n\mathbf{1})(y)\mu_x(dy) = A_\varphi(x)\cdots A_\varphi(S^{-(n-1)}(x)). \tag{8}$$

This implies that the functions h_n defined by

$$h_n(y) = \frac{(V^n\mathbf{1})(y)}{A_\varphi(\pi(y))A_\varphi(S^{-1}\pi(y))\cdots A_\varphi(S^{-(n-1)})\pi(y))},$$

for $y \in Y$, form a fundamental sequence in the sense of uniform convergence. Since (8) means that for $x \in X$

$$\int_Y h_n(y)\mu_x(dy) = 1$$

we also have that

$$\int_Y h(y)\mu_x(dy) = 1,$$

where h is the limit of the sequence h_n.

It is immediate to show that

$$(Vh_n)(\cdot) = A_\varphi(\pi(\cdot))h_{n+1},$$

which implies that

$$Vh = A_\varphi \circ S^{-1} \circ \pi.$$

The uniqueness is an easy consequence of corollary 4.14 in [DG2].
Indeed, if h' is another function satisfying $Vh' = A_\varphi \circ S^{-1} \circ \pi$ and $\int_Y h' d\mu_x = 1$ for $x \in X$, then, for some $\rho < 1$, we have that

$$\frac{V^n h'(y)}{V^n \mathbf{1}(y)} \leq (1 + \rho^n) \frac{V^n h'(z)}{V^n \mathbf{1}(z)}$$

for all $y, z \in Y$.
On the other hand,

$$\frac{V^n h'(y)}{V^n \mathbf{1}(y)} \to \frac{h'(y)}{h(y)},$$

therefore

$$\max_{y \in Y} \frac{h'(y)}{h(y)} \leq \min_{z \in Y} \frac{h'(z)}{h(z)},$$

i.e. $h = h'$. □

Remark 2.8
It can easily be deduced from [DG2] that h is uniformly Hölder continuous on the fibres and is globally continuous on Y. A close inspection of [DG2] shows that it is already this type of potentials (rather than globally Hölder continuous ones) for which the results of [DG2] hold. Within the class of these potentials theorem **2.7** may be restated in the following form.

Any potential φ (with corresponding global transfer operator V) may be changed by a coboundary (more precisely, a coboundary of $\log h$)

$$\varphi' = \varphi + \log h - \log h \circ T,$$

such that the transfer operator V' corresponding to the potential φ' satisfies the relation

$$(V'\mathbf{1})(y) = A(x), \quad x \in X, y \in Y_x.$$

In section **4** we shall restate this condition in terms of the Gibbs family for φ'.□

Remark 2.9
Generally speaking, a change of a potential in a coboundary of a function, which is globally continuous on Y and uniformly Hölder continuous on the fibres, does change the gauge function. Only the cohomology class of the gauge function persists under a change of this type (cf. Prop. 2.6). However, in the special case considered in this subsection, the potential is changed in a coboundary of $\log h$, where h satisfies the norming condition $\int_Y h(y)\mu_x(dy) = 1$ for all $x \in X$. This implies, in view of the explicit formula for g from Prop. 2.6, that also the gauge function itself is not changed in this case.

2.3 An inverse limit construction

In this paragraph we recall the construction of an extension of (Y, T) which is, in particular, a factor of the natural extension of (Y, T) and factorises over the natural extension of (X, S). A *natural extension* of (X, S) is an invertible extension of this system which is the 'smallest one' in the sense that any other invertible extension of (X, S) factorises over (X, S) through the natural one. This terminology goes back to Rokhlin [Ro2] who considered this concept in the measure theoretic setting. There exists a unique natural extension (up to isomorphism). It can be described explicitly as an inverse limit constructed from X and T (see remark **3.11** below). A generalisation of this construction in the relative framework is reproduced below (cf. [DG2]). Thus, we deal with explicit constructions despite we only need some very general properties of the constructed objects. What we consider here is a *natural base extension* which makes the base map invertible leaving the fibre maps unchanged. It can be also considered as a Cartesian square constructed from two extensions of (X, S), namely the natural extension $p_X : (\tilde{X}, \tilde{S}) \to (X, S)$ and the fibred system under consideration $\pi : (Y, T) \to (X, T)$.

Recall that according to **2.1** $Y_n := \{(x, y) \in X \times Y : S^n(x) = \pi(y)\}$, and the maps $\pi_n^Y : Y_n \to Y$ and $i_0 : Y \to Y_0$ are defined by the relations $\pi_n^Y((x, y)) = y$ for $(x, y) \in Y_n$ and $i_0(y) = (\pi(y), y)$.

For any $0 \le n \le m$, we define $T_{m,n} : Y_n \to Y_m$ and $\pi_{n,m}^Y : Y_m \to Y_n$ by $T_{m,n}(x, y) = (x, T^{m-n}(y))$ and $\pi_{n,m}^Y(x, y) = (S^{m-n}(x), y)$, respectively. Note that the map $\pi_{0,n}^Y$ essentially coincides with the projection π_n^Y. Indeed, we have $\pi_n^Y = i_0 \circ \pi_{0,n}^Y$ where i_0 is a homeomorphism. The following assertion is proved in proposition 5.1 of [DG2].

Proposition 2.10
For every $0 \le n \le m$, $T_{m,n}$ is a local homeomorphism and $\pi_{n,m}$ is an open map.$\Box$

Remark 2.11
We define $\tilde{Y}$ as the inverse limit of the spaces $\{Y_n : n \ge 0\}$ with respect to the

maps $\{\pi^{Y}_{n,m+n} : \ m, n \geq 0\}$. Then the following diagram commutes.

$$\begin{tikzcd}
\tilde{Y} \arrow[loop left, "\tilde{T}"] \arrow[r, "p_Y"] \arrow[d, "\tilde{\pi}"'] & Y \arrow[loop right, "T"] \arrow[d, "\pi"] \\
\tilde{X} \arrow[loop left, "\tilde{S}"] \arrow[r, "p_X"] & X \arrow[loop right, "S"]
\end{tikzcd},$$

where $\tilde{X}$ together with a map $\tilde{S} : \tilde{X} \to \tilde{X}$ is the natural extension of $S : X \to X$ (this means that the invertible system $(\tilde{X}, \tilde{S})$ is the inverse limit for $\{X_n \equiv X, n \geq 0\}$ with respect to $S_{n,m+n} : X_{m+n} \to X_n$ sending x to $S^m x$). Also, the limit $\tilde{T} : \tilde{Y} \to \tilde{Y}$ for $\{T_{m+n,n} : m, n \geq 0\}$ is defined in a natural way. Under the assumptions that S and π are open and i_1 is a local homeomorphism, $\tilde{T}$ is a local homeomorphism as well.

Let us also remark that, under our assumptions, both $\tilde{\pi}$ and $\tilde{T}$ are surjective maps. Indeed, it can easily be verified that both of these properties follow from (in fact, are equivalent to) the following one. For any $x \in X$ T maps the fibre Y_x onto $Y_{S(x)}$. The latter assertion is a consequence of the exactness along fibres which was assumed.

Remark 2.12
In the sequel we shall also need the fact that the natural extension $(\hat{Y}, \hat{T})$ of (Y, T) factorises through $(\tilde{Y}, \tilde{T})$. This can be deduced from the above explicit constructions or from the 'abstract nonsense'. Indeed, $(\hat{Y}, \hat{T})$, as an invertible transformation extending (X, S) (through (Y, T)), may be mapped (compatibly with the dynamics) to (X, S) not only through (Y, T), but also through $(\tilde{X}, \tilde{S})$ (because the latter is the 'smallest' invertible extension of (X, S)). Therefore, $(\hat{Y}, \hat{T})$ together with (Y, T), $(\tilde{X}, \tilde{S})$ and (X, S) form a commutative diagram. Again, by the property of the Cartesian square the latter diagram maps to the above diagram (with identical maps for (Y, T), $(\tilde{X}, \tilde{S})$ and (X, S)) which, in particular, connects $(\hat{Y}, \hat{T})$ and $(\tilde{Y}, \tilde{T})$.

Remark 2.13
If $(\tilde{Y}, \tilde{T}, \tilde{X}, \tilde{S}, \tilde{\pi})$ is a fibred system from the above commutative diagram, and $\tilde{\varphi} = \varphi \circ p_Y$ is a lift of a (globally continuous on Y and uniformly Hölder continuous on the fibres) potential φ from Y to $\tilde{Y}$, then the corresponding gauge functions are connected by a similar relation, namely $\tilde{A}_{\varphi} = A_{\varphi} \circ p_X$. This follows from proposition 2.5.

Finally, let us remark that all these constructions make sense and the assertions are valid in the setting of ergodic theory for measure preserving maps of Lebesgue-Rokhlin spaces (see sections **3** and **4**).

3 A fibred version of Rokhlin's formula and regularity of a factor-endomorphism

Rokhlin's formula for the entropy of an endomorphism (see [Ro2]) belongs to the basic results in entropy theory. We deal with its relativised version. It turns out that only under some rather specific additional assumptions (in particular, if the factor-endomorphism is an automorphism) a straightforward fibred analogue of Rokhlin's formula holds. In a more general setting the corresponding formula has to contain some limit operator.

Our considerations concerning the entropy will be carried out in the context of Lebesgue-Rokhlin probability spaces. These standard measure spaces were introduced and studied by V.A. Rokhlin under the name of Lebesgue spaces. The interested reader is referred to [Ro 1] for the basic concepts, notations, results and further references on measure theory (in particular, on Rokhlin's theory of measurable partitions) and entropy theory. All partitions which will be considered in this paragraph are assumed to be measurable.

For a Lebesgue-Rokhlin space (Y, μ) we denote by ϵ_Y and ν_Y, respectively, the partition of Y into distinct points and the trivial partition of Y, so that ϵ_Y and ν_Y are, respectively, the finest and the coarsest partitions of Y. We write $\xi \le \eta$ for partitions ξ, η if η is finer than ξ.

If $T : Y \to Y$ is an endomorphism of a Lebesgue-Rokhlin-space (Y, μ) then a partition η is called *invariant* with respect to T if $T^{-1}\eta \le \eta$. Let $X = (Y/\eta, \mu_\eta)$, $\pi : Y \to X$ and $S : X \to X$, denote the factor space of (Y, μ) with respect to an invariant partition η, the canonical factorisation map (or projection), and the factor-endomorphism (also called the factor map or simply the factor) of T with respect to η. The projection π is measure preserving and semi-conjugates T and S such that $\pi T = S\pi$.

Now we recall the definition and the main properties of relative entropy (of a partition and of an endomorphism). In general, the lower subscript denoting the measure on the space under consideration will be omitted. We use the standard notations $\xi^{(n)} = \bigvee_{k=0}^{n-1} T^{-k}\xi$ and $\xi^- = \bigvee_{k=0}^{\infty} T^{-k}\xi$.

Definition 3.1
Let $H(\xi/\pi^{-1}\epsilon_X) < \infty$. We set

$$h(T|S, \xi) = \lim_{n\to\infty} H(T^{-n}\xi/T^{-(n+1)}\xi^- \vee \pi^{-1}\epsilon_X). \tag{9}$$

We call $h(T|S, \xi)$ the entropy of ξ relative to $T|S$.

The following proposition shows that the previous definition makes sense. Moreover, it gives certain other representations for $h(T|S, \xi)$ which will be useful in order to prove relative versions of some well-known properties of the entropy $h(T, \xi)$ of ξ relative to T (cf. [Ro1]).

Proposition 3.2
Let $H(\xi/\pi^{-1}\epsilon_X) < \infty$. Then the sequence

$$\{H(T^{-n}\xi/T^{-(n+1)}\xi^- \vee \pi^{-1}\epsilon_X),\ n \geq 1\}$$

decreases and the limit in (9) exists. Moreover, we also have that

$$h(T|S,\xi) = \lim_{n\to\infty}(1/n)H(\xi^{(n)}/\pi^{-1}\epsilon_X \vee T^{-n}\xi^-) \tag{10}$$

and

$$h(T|S,\xi) = \lim_{n\to\infty}(1/n)H(\xi^{(n)}/\pi^{-1}\epsilon_X). \tag{11}$$

Remark 3.3
The definition for $h(T|S,\xi)$ in [LW] is equivalent to (11).

Proof: The mean conditional entropy $H(\xi/\eta)$ is known to be nonnegative and to decrease in its second argument (proposition 5.10 in [Ro1]). Hence we have, because of the invariance of ϵ_X, that

$$\begin{aligned} H(T^{-(n-1)}\xi/T^{-n}\xi^- \vee \pi^{-1}\epsilon_X) &= H(T^{-n}\xi/T^{-(n+1)}\xi^- \vee \pi^{-1}T^{-1}\epsilon_X) \\ &\geq H(T^{-n}\xi/T^{-(n+1)}\xi^- \vee \pi^{-1}\epsilon_X) \geq 0 \end{aligned}$$

and also that

$$H(T^{-n}\xi/T^{-(n+1)}\xi^- \vee \pi^{-1}\epsilon_X) \leq H(T^{-n}\xi/T^{-n}\pi^{-1}\epsilon_X) = H(\xi/\pi^{-1}\epsilon_X) < \infty,$$

which proves the first assertion.

By the chain property of the mean conditional entropy (cf. 5.9 in [Ro1]) we have that, for $k = 0, \ldots, n-1$,

$$\begin{aligned} H(T^{-k}\xi^{(n-k)}/T^{-n}\xi^- \vee \pi^{-1}\epsilon_X) &= H(T^{-k}\xi/T^{-(k+1)}\xi^- \vee \pi^{-1}\epsilon_X) \\ &\quad + H(T^{-(k+1)}\xi^{(n-k-1)}/T^{-n}\xi^- \vee \pi^{-1}\epsilon_X). \end{aligned}$$

Thus, we arrive at the identity

$$H(\xi^{(n)}/T^{-n}\xi^- \vee \pi^{-1}\epsilon_X) = \sum_{k=0}^{n-1} H(T^{-k}\xi/T^{-(k+1)}\xi^- \vee \pi^{-1}\epsilon_X).$$

Relation (10) now follows from the above identity, (9), and the property of the Cesaro limit.
We need the following lemma in order to establish (11).

Lemma 3.4
The family $\{H(T^{-(k-1)}\xi/T^{-k}\xi^{(m)} \vee \pi^{-1}\epsilon_X), k \geq 1, m \geq 0\}$ decreases in both k and m. Furthermore, we have that

$$\begin{aligned}
&\lim_{k,m\to\infty} H(T^{-(k-1)}\xi/T^{-k}\xi^{(m)} \vee \pi^{-1}\epsilon_X) \\
&= \inf_{k,m} H(T^{-(k-1)}\xi/T^{-k}\xi^{(m)} \vee \pi^{-1}\epsilon_X) \\
&= h(T|S,\xi).
\end{aligned}$$

Proof: $\{H(T^{-(k-1)}\xi/T^{-k}\xi^{(m)} \vee \pi^{-1}\epsilon_X),\ k \geq 1,\ m \geq 0\}$ decreases in k (m being fixed), because of

$$\begin{aligned}
H(T^{-(k-1)}\xi/T^{-k}\xi^{(m)} \vee \pi^{-1}\epsilon_X) &= H(T^{-k}\xi/T^{-(k+1)}\xi^{(m)} \vee T^{-1}\pi^{-1}\epsilon_X) \\
&\geq H(T^{-k}\xi/T^{-(k+1)}\xi^{(m)} \vee \pi^{-1}\epsilon_X).
\end{aligned}$$

The value also decreases in m (for k being fixed) by proposition 5.10 in [Ro1]. Hence,

$$\begin{aligned}
&\lim_{k,m\to\infty} H(T^{-(k-1)}\xi/T^{-k}\xi^{(m)} \vee \pi^{-1}\epsilon_X) \\
&= \inf_{k,m} H(T^{-(k-1)}\xi/T^{-k}\xi^{(m)} \vee \pi^{-1}\epsilon_X).
\end{aligned}$$

In order to obtain the value of the limit we first let m tend to ∞ and get $H(T^{-(k-1)}\xi/T^{-k}\xi^{-} \vee \pi^{-1}\epsilon_X)$, which tends to to $h(T|S,\xi)$ as $k \to \infty$. □

Let us now prove relation (11). By the chain property of the mean conditional entropy, it follows as before from 5.9 in [Ro1] that

$$H(\xi^{(n)}/\pi^{-1}\epsilon_X) = \sum_{k=0}^{n-1} H(T^{-k}\xi/T^{-(k+1)}\xi^{(n-k-1)} \vee \pi^{-1}\epsilon_X),$$

because we have that, for $k = 0, \dots, n-1$,

$$\begin{aligned}
&H(T^{-k}\xi^{(n-k)}/\pi^{-1}\epsilon_X) \\
&= H(T^{-k}\xi/T^{-(k+1)}\xi^{(n-k-1)} \vee \pi^{-1}\epsilon_X) + H(T^{-(k+1)}\xi^{(n-k-1)}/\pi^{-1}\epsilon_X).
\end{aligned}$$

By lemma **3.4**, the summands on the right hand side converge to $h(T|S,\xi)$ for $k \to \infty$, $n-k \to \infty$, resp. Moreover, each term is bounded from above by $H(\xi/\pi^{-1}\epsilon_X)$ which implies that

$$\begin{aligned}
&\lim_{n\to\infty} (1/n) H(\xi^{(n)}/\pi^{-1}\epsilon_X) \\
&= \lim_{n\to\infty} (1/n) \sum_{k=0}^{n-1} H(T^{-k}\xi/T^{-(k+1)}\xi^{(n-k-1)} \vee \pi^{-1}\epsilon_X) \\
&= h(T|S,\xi).
\end{aligned}$$

□

Corollary 3.5
$h(T|S,\xi)$ is an increasing function of ξ.
Proof: This follows from (11). $\square$

Definition 3.6
The relative entropy $h(T|S) = h_\mu(T|S)$ of an endomorphism T with respect to its factor S is defined by the expression

$$h(T|S) = \sup\{h(T|S,\xi) : \xi \text{ is a partition of } Y \text{ with } H(\xi/\pi^{-1}\epsilon_X) < \infty\}.$$

Remark 3.7
We get the same value of $h(T|S)$ taking sup over all finite partitions ξ of Y as in [LW]. $\square$

The following proposition is a special case of lemma 3.1 in [LW].

Proposition 3.8
If S is a factor of T then

$$h(T) = h(T|S) + h(S).$$

$\square$

Definition 3.9
A measurable partition ξ of a Lebesgue-Rokhlin space (Y,μ) is said to be *a unilateral relative generator* for an endomorphism $T : Y \to Y$ and a T-invariant partition η of Y, if

$$\eta \vee \bigvee_{n\geq 0} T^{-n}\xi = \varepsilon_Y.$$

Proposition 3.10
If an endomorphism T of the space (Y,μ) has a factor-endomorphism S of (X,μ_{ϵ_X}) and a unilateral relative generator of finite mean conditional entropy then

$$h(T|S) = \lim_{n\to\infty} H(T^{-n}\epsilon_Y/T^{-(n+1)}\epsilon_Y \vee \pi^{-1}\epsilon_X). \tag{12}$$

Proof: Let ξ be a generator as in the statement. Note that

$$\begin{aligned} h(T|S,\xi) &= \lim_{n\to\infty} H(T^{-n}\xi/T^{-(n+1)}\xi^{-} \vee \pi^{-1}\epsilon_X) \\ &= \lim_{n\to\infty} H(T^{-n}\xi^{-}/T^{-(n+1)}\xi^{-} \vee \pi^{-1}\epsilon_X) \\ &= \lim_{n\to\infty} H(T^{-n}\epsilon_Y/T^{-(n+1)}\epsilon_Y \vee \pi^{-1}\epsilon_X), \end{aligned}$$

and hence

$$h(T|S) \geq \lim_{n\to\infty} H(T^{-n}\epsilon_Y/T^{-(n+1)}\epsilon_Y \vee \pi^{-1}\epsilon_X).$$

To prove the opposite inequality, we have to show that, for any partition ζ with $H(\zeta/\eta) < \infty$, we have that

$$H(T|S, \zeta) \leq \lim_{n\to\infty} H(T^{-n}\epsilon_Y / T^{-(n+1)}\epsilon_Y \vee \pi^{-1}\epsilon_X).$$

If, for some $k > 0$, we have that $\zeta = \xi^k$, the assertion follows from (11). More generally, if, for some $k > 0$, we have that $\zeta < \xi^k$ then it follows from corollary **3.5**. To complete the proof, we observe, by analogy with section 6 in [Ro1] that the set of partitions ζ with $\zeta > \eta$ and $H(\zeta/\eta) < \infty$ is a metric space, setting $\rho(\zeta_1, \zeta_2) = H(\zeta_1, \zeta_2) + H(\zeta_2, \zeta_1)$. Then we have that the partitions considered above are dense in this space (because ξ is a relative generator). We omit further details and refer to [Sch]. □

Remark 3.11
As should be clear from the proof of proposition **3.10**, under the same assumptions the sequence $\{H(T^{-n}\epsilon_Y / T^{-(n+1)}\epsilon_Y \vee \pi^{-1}\epsilon_X), n \geq 1\}$ decreases. □

Definition 3.12
An endomorphism S of a space X is said to be a *regular factor* of an endomorphism T of a space Y if the partitions $\pi^{-1}\epsilon_X$ and $T^{-1}\epsilon_Y$ are conditionally independent given $T^{-1}\pi^{-1}\epsilon_X$. In this situation $\pi^{-1}\epsilon_X$ is also said to be *a regular (invariant)* partition.

In the assertions 2) – 4) of the next theorem we mean by conditional measures on fibres canonical measures which are associated with any measurable partition in the theory of Lebesgue-Rokhlin spaces. In particular, all fibres are equipped with the measurable structure induced by the total measure space.

Theorem 3.13
Let T be an endomorphism of a Lebesgue-Rokhlin space (Y, μ), (X, S) be its factor with factorisation map π and corresponding T-invariant partition $\pi^{-1}\epsilon_X$ of Y. Suppose that we have that

$$\bigwedge_{n\geq 0} (\pi^{-1}\epsilon_X \vee T^{-n}\epsilon_Y) = \pi^{-1}\epsilon_X. \tag{13}$$

Then the following four conditions are equivalent.

1) $\pi^{-1}\epsilon_X$ is a regular invariant partition;

2) for any $k \geq 1$, the partitions $\pi^{-1}\epsilon_X$ and $T^{-k}\epsilon_Y$ are conditionally independent given $T^{-1}\pi^{-1}\epsilon_X \vee T^{-(k+1)}\epsilon_Y$;

3) for almost all $x \in X$, the map T sends the conditional measure μ_x on $\pi^{-1}(x)$ to the conditional measure $\mu_{S(x)}$ on $\pi^{-1}(S(x))$ (in different words, the system $\{\mu_x, x \in X\}$ of conditional measures is *covariant* with respect to T);

4) for any $k \geq 1$ and almost all $y \in Y$, the map T sends the conditional measure on $\pi^{-1}(\pi(y)) \cap T^{-(k+1)}(T^{(k+1)}(y))$ to the conditional measure on $\pi^{-1}(\pi(T(y))) \cap T^{-k}(T^k(y))$.

Moreover, if T has a unilateral relative generator ξ of finite mean conditional entropy with respect to $\pi^{-1}\epsilon_X$, then any of the properties 1) – 4) is equivalent to the following one.

5) The relative entropy $h(T|S)$ of T with respect to the factor-transformation $S = T/\pi^{-1}\epsilon_X$ satisfies

$$h(T|S) = H(\epsilon_Y/\pi^{-1}\epsilon_X \vee T^{-1}\epsilon_Y). \tag{14}$$

Proof: First we note that for three partitions ξ_1, ξ_2, ξ_3 of a probability space conditional independence of ξ_1 and ξ_3 given ξ_2 is equivalent to the following property. For any ξ_1-integrable function f (we say that $f \in L_1(\xi_1)$) the following relation between conditional expectations holds a.e.

$$\mathrm{E}(f|\xi_2 \vee \xi_3) = \mathrm{E}(f|\xi_2).$$

We call a triple (ξ_1, ξ_2, ξ_3) of partitions with the latter property a *Markov triple.* In view of this remark condition 1) says that $(\pi^{-1}\epsilon_X, T^{-1}\pi^{-1}\epsilon_X, T^{-1}\epsilon_Y)$ is a Markov triple.

Let us deduce 2) from 1). We shall operate with the Markov triple property rather then with the original formulation of conditional independence. Let $f \in L_1(\pi^{-1}\epsilon_X)$. Because of

$$(T^{-1}\pi^{-1}\epsilon_X \vee T^{-(k+1)}\epsilon_Y) \vee (T^{-k}\epsilon_Y) = T^{-1}\pi^{-1}\epsilon_X \vee T^{-k}\epsilon_Y,$$

we have to prove that, for any $k \geq 1$,

$$\mathrm{E}(f|T^{-1}\pi^{-1}\epsilon_X \vee T^{-k}\epsilon_Y) = \mathrm{E}(f|T^{-1}\pi^{-1}\epsilon_X \vee T^{-(k+1)}\epsilon_Y).$$

By 1) we obtain, for any $j \geq 1$, that

$$\begin{aligned}
\mathrm{E}(f|T^{-1}\pi^{-1}\epsilon_X \vee T^{-j}\epsilon_Y) &= \mathrm{E}(\mathrm{E}(f|T^{-1}\pi^{-1}\epsilon_X \vee T^{-1}\epsilon_Y)|T^{-1}\pi^{-1}\epsilon_X \vee T^{-j}\epsilon_Y) \\
&= \mathrm{E}(\mathrm{E}(f|T^{-1}\pi^{-1}\epsilon_X)|T^{-1}\pi^{-1}\epsilon_X \vee T^{-j}\epsilon_Y) \\
&= \mathrm{E}(f|T^{-1}\pi^{-1}\epsilon_X).
\end{aligned}$$

In view of the remark at the beginning of the proof this implies the claim.

To deduce 1) from 2), we note that, for $f \in L_1(\pi^{-1}\epsilon_X)$ and any $k \geq 1$, property 2) implies that

$$\mathrm{E}(f|T^{-1}\pi^{-1}\epsilon_X \vee T^{-k}\epsilon_Y) = \mathrm{E}(f|T^{-1}\pi^{-1}\epsilon_X \vee T^{-(k+1)}\epsilon_Y).$$

Hence, for any $n \geq 1$, we have that

$$\mathrm{E}(f|T^{-1}\pi^{-1}\epsilon_X \vee T^{-1}\epsilon_Y) = \mathrm{E}(f|T^{-1}\pi^{-1}\epsilon_X \vee T^{-n}\epsilon_Y).$$

Passing to the limit as $n \to \infty$ in the above relation and using (13), we obtain that

$$\mathrm{E}(f|T^{-1}\pi^{-1}\epsilon_X \vee T^{-1}\epsilon_Y) = \mathrm{E}(f|T^{-1}\pi^{-1}\epsilon_X),$$

which is equivalent to 1).

Let us now show the equivalence of 1) and 3). Again, we apply the Markov triple reformulation of 1). It implies that, given $x' \in X$, the conditional measure $\mu_{x'}$ on $\pi^{-1}(x')$ w.r.t. the σ-field generated by the restriction of T to $\pi^{-1}(x')$, is in fact a $T^{-1}\pi^{-1}$-measurable function that only depends on $S(x')$. But this fact is equivalent to the assertion that the image of μ_x under T (supported on $T\pi^{-1}(x') = \pi^{-1}(S(x'))$) is the same for all x' with the same image $x = S(x')$. But, for almost all $x \in X$, in view of the fact that T is μ-preserving, μ_x is a convex combination of the images under T of the measures $\mu_{x'}$, $S(x') = x$, the weights in the convex combination being determined by the conditional measure (given S) on the preimages $x' \in S^{-1}(x)$ of x. Hence, in our situation, the image of μ'_x under T coincides with μ_x for almost all $x \in X$ and all $x' \in S^{-1}(x)$. This reasoning works in both directions, thus, 1) and 3) are equivalent.

Equivalence of 2) and 4) follows exactly in the same way as that of 1) and 3).

We show the equivalence of 2) and 5). According to proposition **3.10** and remark **3.11** statement 5) holds if and only if, for every $k \geq 1$, we have that

$$H(T^{-(k-1)}\epsilon_Y/T^{-k}\epsilon_Y \vee \pi^{-1}\epsilon_X) = H(T^{-k}\epsilon_Y/T^{-(k+1)}\epsilon_Y \vee \pi^{-1}\epsilon_X).$$

This relation is equivalent, in view of the measure invariance, to

$$H(T^{-k}\epsilon_Y/T^{-(k+1)}\epsilon_Y \vee T^{-1}\pi^{-1}\epsilon_X) = H(T^{-k}\epsilon_Y/T^{-(k+1)}\epsilon_Y \vee \pi^{-1}\epsilon_X),$$

which holds, according to property (5.10) in [Ro1], if and only if 2) is valid. The proof is completed. □

4 Relative variational principle for fibre expanding maps

Throughout this section let $\mathcal{Y}$ be a fibred system which is fibrewise expanding, exact, and let i_1 be a local homeomorphism (cf. paragraph **2.1**). Now we are going to collect our main results on the relative variational principle. We begin with (cf. remark **2.8**)

Definition 4.1
A bounded measurable function $\varphi : Y \to \mathbb{R}$ is called a *basic potential* (or basic function), if for $x \in X$ and $y \in Y_{S(x)}$ we have

$$(V_x \mathbf{1})(y) = A_\varphi(x).$$

The following assertion is a direct consequence of theorem **2.7** and the remarks **2.8** and **2.9**.

Proposition 4.2
If S is invertible, then, for every potential φ which is continuous on Y and uniformly Hölder continuous on the fibres there exists a cohomologous Hölder continuous basic potential.
Proof: The proof is a direct consequence of theorem **2.7**. $\square$

Definition 4.3
(cf. theorem **3.13**) We call a family of probability measures $(\mu_x)_{x \in X}$ on the fibres a *covariant Gibbs-family* if

$$\mu_x \circ T^{-1} = \mu_{S(x)}$$

for each $x \in X$.

Proposition 4.4
Let $(\mu_x)_{x \in X}$ be a Gibbs family of conditional probability measures on the fibres of a fibred system $\mathcal{Y}$ associated with a potential φ. Then the family $(\mu_x)_{x \in X}$ is covariant if and only if the potential φ is basic.
Proof: The map T sends μ_x to $\mu_{S(x)}$ if and only if for any function $g \in C_{Y_{S(x)}}$ we have $\int_{Y_{S(x)}} g(y')\mu_{S(x)}(dy') = \int_{Y_x} g(T(y))\mu_x(dy)$. On the other hand, substituting $g(T(\cdot))$ instead of f in the identity (6) and making use of $(V_x(g \circ T))(\cdot) = g(\cdot)(V_x \mathbf{1})(\cdot)$, we obtain that

$$\int_{Y_{S(x)}} g(y')(V_x \mathbf{1})(y')\mu_{S(x)}(dy') = A_\varphi(x) \int_{Y_x} g(T(y))\mu_x(dy).$$

Hence, the family $(\mu_x)_{x \in X}$ is covariant if and only if for each $x \in X$ and each $g \in C_{Y_{S(x)}}$ we have that $A_\varphi^{-1}(x) \int_{Y_{S(x)}} g(y')(V_x \mathbf{1})(y')\mu_{S(x)}(dy') = \int_{Y_{S(x)}} g(y')\mu_{S(x)}(dy')$, which is equivalent to the assertion that φ is a basic potential.

Theorem 4.5
Let be $\varphi : Y \to \mathbb{R}$ Hölder continuous. Then, for any S-invariant measure ν on X, we have that

$$\sup_{\substack{\mu \in M(T) \\ \mu \circ \pi^{-1} = \nu}} \left(h_\mu(T|S) + \int_Y \varphi \mathrm{d}\mu \right) = \int_X \log A_\varphi \mathrm{d}\nu. \tag{15}$$

Moreover, the supremum is finite and is attained by a unique probability measure.
Proof: We are going to reduce the proof to the case of an invertible base transformation S. Let $(\tilde{Y}, \tilde{X}, \tilde{T}, \tilde{S}, \tilde{\pi})$ be the fibred system from the natural base extension as described in remark **2.11**. Let us note first that $(\tilde{X}, \tilde{S})$ is the natural extension (in the sense of topological dynamics) of (X, S). In view of the extremality of the natural extension, each measure $\nu \in M(S)$ can be lifted to a unique measure $\tilde{\nu} \in M(\tilde{S})$ satisfying $\tilde{\nu} \circ p_X^{-1} = \nu$, which gives rise to a measure theoretic natural extension. Analogously, each pair of measures μ, ν such that $\mu \in M(T)$, $\nu \in M(S)$, $\mu \circ \pi^{-1} = \nu$ admits a unique lift to the pair $\tilde{\mu} \in M(\tilde{T})$, $\tilde{\nu} \in M(\tilde{S})$ such that $\tilde{\mu} \circ \tilde{\pi}^{-1} = \tilde{\nu}$, $\tilde{\nu} \circ p_X^{-1} = \nu$, $\tilde{\mu} \circ p_Y^{-1} = \mu$. In particular, we have that and $\tilde{Y}$ is a factor of the natural extension $(\hat{Y}, \hat{T})$ of (Y, T) (the latter fact is explained in remark **2.12**).
We need, for each $\mu \in M(T)$ the relation

$$h_\mu(T|S) = h_{\tilde{\mu}}(\tilde{T}|\tilde{S})$$

which shows that the relative entropy persists under passing to the natural base extension. This can be deduced in general case from the definition of the mean relative entropy in paragraph **3** and the extremal property of the natural base extension. The complete proof can be found in [Sch]. Here we restrict ourselves to the case of $h_\nu(S) < \infty$, where $\nu = \mu \circ \pi^{-1}$. According to [Ro2], $h_{\tilde{\nu}}(\tilde{S}) = h_\nu(S)$ and $h_{\hat{\mu}}(\hat{T}) = h_\mu(T)$. The latter relation implies that $h_{\tilde{\mu}}(\tilde{T}) = h_\mu(T)$, because T is a factor of $\tilde{T}$, and, according to remark **2.12**, $\tilde{T}$ is a factor of $\hat{T}$. On the other hand, the addition theorem (proposition **3.8**) for the relative entropy gives that $h_\mu(T) = h_\nu(S) + h_\mu(T|S)$ and $h_{\tilde{\mu}}(\tilde{T}) = h_{\tilde{\nu}}(\tilde{S}) + h_{\tilde{\mu}}(\tilde{T}|\tilde{S})$, which implies the claim provided that $h_\nu(S) < \infty$. Set $\tilde{\varphi} = \varphi \circ p_Y$. We have, because of $\tilde{\mu} \circ p_Y^{-1} = \mu$, that

$$\int_Y \varphi(y) d\mu(y) = \int_{\tilde{Y}} \tilde{\varphi}(\tilde{y}) d\tilde{\mu}(\tilde{y}).$$

It is clear that each measure $\tilde{\mu} \in M(\tilde{T})$ with $\tilde{\mu} \circ \tilde{\pi}^{-1} = \tilde{\nu}$ arises, as a (unique) lifting of the measure $\mu \in M(T)$ defined by $\mu = \tilde{\mu} \circ p_Y^{-1}$. This allows us to conclude that

$$\sup_{\substack{\tilde{\mu} \in M(\tilde{T}) \\ \tilde{\mu} \circ \tilde{\pi}^{-1} = \tilde{\nu}}} \left(h_{\tilde{\mu}}(\tilde{T}|\tilde{S}) + \int_{\tilde{Y}} \tilde{\varphi} \mathrm{d}\tilde{\mu} \right) = \sup_{\substack{\mu \in M(T) \\ \mu \circ \pi^{-1} = \nu}} \left(h_\mu(T|S) + \int_Y \varphi \mathrm{d}\mu \right),$$

where $\tilde{\nu} \circ p_X^{-1} = \nu$.

We now consider the righthand side of (15). Let $\tilde{\varphi} = \varphi \circ p_Y$ be the lift of the potential φ to $\tilde{Y}$. Then, according to remark **2.13**, for the corresponding gauge functions the relation $A_{\tilde{\varphi}} = A_\varphi \circ p_X$ holds. Thus, we have that

$$\int_X \log A_\varphi \mathrm{d}\nu = \int_{\tilde{X}} \log A_{\tilde{\varphi}} \mathrm{d}\tilde{\nu},$$

because of $\tilde{\nu} \circ p_X^{-1} = \nu$.

It follows from the previous part of the proof that it suffices to establish (15) for invertible S. Moreover, by remarks **2.8** and **2.9** we can reduce the proof to the case where φ is a basic potential (with the same gauge function). In this case we can apply the following lemma to the invariant measures μ' of the form $\mu' = \mu'_x \nu(\mathrm{d}x)$, where $\nu \in M(S)$ is fixed and $(\mu'_x)_{x \in X}$ is a covariant Gibbs family determined by μ'. In the following lemma a measurable partition $\pi^{-1}\epsilon_X \vee T^{-1}\epsilon_Y$ of a Lebesgue-Rokhlin space (Y, μ') is considered which is generated by the map $i_1 : y \mapsto (\pi(y), T(y))$. For each $y \in Y$ let $Y_y = \{y' \in Y : \pi(y') = \pi(y), T(y') = T(y)\}$ be an element of this partition containing y and let μ'_y be the conditional measure on Y_y corresponding to μ' (recall, that according to Rokhlin's theory of 'canonic systems of measures' the function $y \mapsto \mu'_y$ is a constant (mod μ) on each element $Y_{y'}$). In view of our assumptions, each set Y_y is finite. Hence, each measure μ'_y is a discrete measure on a finite set containing Y, and $y \mapsto \mu'_y(\{y\})$ is a well-defined measurable function on Y.

Lemma 4.6
Let S be invertible and φ be basic. Assume that there exists a measure $\mu \in M(T)$ such that the conditional measures μ_y on the fibres Y_y satisfy for almost all $y \in Y$ the relation

$$\mu_y(\{y\}) = \exp[\varphi(y)] A_\varphi^{-1}(\pi(y)).$$

Then μ is a relative equilibrium measure for T, $\nu = \mu \circ \pi^{-1}$ and φ. Moreover, any other relative equilibrium measure for $\mathcal{Y}$, ν and φ has the same conditional measures with respect to $\pi^{-1}\epsilon_X \vee T^{-1}\epsilon_Y$.

Proof: First note that T has a unilateral relative generator of finite relative entropy (in fact, even a finite relative generator) , since $\mathcal{Y}$ is fibrewise expanding. As S is invertible, for each $\mu' \in M(T)$ $(S, \mu' \circ \pi^{-1})$ is a regular factor of (T, μ') (because conditional independence holds in this case by trivial reasons), hence theorem **3.13** applies. Let $\mu' \in M(T)$ be such that $\mu' \circ \pi^{-1} = \nu$. As explained above, both μ and μ' give rise to the conditional measures μ_y of μ'_y on Y_y for $y \in Y$. Using equation (14) we obtain

$$\begin{aligned}
& h_{\mu'}(T|S) + \int \varphi \mathrm{d}\mu' \\
= \; & H_{\mu'}(\epsilon_Y / \pi^{-1}\epsilon_X \vee T^{-1}\epsilon_Y) + \int \varphi \mathrm{d}\mu' \\
= \; & \int_Y \left[H_{\mu'_y}(\epsilon_Y / \pi^{-1}\epsilon_X \vee T^{-1}\epsilon_Y) + \int_{Y_y} \varphi(y') \mathrm{d}\mu'_y(y') \right] \mathrm{d}\mu'(y) \\
= \; & -\int_Y \sum_{y' \in Y_y} \mu'_y(\{y'\}) \Big(\log[\mu'_y(\{y'\})] - \varphi(y') \Big) \mathrm{d}\mu'(y)
\end{aligned}$$

$$= -\int_Y \sum_{y'\in Y_y} \mu_y'(\{y'\}) \log \frac{\mu_y'(\{y'\})}{\mu_y(\{y'\})} \mathrm{d}\mu(y) + \int_X \log A_\varphi(x) \mathrm{d}\nu(x)$$
$$\leq \int_X \log A_\varphi(x) \mathrm{d}\nu(x)$$

with equality if and only if $\mu_y'(y') = A_\varphi^{-1}(\pi(y)) \exp \varphi(y')$ μ'-almost surely (this is the classical information inequality applied to the two probability measures μ' and μ on the finite set Y_y). Thus μ is a relative equilibrium measure and any other relative equilibrium measure must have the same conditional measures over Y_y for almost all $y \in Y$. □

Corollary 4.7
Let φ be a Hölder continuous basic potential. Then there exists a unique covariant Gibbs-family $\{\mu_x\}$. The measure $\mu := \mu_x \nu(\mathrm{d}x)$ gives the supremum in (15). □

Corollary 4.8
Let φ be a Hölder continuous potential. Then there exists a unique relative equilibrium measure for $\mathcal{Y}$, ν and φ. □

Corollary 4.9
If ν maximises the pressure $\mathrm{P}_S(\log A_\varphi)$ of $\log A_\varphi$ then it follows that $\mu_x \nu(\mathrm{d}x)$ maximises $\mathrm{P}_T(\varphi)$. □

5 Equilibrium conditional measures for basic potentials, examples

That a potential is *basic* means that V_x preserves constant functions, equivalently that the constant function 1 is an eigenfunction with eigenvalue $A_\varphi(x)$ for the operator V_x. Then, by proposition 4.1 in [DG1], a Gibbs family $\{\mu_x : x \in X\}$ for (φ, A_φ) is covariant and every S-invariant measure ν on X defines a T-invariant measure $\mu(E) = \int_X \mu_x(E)\nu(dx)$ on Y. Conversely, every covariant Gibbs family μ_x implies that V_x preserves constant functions. Thus, from the point of view of thermodynamics, it is important to characterise those potentials which are cohomologous to basic potentials. In general, there are potentials which do not admit a cohomologous basic potential.

Example 5.1
Let $\widehat{T}$ denote a polynomial mapping of $\mathbb{C}^2$. Such a mapping can be written in the form

$$\widehat{T}(x, y) = (p(x, y), q(x, y)),$$

where p and q are polynomials in $x, y \in \mathbb{C}$. It is called (d, d')-regular in [H2] where $d \in \mathbb{Q}$ and $d' \in \mathbb{N}$, if there are constants $k_1, k_2 > 0$ and $r \geq 0$ such that for every $z \in \mathbb{C}^2$, $\|z\| \geq r$

$$k_1 \, \|z\|^p \leq \|\widehat{T}(z)\| \leq k_2 \, \|z\|^q.$$

In case that $d = d'$, $\widehat{T}$ is called strict. A special case are skew products when p does not depend on y. Then $\pi \circ \widehat{T} = p \circ \pi$ where $\pi : \mathbb{C}^2 \to \mathbb{C}$, denotes the projection map onto the first factor.

According to [H1], a point $z \in \mathbb{C}^2$ is called weakly normal if there exists an open neighbourhood V of z and a family $\{\mathcal{K}_x : x \in V\}$ of at least one-dimensional complex analytic sets $\mathcal{K}_x$ such that $x \in \mathcal{K}_x$ and the family $\{\widehat{T}^n|_{\mathcal{K}_x} : \; n \geq 0\}$ is normal in x. The complement of the set of normal points is called the Julia set of $\widehat{T}$ and is denoted by $J(\widehat{T})$. It is shown in [H1] that for regular polynomials $J(\widehat{T})$ is compact and fully invariant. In particular, it follows that a regular skew product restricted to $J(\widehat{T})$ is a fibred system, but not a skew product in general. It is worth mentioning that $J_x = \pi^{-1}(x) \cap J(\widehat{T})$ is the fibre over x, and (for certain maps $\widehat{T}$ at least) is the Julia set of $\widehat{T}^n_x$ in case x is periodic with period n.

Let $\widehat{T} : \mathbb{C}^2 \to \mathbb{C}^2$ be a skew product and $T = \widehat{T}|_{J(\hat{T})}$ be its restriction to the Julia set $J = J(\widehat{T})$. Denote by J_x^* the Julia set for the family of maps

$$q_{p^n(x)} \circ q_{p^{n-1}(x)} \circ \cdots \circ q_x \qquad (n \geq 0).$$

Then $J = \bigcup_{x \in J(p)} \{x\} \times J_x^*$ if each q_x for $x \in J(p)$ is hyperbolic (see [H3]). If in addition, p is a hyperbolic polynomial, then T is hyperbolic as well (on J) (see [H1] and Theorem 2.3.1 in [H3]). In particular, these maps are fibrewise expanding and topologically exact along fibres. Moreover, i_1 is a local homeomorphism. Hence we obtain from theorem **4.5** together with theorem **4.7** and corollary **4.9**.

(*) For every Hölder continuous basic potential φ there exists a unique equilibrium state m for φ. m has a unique Hölder continuous disintegration $\{m_x : x \in J(p)\}$ with respect to $\pi^{-1}\mathcal{B}_{\mathbb{C}}$ which is absolutely continuous with respect to the Gibbs family $m \circ \pi^{-1}$ for φ. Moreover, the measure m_X is the unique equilibrium state for $\log A_\varphi$ and m is the unique relative equilibrium state for T, m_X and φ. □

It is possible to extend this example into different directions.

Example 5.2
Consider a regular polynomial mapping

$$\widehat{T} : \mathbb{C}^2 \to \mathbb{C}^2$$

of the form

$$\widehat{T}(x, y) = (p(x), q_x(y)) = (x^e + f, \; y^d + k(x)),$$

where $2 \leq d, e \in \mathbb{N}$ and $k(x)$ is any polynomial. Following [H3], we assume that for $x \in J(p)$ there exists a real-analytic Jordan domain U_x such that

- $x \mapsto U_x$ is continuous
- $q_x(U_x) \subset\subset U_{p(x)}$.
- $J(p)$ is a hyperbolic or parabolic Jordan curve.

It follows from [H2] and [H3] that $T = \widehat{T}|_J$ is uniformly expanding in case p is hyperbolic. Since the Julia set J is fully invariant (i.e. all inverse branches of fibre maps land in J again), $\mathcal{Y} = (J, T, J(p), p, \pi)$ is fibrewise expanding. It is also topologically exact along fibres, for this is true for periodic fibres (i.e. for fibres over periodic points in $J(p)$) and extends by the continuity of fibres. This follows in the expansive case as in [H3].
Now let $\varphi : J \to \mathbb{R}$ be a Hölder continuous basic potential.
Again, theorems **4.5** and **4.7** together with corollary **4.9** yield **(*)** □

Example 5.3
Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be a fibre expanding and exact skew product such that the number of preimages of points is fibrewise constant. Assume that $\varphi = 0$, i.e. we are considering topological entropy. Examples are provided by skew products of the form

$$T(x, y) = (p(x), y^d + k(x)).$$

We find that

$$V_x 1(y) = \sum_{\substack{T(y')=y \\ \pi(y')=x}} 1$$

equals some constant D, because the number of preimages of points is constant. Therefore φ is a basic potential and, by theorem **4.5**, we obtain a unique measure of maximal entropy for p on $J(T)$ by

$$\int_Y g(y)\mu(dy) = \int_X \int_{Y_x} g(y)\mu_x(dy)\nu(dx)$$

where μ_x is the covariant Gibbs family and ν is the unique measure of maximal entropy for S (by [L] and [M]). Thus, there exists a unique measure of maximal entropy of skew products of doughnut-type on J (cf. example **5.2**).

Results of this type appear frequently in the literature. For example, it has been shown in [J] that the measure of maximal entropy for a strict skew product in $\mathbb{C}^2$ derived from the Green current admits a disintegration given by the slices of the current on the vertical lines. This is a special case of the above statement.□

Example 5.4
Let G denote a finitely generated semigroup of hyperbolic rational functions $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ (see [S]). Let f_i, $(1 \le i \le s)$, denote a set of generators of G and $\Sigma = \{1, \ldots, s\}^{\mathbb{N}}$. Define a map

$$\hat{T} : \Sigma \times \overline{\mathbb{C}} \to \Sigma \times \overline{\mathbb{C}}$$

by

$$\hat{T}((\xi_1, \xi_2, \ldots), z) := ((\xi_2, \xi_3, \ldots), f_{\xi_1}(z)).$$

It has been shown in [S], that for each $\xi \in \Sigma$ the Julia set

$$J_\xi = \{z \in \overline{\mathbb{C}} : \ z \text{ is not normal for } f_{\xi_n} \circ \cdots \circ f_{\xi_1}\}$$

defines the Julia set

$$J(\hat{T}) = \bigcup_{\xi \in \Sigma} \{\xi\} \times J_\xi.$$

$J(\hat{T})$ is compact and $\hat{T}$-invariant. Because of hyperbolicity, $T = \hat{T}_{|J(T)}$ is fibre expanding and exact, and the shift transformation in the base is expanding. Hence we obtain **(*)** in the same way as in example **5.1**.

Similar results can be obtained for Sester's [Se] approach. It is easily seen that his hyperbolic dynamics is fibrewise expanding and topologically exact. The conclusion here holds under further assumptions on the base. $\square$

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