Transfer operators and period functions for subgroups of $PSL(2, \mathbb{Z})$

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0. Introduction: In the approach through the transfer operator belonging to the geodesic flow on the modular surface, Selberg's zeta function for $PSL(2, \mathbb{Z})$ can be expressed as the Fredholm determinant of a meromorphic family of nuclear operators $\mathcal{L}_\beta$. They generalize the transfer matrices of statistical mechanics which serve to calculate so called partition functions. Since the nontrivial zero's of Selberg's function are closely related to the eigenvalues of the Laplace-Beltrami operator on the modular surface, the transfer operator $\mathcal{L}_\beta$ connects the geodesic flow trough its classical properties to the spectrum of the quantized system. Indeed, it turned out that also the eigenfunctions of $\mathcal{L}_\beta$ to eigenvalue $\lambda = 1$ can be directly related to the Maass wave forms for $PSL(2, \mathbb{Z})$. These eigenfunctions generalize, as shown by J. Lewis and D. Zagier, the period polynomials of Eichler, which also show up as eigenfunctions for $\mathcal{L}_\beta$ with eigenvalue $\lambda = 1$ for certain β-values.
In the present note we report on first results to extend the above results to subgroups of the full modular groups $PSL(2, \mathbb{Z})$.
In detail the paper is organized as follows: we briefly recall the definition of the Ruelle zeta function for the geodesic flow for a Fuchsian group Γ with some representation $\chi : \Gamma \to \text{end } V$ and how it is related to the transfer operator. Next we consider subgroups $\Gamma \subset PSL(2, \mathbb{Z})$ of finite index and the symbolic dynamics of the geodesic flow on the surface $\Gamma/\mathbb{H}$. This allows us to construct the transfer operator $\mathcal{L}_\beta^\Gamma$ for Γ. We show that $\mathcal{L}_\beta^\Gamma$ is closely related to the transfer operator for $PSL(2, \mathbb{Z})$ together with its representation χ_{ind} induced from the trivial representation of Γ. In the next chapter we consider the special congruence subgroups Γ_o (2) , Γ^o (2) , Γ_ϑ and $\Gamma(2)$ and their transfer operators. Wie briefly recall Eichler's automorphic integrals and the period polynomicals of holomorphic cusp forms for a general Fuchsian group and their cocycle relations, which in the case $\Gamma = PSL(2, \mathbb{Z})$ have been related by Zagier to a functional equation of Lewis. We derive the analogous functional equation for the group Γ_o (2) and discuss the solutions of this equation. It turns out that these solutions fall into two classes, so called old and new solutions. They obviously correspond to old and new forms in Atkin-Lehner's theory. Indeed in the case of old solution we can establish a relation with the old forms for these simple congruence groups whereas for the new solutions besides some polynomial solutions not much is known up to now besides numerical hints. But even in case of the polynomials we do not know the explicit relation to the new cusps forms for instance for the group Γ_o (2).

I. Ruelle's zeta function and the transfer operator

On the hyperbolic plane $\mathbb{H} = \{z = x + iy : y > 0\}$ with metric $ds^2 = \frac{1}{y^2}\,(dx^2 + dy^2)$ acts the modular group

$$\Gamma(1) = PSL(2, \mathbb{Z}) = \{g = \left(\begin{array}{c} a\ b \\ c\ d \end{array}\right) : a, b, c, d \in \mathbb{Z},\ det\ g = 1\}/(\pm\ 1)$$

by Möbius transformations

$$gz = \frac{az + b}{cz + d} \quad . \tag{1}$$

In the following we are especially interested in subgroups $\Gamma \subset \Gamma(1)$ of finite index $[\Gamma(1) : \Gamma] = \mu < \infty$. The geodesics on the surface $\Gamma/\mathbb{H}$ are just the projections onto $\Gamma/\mathbb{H}$ of the geodesics on $\mathbb{H}$ which are half circles based on the real line or straight lines parallel to the imaginary axis.

The geodesic flow $\phi_t : S_1\,\Gamma/\mathbb{H} \longrightarrow S_1\,\Gamma/\mathbb{H}$ is the flow with constant velocity along the geodesics on $\Gamma/\mathbb{H}$. An orbit γ of ϕ_t is periodic iff $\pi\gamma$ is a closed geodesic where $\pi : S_1\Gamma/\mathbb{H} \longrightarrow \Gamma/\mathbb{H}$ denotes the projection of the unit tanget bundle $S_1\Gamma/\mathbb{H}$ onto $\Gamma/\mathbb{H}$. For unit velocity the period $T(\gamma)$ of a periodic orbit γ is identical to the length $l(\pi\gamma)$ of the closed geodesic $\pi\gamma$ which we denote simply by $l(\gamma)$.

Consider some finite dimensional representation $\chi : \Gamma \longrightarrow end\ V$. Ruelle's dynamical zeta function $\zeta^\chi(\beta)$ for the geodesic flow ϕ_t which representation χ is then defined for $Re\ \beta > 1$ as

$$\zeta^\chi(\beta) = \prod_{\gamma\ prime} det\left(1 - \chi(\sigma_\gamma)e^{-\beta l(\gamma)}\right)^{-1} \tag{2}$$

where $\sigma_\gamma \in \Gamma$ has the property that it fixes a lift $\tilde{\gamma}$ of the geodesic γ in $\mathbb{H}$.

The generalized Selberg zeta function $Z^{\Gamma,\chi}$ of Γ with representation χ is then given iven for $Re\beta > 1$ as

$$Z^{\Gamma,\chi}\,(\beta) = \prod_{k=0}^{\infty}\ \zeta^\chi(\beta + k)^{-1} \tag{3}$$

The classical approach to the analytic properties, the zero's and poles and to the functional equation of this function is through Selberg's trace formula [14]. Another approach, which grew out from the thermodynamic formalism and its applications in the ergodic theory of dynamical dystems [12], uses the so called transfer operator of the geodesic flow ϕ_t . The main idea behind this appoach is the following: consider a Poincaré section $\sum$ transversal to ϕ_t and the induced Poincaré map $P : \sum \to \sum$. Obviously an orbit γ is periodic under ϕ_t iff the point $x \in \gamma \cap \sum$ is periodic under the Poincaré map P, that is $P^n x = x$ for some $n \geqslant 1$. The period $l(\gamma)$ can be expressed through the recurrence time function $r : \sum \to \mathbb{R}_+$ with $Px = \phi_{r(x)}x$ as

$$l(\gamma) = \sum_{k=0}^{n-1}\ r(P^k x) \tag{4}$$

if $P^n x = x$

By simple calculation one then shows [13]

<u>Lemma:</u> The Ruelle function $\zeta^\chi(\beta)$ can be written as

$$\zeta^\chi(\beta) = exp\ \sum_{n=1}^{\infty}\ \frac{1}{n}\ Z_n^\chi\ (P, \beta)$$

with $Z_n^\chi\ (P, \beta)$ the partition functions

$$Z_n^\chi\ (P, \beta) = \sum_{x \in Fix P^n}\ trace\ \chi\ (\sigma_x)\ exp - \beta\ \sum_{k=0}^{n-1}\ r\ (P^k x)$$

where $\sigma_x \in \Gamma$ fixes the closed orbit γ through x.
The main problem now is to calculate the partition function Z_n. Long time ago there has been developped in Statistical Mechanics the so called transfer matrix method to transform this combinatorial problem into an algebraic one: one constructs a matrix or more generally a trace class operator whose traces determine the numbers $Z_n\,(\beta)$.
This program obviously demands the explicit knowledge of the Poincaré map P, its fixed points and the recurrence time function $r : \sum \to \mathbb{R}_+$.
Surprisingly, these quantities could be determined explicitly for the modular group and the geodesic flow on the modular surface $\Gamma(1)/\mathbb{H}$ [11].
The Poincaré section $\sum$ can be choosen in a certain coordinate system of $S_1\ \Gamma(1)/\mathbb{H}$ as $\sum = I_2 \times \mathbb{Z}_2$, where I_2 denotes the unit square in $\mathbb{R}^2$. The Poincaré map $P : I_2 \times \mathbb{Z}_2 \circlearrowleft$ then reads

$$P(x_1, x_2, \varepsilon) = \left(\frac{1}{x_1}\ mod\ 1\ ,\ \frac{1}{x_2 + \left[\frac{1}{x_1}\right]}, -\varepsilon \right) \tag{5}$$

with $\left[\frac{1}{x_1}\right]$ the largest integer smaller than $\frac{1}{x_1}$.
The recurrence time function r turns out to be the function

$$r(x_1, x_2, \varepsilon) = -log\ x_1^2\ \ .$$

From these data's the transfer operator $\mathcal{L}_\beta$ for the map P restricted to the expanding directions which in our case are just (x_1, ε) , $\varepsilon\ = \pm 1$, can be constructed in a standard manner and reads [4], [3]

$$\tilde{\mathcal{L}}_\beta^\chi\ \underline{f}\ (x_1, \varepsilon) = \sum_{n=1}^{\infty} \left(\frac{1}{x_1 + n} \right)^{2\beta}\ \chi(QT^{n\varepsilon}) \underline{f}\ (\frac{1}{x+n}, -\varepsilon) \tag{6}$$

where T and Q denote the generators $Tz = z + 1$, $Qz = -\frac{1}{z}$ of the modular groups $\Gamma(1)$.
Chosing the function $\underline{f} : I_2 \times \mathbb{Z}_2 \to V$ holomorphic for instance in the disc $D = \{z : |z - 1| < \frac{3}{2}\}$, the operator $\tilde{\mathcal{L}}_\beta^\chi$ is a well defined operator for $Re\beta > \frac{1}{2}$ in the Banach space $B(D \times \mathbb{Z}_2,\ V)$ of functions holomorphic in D with values in the linear space V. Indeed, one can show the following properties [4]

- $\tilde{\mathcal{L}}_\beta^\chi$ is a meromorphic family of nuclear operators with possible poles only at $\beta = \beta_k = \frac{1-k}{2}$, $k = 0, 1, \cdots$
- $Z_n^\chi\ (\beta)\ =\ trace\ (\tilde{\mathcal{L}}_\beta^\chi)^n - (-1)^n\ trace\ (\tilde{\mathcal{L}}_\beta^\chi)^n$
- $\zeta^\chi\ (\beta)\ =\ det\ (1 - \tilde{\mathcal{L}}_{\beta+1}^\chi) / det\ (1 - \tilde{\mathcal{L}}_\beta^\chi)$
- $Z^{\Gamma(1),\chi}(\beta)\ =\ det\ (1 - \tilde{\mathcal{L}}_\beta^\chi)$

This shows, that the zero's and poles of Selberg's zeta function are closely related to spectral properties of the transfer operator $\tilde{\mathcal{L}}_\beta^\chi$. An analogous interpretation of zero's has been given, as is well known, for the zeros of the zeta function of Artin and Weil defined for projective algebraic varieties over finite fields by Dwork and Deligne [7]. If also the zero's of Riemann's zeta function allow for such an interpretation is being discussed at the moment with great intensity.
Quite a lot is known also about the eigenfunctions with eigenvalue $\lambda = 1$ of the transfer operator $\tilde{\mathcal{L}}_\beta^\chi$ for the trivial representation $\chi = 1$. In this case $\tilde{\mathcal{L}}_\beta^\chi$ has the form

$$\tilde{\mathcal{L}}_\beta^\chi = \begin{pmatrix} D & \mathcal{L}_\beta \\ \mathcal{L}_\beta & 0 \end{pmatrix}$$

with $\mathcal{L}_\beta\, f(z) = \sum\limits_{n=1}^{\infty} (\frac{1}{z+n})^{z\beta}\, f(\frac{1}{z+n})$, $Re\beta > \frac{1}{2}$, $f \in B(D)$.

Then Selberg's function $Z^{\Gamma(1)}(\beta)$ can be written as

$$Z^{\Gamma(1)}(\beta) = det\, (1 - \mathcal{L}_\beta)\; det\, (1 + \mathcal{L}_\beta) \tag{7}$$

Hence one has to understand the eigenfunctions of $\mathcal{L}_\beta$ with eigenvalues $\lambda_\beta = \pm 1$. If f_β is such an eigenfunction then it satisfies the functional equation [9]

$$\lambda_\beta f_\beta(z) = \lambda_\beta f_\beta(z+1) + (\frac{1}{z+1})^{2\beta} f_\beta(\frac{1}{z+1}) \quad , \quad \lambda_\beta = \pm 1 \quad . \tag{8}$$

This equation has been derived by J. Lewis in his study of the Maass wave forms of the modular group by methods in harmonic analysis. One can show also that any solution of this Lewis equation holomorphic in the complex plane cut along $(-\infty,\ 1)$ with a certain growth condition at infinity defines an eigenfunction of $\mathcal{L}_\beta$ with eigenvalue λ_β [3],[10].
Work by Zagier, Lewis, Chang and Mayer [3], [5] has shown the following results for the eigenfunctions f_β with eigenvalue $\lambda_\beta = \pm 1$:

- for $\beta = -n$, $n = 1, 2, \cdots$, corresponding to the trivial zero's of $Z(\beta)$, the functions $f_\beta(z-1)$ are the odd ($\lambda_\beta = 1$) respectively even parts ($\lambda_\beta = -1$) of Eichler's period polynomials $p_Q\ (z)$ of the holomorphic cusps forms of weight $2n+2$ of $\Gamma(1)$ respectively the odd/even parts of Zagier's period functions $p_Q\ (z)$ of the holomorphic Eisenstein series of weight $2n+2$ of $\Gamma(1)$.
- for $Re\beta = \frac{1}{2}$, corresponding to the so called spectral zero's of $Z(\beta)$, the functions $f_\beta(z-1)$ are the Lewis transforms of the even ($\lambda_\beta = 1$) respectively odd Maass cusp forms $\varphi_\beta(z)$ of $\Gamma(1)$ obeying $-\triangle\varphi(z) = \beta(1-\beta)\varphi(z)$ and $\varphi(-x, y) = \pm\varphi(x, y)$
- for $Re\beta > 0$ such that $\zeta(2\beta) = 0$, corresponding to the nontrivial zero's of Riemann's zeta function $\zeta(z)$, the function $f_\beta(z-1)$ is the Lewis transform of the nonholomorphic Eisenstein series $E(\beta, z)$ of $\Gamma(1)$.
  The Lewis transforms relating Maass- and Eisenstein wave forms to period functions read [9]
  $$f_\beta(z-1) \;=\; \phi_\beta\ (z) \;=\; z\int_0^\infty y^\beta\ \varphi_\beta(iy)(z^2+y^2)^{-\beta-1} dy \tag{9}$$
  for even forms φ_β and [?]
  $$f_\beta(z-1) \;=\; \phi_\beta\ (z) \;=\; \int_0^\infty \frac{\partial}{\partial x}\varphi_\beta(iy)(z^2+y^2)^{-\beta} y^\beta\ dy \tag{10}$$
  for odd forms φ_β .
- for $\beta = 1$ the function $f_\beta(z-1) = \frac{1}{z}$ is the Lewis transform of the Maass form $\varphi_\beta(z) = \ const$.

Remark: The transfer operator $\mathcal{L}_\beta$ for $\beta = 1$ is the well known Perron-Frobenius operator of ergodic theory for the Gau"z-map. It eigenfunction $f_\beta\ (z) = \frac{1}{z+1}$ is up to normalization the invariant density of the Gau"z measure.

II. The transfer operator for subgroups $\Gamma \subset \Gamma(1)$

Since $[\Gamma(1):\Gamma]=\mu<\infty$ the surface $\Gamma/\mathbb{H}$ is a μ-fold, in general branched, covering of the modular surface $\Gamma(1)/\mathbb{H}$. This allows [?] one to lift both the Poincare surface and the Poincare map from $S_1\Gamma(1)/\mathbb{H}$ to $S_1\Gamma/\mathbb{H}$: $\sum_\Gamma = I_1\times\mathbb{Z}_2\times\Gamma\setminus\Gamma(1)$ and $P_\Gamma:\sum_\Gamma\to\sum_\Gamma$ is given by

$$P_\Gamma\ (x_1,x_2,\varepsilon,[g])\ =\ (\frac{1}{x_1}mod\ 1\ ,\ \ \frac{1}{x_2+[\frac{1}{x_1}]}\ ,\ -\varepsilon,\ [gT^{n\varepsilon}Q]) \tag{11}$$

where $n=n(x_1)=[\frac{1}{x_1}]$.
From this one next constructs the transfer operator $\tilde{\mathcal{L}}_\beta^\Gamma$ in the usual way [4]

$$\tilde{\mathcal{L}}_\beta^\Gamma f(z,\varepsilon,[g])=\sum_{n=1}^\infty\left(\frac{1}{z+n}\right)^{2\beta}f\left(\frac{1}{z+n}\ ,-\varepsilon\ ,\ [gQT^{n\varepsilon}]\right)\qquad. \tag{12}$$

Denoting by $\chi^\Gamma:\Gamma(1)\to\mathcal{C}^\mu$ the representation of $\Gamma(1)$ induced from the trivial representation of the subgroup Γ the operator $\tilde{\mathcal{L}}_\beta^\Gamma$ can be written as

$$\tilde{\mathcal{L}}_\beta^\Gamma\ \underline{f}(z,\varepsilon)=\sum_{n=1}^\infty\left(\frac{1}{z+n}\right)^{2\beta}\chi^\Gamma\ (QT^{n\varepsilon})\underline{f}\left(\frac{1}{z+n}\ ,\ -\varepsilon\right) \tag{13}$$

with $\underline{f}:D\times\mathbb{Z}_2\to\mathbb{C}^\mu$ holomorphic in $z\in D$.

Comparing this with expression (6) we see, that $\tilde{\mathcal{L}}_\beta^\Gamma$ is identical $\tilde{\mathcal{L}}_\beta^{\Gamma(1),\chi\Gamma}$ the transfer operator for $\Gamma(1)$ with representation χ^Γ . Hence if $\chi^\Gamma=\bigoplus\ \chi_i$ is the decomposition of χ^Γ into irreducible representations χ_i this induces a decomposition of $\tilde{\mathcal{L}}_\beta^\Gamma$ into a direct sum of transfer operators $\bigoplus\limits_i\ \tilde{\mathcal{L}}^{\Gamma(1),\chi_i}$.

The transfer operator $\tilde{\mathcal{L}}_\beta^\Gamma$ has the following properties [4]

- $\tilde{\mathcal{L}}_\beta^\Gamma$ is a meromorphic family of nuclear operators with possible poles only at $\beta=\beta_k=\frac{1-k}{2}$, $k=0,1,2\cdots$.
- $Z^\Gamma(\beta)=det\ (1-\tilde{\mathcal{L}}_\beta^\Gamma)=det\ (1-\tilde{\mathcal{L}}_\beta^{\Gamma(1),\chi^\Gamma})\ =\ Z^{\Gamma(1),\chi^\Gamma}(\beta)$ which is a special case of the Artin-Venkov-Zograf formula.
- If $\chi^\Gamma\ (T^2)=1$, then $\tilde{\mathcal{L}}_\beta^\Gamma$ can be written as
  $\tilde{\mathcal{L}}_\beta^\Gamma\ =\ \begin{pmatrix}0 & \tilde{\mathcal{L}}_\beta^\Gamma\\ \tilde{\mathcal{L}}_\beta^\Gamma & 0\end{pmatrix}$
  where the operator $\tilde{\mathcal{L}}_\beta^\Gamma$ acts as

$$\tilde{\mathcal{L}}_\beta^\Gamma\ \underline{f}(z)\ =\ \sum_{n=1}^\infty\left(\frac{1}{z+n}\right)^{2\beta}\chi^\Gamma(QT^n)\ \underline{f}\ \left(\frac{1}{z+n}\right)\qquad. \tag{14}$$

Therefore the Selberg function can be written as

$$Z^\Gamma(\beta)\ =\ det\ (1-\tilde{\mathcal{L}}_\beta^\Gamma)\ det\ (1+\tilde{\mathcal{L}}_\beta^\Gamma) \tag{15}$$

As an example consider the congruence subgroups $\Gamma\ =\ \Gamma_o\ (2)$, $\Gamma^o\ (2)$ or Γ_ϑ . In all these cases one finds:

$$\chi^\Gamma\ =\ \chi_1\ \oplus\ \chi_2$$

where χ_1 denotes the trivial 1-dim. representation, and χ_2 denotes the irreducible 2-dim. representation of $\Gamma(1)$.

For the principal congruence subgroup $\Gamma(2)$ one finds

$$\chi^{\Gamma} = \chi_1 \oplus \chi_1' \oplus \chi_2 \oplus \chi_2$$

where χ_1' is the nontrivial irreducible representation of $\Gamma(1)$.

From this one can see already that for $\Gamma(2)$ the transfer operator $\tilde{\mathcal{L}}_\beta^{\Gamma(2)}$ decomposes into operators corresponding to the groups $\Gamma(1)$ and $\Gamma_o(2)$. The only new part is the one corresponding to the representation χ_1' . We will see later how this fits nicely into Atkin and Lehner's theory of old and new forms.

III. The period functions for $\Gamma \subset \Gamma(1)$

In the following we restrict our discussion to the case $\Gamma = \Gamma_o(2)$, we remark however, that in principle we can treat any subgroup $\Gamma \subset \Gamma(1)$ of finite index.
The group $\Gamma_o(2)$ has the generators $e_1 = T$, $e_2 = QT^{-2}Q$ and $e_3 = T^{-1}QT^{-2}Q$ with relations $e_1e_2e_3 = 1$ and $e_3^2 = 1$.

In Eichler's theory [8] of automorphic integrals and period polynomials to every holomorphic cusp form φ_k of weight k of a Fuchsian group Γ can be defined a family p_σ , $\sigma \in \Gamma$, of polynomials of degree $\leq k-2$ with

$$p_\sigma(z) = \frac{1}{(k-2)!} \int_{\tau_0}^{\tau_0} (az + b + (c\tau + d))^{k-2} \varphi_k(\tau)\, d\tau + \Theta(z)\,|_{2-k}\ \sigma - \Theta(z) \tag{16}$$

where Θ is an arbitrary polynomical of degree $\leq k-2$ and the slash operator $|_\sigma$ is defined as

$$f(z)\,|_k\ \sigma = (cz+d)^{-k}\, f(\sigma z) \quad . \tag{17}$$

when $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

The polynomials p_σ fulfill the following cocycle relations

$$p_{\sigma_1\sigma_2} = p_{\sigma_1}\,|_{2-k}\ \sigma_2 + p_{\sigma_2} \tag{18}$$

which show that the polynomials for the generators of Γ determine the family p_σ completely. Since for T the polynomial can be chosen $p_\tau \equiv 0$ [8] it follows from the relations for $\Gamma_o(2)$ that p_{e_3} determines p_{e_2} . The relation $e_3^2 = 1$ leads to the following equation for p_{e_3} :

$$p_{e_3}(z) + p_{e_3}\left(\frac{z+1}{-2z-1}\right)(-2z-1)^{k-2} = 0 \tag{19}$$

From Eichler's theory it follows that the polynomial solutions of this functional equation are just the period polynomials p_{e_3} for the group $\Gamma_o(2)$.

Trivial solutions of this equation are obviously provided by the period polynomials $p_Q\ (z)$ of the modular group $\Gamma(1)$ through [6]

$$p_{e_3}\ (z) = p_Q\ (z) + z^{k-2} p_Q\ \left(-\frac{2z+1}{z}\right) \quad . \tag{20}$$

In the work of J. Lewis and D. Zagier for $\Gamma(1)$ there is a close relation of the corresponding cocycle relations for p_Q and the functional equation of Lewis. To extend their approach to general subgroups, especially for $\Gamma_o\ (2)$, we need the analogous functional equation for this group. This we will discuss next.

Consider an eigenfunction $\underline{f}_\beta$ of the operator $\mathcal{L}_\beta^{\Gamma_o\ (2)}$ with $\mathcal{L}_\beta^{\Gamma_o\ (2)}\ \underline{f}_\beta\ (z) = \lambda_\beta\ \underline{f}_\beta\ (z)$. For $\lambda_\beta = \pm 1$ we find after a simple calculation for the function $\underline{\phi}_\beta\ (z) =\ \underline{f}_\beta\ (z-1)$ the functional equation [6]:

$$\lambda_\beta\ [\underline{\phi}_\beta\ (z) - \chi^{\Gamma_o\ (2)}\ (QTQ)\ \underline{\phi}_\beta\ (z+1)]\ -\ z^{-2\beta}\ \chi^{\Gamma_o\ (2)}\ (QT)\ \underline{\phi}_\beta\ (1+\frac{1}{z})\ =\ \underline{0}\ . \tag{21}$$

Inserting the representation $\chi^{\Gamma_o(2)}$ this leads to the following equations for $\underline{\phi}_\beta\ =\ \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$, where we suppressed the dependence of the ϕ_i on β :

$$\begin{aligned} \lambda_\beta\ [\phi_1\ (z)\ -\ \phi_3\ (z+1)]\ -\ z^{-2\beta}\ \phi_3\ (1+\frac{1}{z})\ &=\ 0 \\ \lambda_\beta\ [\phi_2\ (z)\ -\ \phi_2\ (z+1)]\ -\ z^{-2\beta}\ \phi_1\ (1+\frac{1}{z})\ &=\ 0 \\ \lambda_\beta\ [\phi_3\ (z)\ -\ \phi_1\ (z+1)]\ -\ z^{-2\beta}\ \phi_2\ (1+\frac{1}{z})\ &=\ 0 \end{aligned} \tag{22}$$

This shows that the function $\hat{\phi}\ (z)\ :=\ \phi_1\ (z) + \phi_2\ (z) + \phi_3\ (z)$ solves Lewis equation for the group $\Gamma(1)$

$$\lambda_\beta\ \hat{\phi}\ (z) = \lambda_\beta\ \hat{\phi}\ (z+1)\ +\ \frac{1}{z^{2\beta}}\ \hat{\phi}\ (1+\frac{1}{z}) \quad . \tag{23}$$

It is also easy to see [6] that the solution of the above system of equations can be reduced to the solution of the following equation for the component $\phi_2\ (z)$:

$$\lambda_\beta\ [\phi_2\ (z)\ -\ \phi_2\ (z+1)]\ =\ (2z+1)^{-2\beta}\ \left[\lambda_\beta\ \phi_2\ \left(\frac{z}{2z+1}\right)\ +\ \phi_2\ \left(\frac{z+1}{2z+1}\right)\right] \tag{24}$$

Namely, if ϕ_2 solves this equation then

$$\underline{\phi}\ (z)\ =\ \begin{pmatrix} \lambda_\beta \left[\phi_2\ \left(\frac{1}{z-1}\right)\ -\ \phi_2\ \left(1+\frac{1}{z-1}\right)\right]\ (z-1)^{-2\beta} \\ \phi_2\ (z) \\ \lambda_\beta\ z^{-2\beta}\ \phi_2\ (\frac{1}{z}) \end{pmatrix} \tag{25}$$

solves equation (21) identically.

There is a class of solutions of equation (24) and hence of equation (21) closely related to the solutions of Lewis equation (23) for $\Gamma(1)$ [6].

Proposition: Let $\phi\ (z)$ be a solution of Lewis equation (23) for $\Gamma\ (1)$ for some β and $\lambda_\beta\ =\ \pm\ 1$. Then $\phi\ (z)$ and $\phi\ (2z)$ solve equation (24) for the same β and λ_β .
The solutions $\underline{\phi}$ for equation (21) have the form

$$\underline{\phi}_1\ (z)\ =\ \begin{pmatrix} \phi(z) \\ \phi(z) \\ \phi(z) \end{pmatrix} \qquad \textit{respectively} \qquad \underline{\phi}_2\ (z)\ =\ \begin{pmatrix} 2^{-2\beta}\ (\phi\left(\frac{z-1}{2}\right) - z^{-2\beta}\ \phi\left(\frac{z-1}{z}\right) \\ \phi\ (2z) \\ 2^{-2\beta}\ \phi\ \left(\frac{z}{2}\right) \end{pmatrix} .$$

To understand these old solutions we recall briefly the theory of old and new forms of Atkin and Lehner [1] for $\Gamma_o\ (2)$ respectively $\Gamma^o\ (2)$ [2].

If $g(z)$ is a modular form for $\Gamma\ (1)$ then $g\ (z)$, $g\ (2z)$ are modular forms for $\Gamma_o\ (2)$, $g\ (z)$ and $g\ \left(\frac{z}{2}\right)$ are modular forms for $\Gamma^o\ (2)$. If $h\ (z)$ is a modular form for $\Gamma_o\ (2)$ respectively for $\Gamma^o\ (2)$ then $h\ \left(\frac{z}{2}\right)$ is a modular form for $\Gamma^o\ (2)$ respectively $h\ (2z)$ is a modular form for $\Gamma_o\ (2)$.

Obviously the solutions of the above Proposition are "old" solutions in this sense.

Let us remark only without giving details that those solutions $\phi\ (z)$ of Lewis equation (23) for $PSL\ (2,\mathbb{Z})$ which fulfill certain asymptotic relations at infinity and hence determine also eigenfunctions of the transfer operator determine through the formulas in the Proposition also eigenfunctions of the transfer operator $\mathcal{L}_\beta^{\Gamma_o\ (2)}$. They hence are called old eigenfunctions being determined by the eigenfunctions of $\mathcal{L}_\beta^{\Gamma_o\ (1)}$.

Besides the old eigenfunctions of the transfer operator respectively old solutions of Lewis equation there should exist also new solutions. Since we do not know how the theory of Zagier and Lewis for $\Gamma\ (1)$ can be extended to $\Gamma_o\ (2)$ we looked in a first step after polynomial solutions of equation (24). Indeed for $\beta = -n$, we found such solutions numerically [6]: for $\lambda_\beta = 1$ they have the form

$$\underline{\phi}_{odd}\ (z)\ =\ \begin{pmatrix} -\phi_o\ (z) - e\ \phi_o\ \left(\frac{z}{2}\right) 2^n \\ \phi_o\ (z) \\ e\ \phi_o\ \left(\frac{z}{2}\right)\ 2^n \end{pmatrix} \tag{26}$$

where $\phi_o\ (z)$ is an odd polynomial solving equation (24) of degree $<\ 2n$.
For $\lambda_\beta = -1$ on the other hand we found solutions of the form

$$\underline{\phi}_{even}\ (z)\ =\ \begin{pmatrix} -\phi_e\ (z) - e\ \phi_e\ \left(\frac{z}{2}\right)\ 2^n + c\ p_{2n}^+\ (z) \\ \phi_e\ (z) \\ e\ \phi_e\ \left(\frac{z}{2}\right)\ 2^n \end{pmatrix} \tag{27}$$

with $\phi_e\ (z)$ an even polynomial solution of equation (24), $0 \neq c \in \mathbb{Q}$ and $e\ =\ \pm 1$. The polynomial $p_{2n}^+\ (z)$ is the even part of the period function of the holomorphic Eisenstein series $E_{2n+2}\ (z)$ [4].

One sees that for the odd solutions the sum of the components of $\underline{\phi}_{odd}$ vanishes and hence the solutions cannot be old solutions. Furthermore the solutions $\underline{\phi}_{odd}$, $\underline{\phi}_{even}$ exist only for those weights $2n + 2$ for which there exist new holomorphic cusp forms for the group $\Gamma_o\ (2)$ and the number of solutions found for small values of n numerically coincide exactly with this number of new forms. This obviously supports the expectation that these solutions $\underline{\phi}_{odd}$, $\underline{\phi}_{even}$ are in $1-1$ correspondence with the new cusp forms of $\Gamma_o\ (2)$. The form of these solutions show that the third component of $\underline{\phi}_{even}$ and $\underline{\phi}_{odd}$ describes the group $\Gamma^o\ (2)$, since with $\varphi(z)$ an automorphic form for $\Gamma_o\ (2)$ the function $\varphi\ \left(\frac{z}{2}\right)$ is an automorphic form for $\Gamma^o\ (2)$. One furthermore could conjecture that the first component in $\underline{\phi}$ is related to the group Γ_ϑ , since as we have seen, the three groups $\Gamma_o\ (2)$, $\Gamma\ (2)$ and Γ_ϑ lead to the same transfer operator and hence to the same equation (21) for $\underline{\phi}$.

What is known about the relation between the second component ϕ_2 of $\underline{\phi}$ and the automorphic forms or functions for the group $\Gamma_o\ (2)$? Surprisingly, for the old forms this relation seems to be given by the transformation determined by J. Lewis for Maass forms respectively the Eichler formula for holomorphic cusps forms for $\Gamma(1)$: in both cases a simple calculation shows that [6]

$$if\ p_Q\ (z)\ =\ \int_0^{i\infty} \varphi_{2n+z}\ (z')(z-z')^{2n}\ dz'\ =\ T_n\ \varphi_{2n+z}\ (z) \tag{28}$$

is Eichler's transformation for cusp forms φ_{2n+2} then

$$T_n\ \tilde{\varphi}_{2n+2}\ (z)\ =\ c_n\ p_Q\ (2z) \qquad for \qquad \tilde{\varphi}\ (z)\ =\ \varphi\ (2z) \tag{29}$$

respectively

$$T_n\ \tilde{\varphi}_{2n+2}\ (z)\ =\ d_n\ p_Q\ \left(\frac{z}{2}\right) \qquad for \qquad \tilde{\varphi}\ (z)\ =\ \varphi\ \left(\frac{z}{2}\right) \tag{30}$$

where c_n and d_n are some constants. Similarly one finds for the transformation of Lewis

$$\phi_\beta\ (z)\ =\ (L_\beta\ \varphi_\beta)\ (z)\ =\ z\ \int_0^\infty \varphi_\beta\ (iy)\ (z^2+y^2)^{-\beta-1}\ \mu^\beta\ dy \tag{31}$$

for Maass-wave forms or Eisenstein forms:

$$(L_\beta\ \tilde{\varphi}_\beta)\ (z)\ =\ c'_n\ \phi_\beta\ (2z) \qquad for \qquad \tilde{\varphi}_\beta\ (z)\ =\ \varphi_\beta\ (2z) \tag{32}$$

respectively

$$(L_\beta\ \tilde{\varphi}_\beta)\ (z)\ =\ d'_n\ \phi_\beta\ (2z) \qquad for \qquad \tilde{\varphi}_\beta\ (z)\ =\ \varphi_\beta\ \left(\frac{z}{2}\right) \qquad . \tag{33}$$

with c'_n , d'_n some constants.

Concerning the relation between new forms and new solutions respectively new eigenfunctions of the transfer operator for $\Gamma_o\ (2)$ unfortunately nothing is known at the moment.
It is also not known how the solutions ϕ_2 of Lewis equation for $\Gamma_o\ (2)$ are related to the solutions of the cocycle equation (19) for the period polynomials of the group $\Gamma_o\ (2)$. Numerical investigations of the polynomial solutions show however that the number of such solutions is identical to the number of solutions of equation (24) at least for small values of n. This obviously supports our expectation that these two equations are closely related to each other.

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