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THE TRANSFER OPERATOR APPROACH TO SELBERG'S ZETA FUNCTION AND MODULAR AND MAASS WAVE FORMS FOR $PSL(2,\mathbb{Z})$ *

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Abstract. In this paper we discuss the transfer operator approach to Selberg's zeta function for $PSL(2,\mathbb{Z})$. Since this function can be expressed as the Fredholm determinant $det(1-\tilde{\mathcal{L}}_\beta)$ of the transfer operator $\tilde{\mathcal{L}}_\beta$, $\beta \in \mathbb{C}$ for the geodesic flow on the modular surface, the zeros and poles of the Selberg function are closely related to those β-values, where $\tilde{\mathcal{L}}_\beta$ has an eigenvalue $\lambda = 1$ respectively where $\tilde{\mathcal{L}}_\beta$ has poles. It turns out that the corresponding eigenfunctions of $\tilde{\mathcal{L}}_\beta$ for eigenvalues $\lambda = 1$ are closely related to both holomorphic and non-holomorphic modular forms respectively the Maass wave forms. Therefore these eigenfunctions, which by definition of $\tilde{\mathcal{L}}_\beta$ are holomorphic functions, are by themselves interesting quantities for the group $PSL(2,\mathbb{Z})$: indeed special cases are the period polynomials and functions of the Manin-Eichler and Shimura theory of periods for this group. Another special example of such an eigenfunction is the well known density of Gauss's measure for the continued fraction expansion. The transfer operator approach hence in a surprising way combines several aspects of the theory of modular and Maass wave forms for the modular group, which up to now were not directly related.

Key words. quantum chaos, Selberg zeta function, dynamical zeta function, transfer operator, functional equation, modular forms, Maass wave forms, period polynomial, period function.

AMS(MOS) subject classifications. 58F20, 58F17, 81Q50, 11F72, 11F03, 11M41, 11F11, 39B32

1. Introduction.

Selberg's work on Fuchsian groups, his trace formula and his zeta function are interpreted in the current literature on quantum chaos [G] as prime examples for relating classical and quantum physics: the Selberg zeta function indeed is a special case of a dynamical zeta function associated to the geodesic flow on the unit tangent bundle of a surface of constant negative curvature defined by the action of a Fuchsian group Γ on hyperbolic 2-space. This kind of zeta function can be considered a generating function for the classical length spectrum, that means the periods of the closed orbits of this flow and hence in a certain sense is a purely classical object. Through the trace formula these periods get related to the spectrum of the Laplace-Beltrami operator on the surface, which traditionally describes the quantum mechanical behaviour of the free motion of a massive particle on the same surface. In terms of the Selberg zeta function the content of the trace formula is reflected by the location of the 'nontrivial' zeros of this function being directly related to the eigenval-

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ues of the Laplace-Beltrami operator. Much more involved is the relation between classical physics and the stationary states of the corresponding quantized system if quite generally the classical system is chaotic in the sense of having strong ergodic properties. Even for systems like the aforementioned free geodesic flows on constant negative curvature surfaces the results are rather weak in general. Only for so called arithmetic surfaces, where well developed methods of analytic number theory like the theory of Hecke operators and L-functions are available, stronger results could be proved quite recently [Sa]. For general Fuchsian groups however these methods are not available. Hence it is necessary to develop a different, if possible more dynamical approach to this circle of problems. A starting point for such an approach could be the transfer operator method. This method is well known in classical statistical mechanics and is one of the main ingredients of the thermodynamic formalism in the ergodic theory of hyperbolic dynamical systems like the Axiom-A systems. Quite soon in the development of this method it was realized by D. Ruelle [Ru1] [Ru2] [Ru3] that the transfer operators of such systems can be used for the analytic extension of their dynamical zeta functions, especially in the case when the dynamical systems are real analytic. Natural examples of such systems are the geodesic flows on surfaces of constant negative curvature. By expressing the zeta function through Fredholm determinants of such transfer operaters the zeros or poles of this function are directly related to spectral properties of these operators. There is obviously a well known historical example for this approach; namely Dwork's proof of rationality of the Artin-Weil zeta function for counting rational points in algebraic varieties over finite fields [D]. The operator Dwork defined in certain spaces of holomorphic functions over complex p-adic numbers can be interpreted as the transfer operator for the Frobenius map considered a discrete time dynamical system on the algebraic variety. For a very readable description of Dwork's precohomological approach to the Artin-Weil conjectures see the review article by P. Robba [Ro]. Like in Dwork's treatment it is in general not possible to extract from the representation of the dynamical zeta function as a quotient of Fredholm determinants of transfer operators the location of its zeros and poles, since a lot of cancellations in the different Fredholm determinants take place in general, which was clarified later by the cohomological approach of Grothendieck and Deligne [G1] [De]. The situation however is different when the most favourable situation occurs where the zeta function is just the Fredholm determinant of a single transfer operator: this happens for instance in the case of the modular group $PSL(2, \mathbb{Z})$. There the divisor of the Selberg function is determined by all the eigenvalues respectively the poles of the Fredholm determinant. Hence this system is an ideal example for testing how far the transfer operator method can be used to derive the properties of the Selberg zeta function from this operator and hence from a classical object. It turns out that this approach can give indeed much more than only the analytic properties of the Selberg function. All the eigenfunctions of the transfer operator to the eigenvalue $\lambda = 1$ are related to classical modular functions for the group $PSL(2, \mathbb{Z})$ as for instance the holomorphic modular forms, the Maas wave forms and presumably also the nonholomorphic Eisenstein series describing the scattering theory for the group. Since the Maass cusp forms are just the stationary states of the quantized geodesic flow we see that these states can be determined in principle from a classical object like the transfer operator.

In detail the paper is organized as follows: we briefly recall the definition and the analytic properties of the transfer operator $\tilde{\mathcal{L}}_\beta$ for $PSL(2,\mathbb{Z})$ and its relation to the generalized Perron-Frobenius operator $\mathcal{L}_\beta$ for the continued fraction map of Gauss. We discuss next the poles and the residues of the Fredholm determinants $det(1 \pm \mathcal{L}_\beta)$ and determine the singularities of the eigenvalues of $\mathcal{L}_\beta$ at the corresponding β-values. We decompose $\mathcal{L}_\beta$ into a part regular at the poles of $\mathcal{L}_\beta$ and a finite rank operator containing the singularities and show how the spectra of these two operators are related to the spectrum of $\mathcal{L}_\beta$. Of special interest are obviously those β-values for which $\mathcal{L}_\beta$ has the numbers $\lambda = 1$ or $\lambda = -1$ as eigenvalues, since these are closely related to the zeros of the Selberg function. We determine the eigenfunctions of $\mathcal{L}_\beta$ to the eigenvalues ± 1 for the β-values $\beta = \beta_k = \frac{1-k}{2}$, $k = 2N+1$ and $N \in \mathbb{N}$, which correspond to the so called 'trivial' zeros of Selberg's function. It is shown that the eigenfunctions at $\beta = \beta_k$ are the period polynomials of the holomorphic modular cusp forms of weight $k+1$, both their even and odd parts, and the period functions of the holomorphic Eisenstein series, again both the even and odd parts. We show that these eigenfunctions are just enough in number to give the correct order of the zeros at the β-values $\beta_k = \frac{1-k}{2}$, $k = 3, 5, 7, \cdots$. We show also that there are enough eigenfunctions with eigenvalues $\lambda = \pm 1$ to describe correctly the behaviour of the Selberg function at the points $\beta = \frac{1}{2}$, 0 and the negative half integers. By using a recent result by J. Lewis where he established an explicit relation between Maass cusp forms and certain holomorphic 'period functions' fulfilling a simple functional equation, we establish a relation between all the eigenfunctions of the Laplace-Beltrami operator for $PSL(2,\mathbb{Z})$ and the eigenfunctions of $\mathcal{L}_\beta$ with eigenvalue $\lambda = \pm 1$ vanishing at infinity for $\Re\beta > 0$. We finally show that a certain function $h_\beta(z)$ introduced recently by D. Zagier [Z1] which for all $\beta \neq 1$ fulfills Lewis functional equation indeed is identical for $\beta = \beta_k$, $k = 3, 5, 7, \cdots$, that means for those β-values which constitute the trivial zeros $\zeta(2\beta) = 0$ of Riemanns zeta function, to the odd parts of the period functions of the holomorphic Eisenstein series of weight $k+1$ and hence is an eigenfunction of $\mathcal{L}_\beta$ with eigenvalue $\lambda = -1$.

For β-values with $\zeta(2\beta) = 0$ corresponding to the nontrivial zeros of Riemanns function the analytic extension of $h_\beta(z)$ is an eigenfunction of $\mathcal{L}_\beta$ with eigenvalue $\lambda = 1$. However the explicit form of these functions is rather complicated and their relation to the nonholomorphic Eisenstein series to the same β-value is not known in detail, even if such a relation is certainly expected.

2. The thermodynamic formalism.

2.1. The modular surface. Consider the Poincaré upper half-plane $\mathcal{H} = \{x + iy | x, y \in \mathbb{R}, y > 0\}$ with Poincaré metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ and the quotient space $\mathcal{H}/\Gamma$, where Γ is a discrete subgroup of $PSL(2,\mathbb{R})$ acting on $\mathcal{H}$ as $z' = \frac{az+b}{cz+d}$ if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This is a hyperbolic surface with constant negative curvature -1. The geodesics on this surface are either straight lines parallel to the y axis or circles orthogonal to the x-axis, projected down to $\mathcal{H}/\Gamma$. The flow along these geodesics with unit speed is well known to be chaotic: starting from the same point in different directions there is an exponential fast separation, that means a positive Lyapunov exponent and hence sensitive dependence on initial conditions [M]. This geodesic flow corresponds to free classical motion of a particle on the surface. The standard quantization of this motion is just described by the Laplace-Beltrami operator on the surface

$$-\triangle_{LB} = -y^2\,(\,\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,)\,,$$

which hence describes the quantized free motion of a particle on the surface. The main question in quantum chaos now is to relate classical and quantum motion especially for those systems which classically are chaotic. In the following we address this question in the special case of the modular surface $\mathcal{H}/PSL(2,\mathbb{Z})$, where $PSL(2,\mathbb{Z})$ denotes the group $SL(2,\mathbb{Z})\ mod\ \pm 1$ whose elements $\gamma \in SL(2,\mathbb{Z})$ we identify with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for $a,b,c,d \in \mathbb{Z}$, $ad-bc=1$.

2.2. The transfer operator for $PSL(2,\mathbb{Z})$. The ergodic properties of the geodesic flow on the space $\mathcal{H}/PSL(2,\mathbb{Z})$ have been studied in [Se] and [AF] by constructing the symbolic dynamics for this flow: the geodesic flow for this surface is more or less isomorphic to the suspension flow over the map $T : I\times I\times \mathbb{Z}_2 \to I\times I\times \mathbb{Z}_2$,

$$T(x,y,\varepsilon) = (\,\frac{1}{x}\,mod\,1\,,\frac{1}{y+[\frac{1}{x}]},-\varepsilon) \tag{2.1}$$

under the roof function $\rho(x,y,\varepsilon) = -log\,x^2$ and hence closely related to the continued fraction map

$$T_G(x) = \frac{1}{x}\,mod\,1 \tag{2.2}$$

on the unit interval I as found already by E. Artin in [A]. From this it follows immediately, as discussed in general in [M1], that the transfer operator $\tilde{\mathcal{L}}_\beta$ for the geodesic flow on the modular surface has the form $\tilde{\mathcal{L}}_\beta : B(D)\oplus B(D) \to B(D)\oplus B(D)$ with

$$\tilde{\mathcal{L}}_\beta f(z,\varepsilon) := \sum_{n=1}^{\infty}(\frac{1}{n+z})^{2\beta} f(\frac{1}{n+z},-\varepsilon)\,,\quad \varepsilon \pm 1\,, \tag{2.3}$$

where $B(D)$ denotes the Banach space of holomorphic functions on the disc

$$z \in D := \left\{ z : z \in \mathbb{C},\ |\ z-1\ |< \frac{3}{2} \right\}\,,$$

with the sup norm. For $\Re\beta > \frac{1}{2}$, $\tilde{\mathcal{L}}_\beta$ in (2.3) defines a nuclear operator of order zero holomorphic in β and has a holomorphic Fredholm determinant

$det(1-\tilde{\mathcal{L}}_\beta)$. Obviously the operator $\tilde{\mathcal{L}}_\beta$ can be written as

$$\tilde{\mathcal{L}}_\beta = \begin{pmatrix} 0 & \mathcal{L}_\beta \\ \mathcal{L}_\beta & 0 \end{pmatrix}, \tag{2.4}$$

with $\mathcal{L}_\beta : B(D) \to B(D)$ the generalized Perron-Frobenius operator [LM] for the Gauss map $T_G : I \to I$

$$\mathcal{L}_\beta f(z) := \sum_{n=1}^{\infty} (\frac{1}{n+z})^{2\beta} f(\frac{1}{n+z}), \quad \Re\beta > \frac{1}{2}. \tag{2.5}$$

The Fredholm determinant $det(1-\tilde{\mathcal{L}}_\beta)$ hence can be written for $\Re\beta > \frac{1}{2}$ as

$$det(1-\tilde{\mathcal{L}}_\beta) = det(1+\mathcal{L}_\beta)\, det(1-\mathcal{L}_\beta),$$

where $det(1 \pm \mathcal{L}_\beta)$ are the Fredholm determinants of the generalized Perron-Frobenius operator $\mathcal{L}_\beta$. The analytic extension of $\tilde{\mathcal{L}}_\beta$ and its Fredholm determinant follow immediately from the one for the operator $\mathcal{L}_\beta$ and its Fredholm determinants $det(1 \pm \mathcal{L}_\beta)$ which were discussed in detail in [M2]. Indeed, writing $\mathcal{L}_\beta$ as

$$\begin{aligned} \mathcal{L}_\beta f(z) &= \sum_{n=1}^{\infty} (\frac{1}{n+z})^{2\beta} \left[f(\frac{1}{n+z}) - \sum_{l=0}^{k} \frac{f^{(l)}(0)}{l\,!} (\frac{1}{z+n})^l \right] \\ &\quad + \sum_{n=1}^{\infty} (\frac{1}{n+z})^{2\beta} \sum_{l=0}^{k} \frac{f^{(l)}(0)}{l\,!} (\frac{1}{z+n})^l, \end{aligned} \tag{2.6}$$

it was shown there that

$$\begin{aligned} \mathcal{L}_\beta f(z) &= \sum_{n=1}^{\infty} (\frac{1}{n+z})^{2\beta} \left[f(\frac{1}{n+z}) - \sum_{l=0}^{k} \frac{f^{(l)}(0)}{l\,!} (\frac{1}{z+n})^l \right] \\ &\quad + \sum_{l=0}^{k} \frac{f^{(l)}(0)}{l\,!} \zeta(2\beta + l, z+1) \end{aligned} \tag{2.7}$$

defines a meromorphic extension of $\mathcal{L}_\beta$ to the half plane $\Re\beta > -\frac{k}{2}$, $k \in I\!N_0$. Denoting the first sum in (2.7) by $\mathcal{L}_\beta^{(k)} f(z)$ and the second one by $\mathcal{A}_\beta^{(k)} f(z)$, then $\mathcal{L}_\beta^{(k)}$ is a nuclear operator analytic in $\Re\beta > -\frac{k}{2}$. The operator $\mathcal{A}_\beta^{(k)}$ is nuclear and meromorphic in the entire β-plane with simple poles at the β-values $\beta_l = \frac{1-l}{2}$, $l = 0, 1, \cdots, k$. The residue of $\mathcal{A}_\beta^{(k)}$ at β_l is the rank 1 operator $\mathcal{N}^{(l)}$ where

$$\mathcal{N}^{(l)} f(z) = \frac{1}{2} \frac{f^{(l)}(0)}{l\,!}.$$

For the following discussion it will be helpful to consider also the operator $\bar{\mathcal{A}}_\beta^{(k)} : B(D) \to B(D)$ defined as

$$\bar{\mathcal{A}}_\beta^{(k)} f_\beta(z) := \sum_{l=1}^{k-1} \frac{f_\beta^{(l)}(0)}{l\,!} \zeta(2\beta + l, z + 1)\,, \tag{2.8}$$

which is a finite rank operator meromorphic in the entire β-plane with simple poles at $\beta_l = \frac{1-l}{2}$, $l = 1, \cdots, k-1$. The operator is regular at $\beta = \frac{1}{2}$ and $\beta = \beta_k = \frac{1-k}{2}$ and will be used to discuss the spectral properties of $\mathcal{L}_\beta$ for β approaching β_k.

2.3. The Selberg zeta function for $PSL(2, \mathbb{Z})$. For $\Gamma \subset PSL(2, \mathbb{R})$ a discrete Fuchsian subgroup acting discontinuously on $\mathcal{H}$ denote by $\phi_t : S_1\mathcal{H}/\Gamma \to S_1\mathcal{H}/\Gamma$ the geodesic flow on the unit tangent bundle of the surface $\mathcal{H}/\Gamma$. The Ruelle-Smale dynamical zeta function for ϕ_t is defined as [Ru3]

$$\zeta_{SR}(s) = \prod_\gamma (1 - e^{-s\,l(\gamma)})^{-1} \tag{2.9}$$

where the product is over the closed primitive orbits γ of ϕ_t whose prime period is $l(\gamma)$. It was realized presumably first by Sinai that Selberg's zeta function [S] for the Fuchsian group Γ

$$Z_S(s) = \prod_{\{\gamma\}} \prod_{k=0}^{\infty} (1 - \mathcal{N}(\gamma)^{-(s+k)})\,,$$

where the product is over the equivalence classes $\{\gamma\}$ of the hyperbolic elements in Γ with norm $\mathcal{N}(\gamma)$, can be interpreted also in a dynamical way and coincides indeed with the function

$$Z_S(s) = \prod_{k=0}^{\infty} \zeta_{SR}(s + k)^{-1}\,,$$

with ζ_{SR} the aforementioned dynamical Smale-Ruelle function for the geodesic flow for the surface $\mathcal{H}/\Gamma$. By applying the thermodynamic formalism á la Ruelle [Ru4] and especially the transfer operator approach to the Smale-Ruelle zeta function it was shown in [M3], that the Selberg zeta function for $PSL(2, \mathbb{Z})$ can be expressed in terms of the transfer operator $\tilde{\mathcal{L}}_\beta$ for the geodesic flow for the modular surface in (2.3) simply as

$$Z_S(\beta) = det(1 - \tilde{\mathcal{L}}_\beta)\,, \quad \Re\beta > \frac{1}{2}\,, \tag{2.10}$$

respectively in terms of the Perron-Frobenius operator $\mathcal{L}_\beta$ in (2.7) as

$$Z_S(\beta) = det(1 - \mathcal{L}_\beta)\, det(1 + \mathcal{L}_\beta)\,, \quad \Re\beta > \frac{1}{2}\,. \tag{2.11}$$

Hence, in principle it should be possible to derive all the properties of the Selberg function $Z_S(\beta)$ from properties of the Perron-Frobenius operator $\mathcal{L}_\beta$. The standard approach to this function is through Selberg's trace formula for $PSL(2,\mathbb{Z})$ [S], which Selberg derived without any reference to the dynamical properties of the geodesic flow associated to such a Fuchsian group. Since Selberg's zeta function connects in a surprising way classical and quantum mechanics for a free particle on such surfaces of constant negative curvature, this function is of utmost interest in all the recent discussions related to quantum chaos. The possibility to replace Selberg's approach, which is based mainly on harmonic analysis and group theory, by something which uses mainly the dynamics of the classical system, is by itself of great interest [Sa]. Indeed, it turns out, that the transfer operator method not only connects through the Fredholm determinants $det(1 \pm \mathcal{L}_\beta)$ the poles and zeros of $Z_S(\beta)$ with the spectrum of $\mathcal{L}_\beta$ but it provides also through the eigenfunctions of $\mathcal{L}_\beta$ a new point of view on different aspects of the theory of modular forms for $PSL(2,\mathbb{Z})$ and especially the Maass cusp forms for this group, as we will show later.

3. The operator $\mathcal{L}_\beta$ and its Fredholm determinants. For $\beta = \beta_k = \frac{1-k}{2}$, $k \in \mathbb{N}_0$ the operator $\mathcal{L}_\beta$ has a simple pole of order 1 with residue the rank 1 operator

$$\mathcal{N}^{(k)} f(z) = \frac{1}{2}\frac{f^{(k)}(0)}{k\,!}\,. \tag{3.1}$$

To understand the behavior of the Fredholm determinants $det(1 \pm \mathcal{L}_\beta)$ at these β values one obviously has to study the behavior of the eigenvalues λ_β of $\mathcal{L}_\beta$ for β approaching the value $\beta = \beta_k$. We will call λ and f a regular eigenvalue respectively eigenfunction of $\mathcal{L}_\beta$ at $\beta = \beta_k$ if $\lambda = \lim_{\beta\to\beta_k}\lambda_\beta$ respectively $f = \lim_{\beta\to\beta_k} f_\beta$ exist, where λ_β is an eigenvalue with eigenfunction f_β of the nuclear operator $\mathcal{L}_\beta$ for $\beta \neq \beta_k$. If $\lim_{\beta\to\beta_k}\lambda_\beta$ diverges we call the corresponding λ a singular eigenvalue. The eigenfunctions $f_\beta \in B(D)$ should be normalized in such a way that when expanded in a power series around $z = 0$ in the singular case a singularity will show up only in the constant term $f_\beta(0)$.

3.1. The residues of Fredholm determinants. Our first result determines the residues of possible poles of Fredholm determinants of the operators $\mathcal{L}_\beta$ and $\mathcal{A}_\beta^{(k)}$:

Proposition 1. *The following Fredholm determinants have a pole of at most first order at $\beta = \beta_k$, $k \in \mathbb{N}_0$:*
(i) $det(1 \pm \mathcal{A}_\beta^{(0)})$ with residue $\pm\frac{1}{2}$ for $k = 0$,

(ii) $det(1 \pm \mathcal{L}_\beta)$ with residue $\pm\frac{1}{2}det(1 \pm \mathcal{L}_{\beta_k}^{(0)})$ for $k = 0$,

(iii) $det(1 \pm \mathcal{A}_\beta^{(k)})$ with residue $\frac{1}{2\,k}\,det(1 \pm \bar{\mathcal{A}}_{\beta_k}^{(k)})$ for $k \in \mathbb{N}$,

(iv) $det(1 \pm \mathcal{L}_\beta)$ with residue $\frac{1}{2\,k}\,det(1 \pm \mathcal{L}_{\beta_k}^{(k)})det(1 \pm \bar{\mathcal{A}}_{\beta_k}^{(k)})$ for $k \in \mathbb{N}$.

The proof of Proposition 1 we give in several steps in the form of Lemmas. First we need two formulas of Grothendieck [G2]:

Lemma 1. *For $\mathcal{B}$ respectively $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_n$ trace-class operators in a Banach space B, the following formulas hold*
(i)

$$det(1 \pm \mathcal{B}) = \sum_{n=0}^{\infty} (\pm)^n \, trace \, \wedge_n \mathcal{B} \,, \tag{3.2}$$

(ii) for $n \geq 2$

$$\begin{aligned} n \, trace\,[\,\wedge_{i=1}^{n} \mathcal{B}_i\,] \quad &= \quad -\sum_{i=1}^{n-1} trace\,[\,\mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_i \mathcal{B}_n \wedge \cdots \wedge \mathcal{B}_{n-1}\,] \\ &\qquad + trace\,[\,\wedge_{i=1}^{n-1} \mathcal{B}_i\,]\; trace\, \mathcal{B}_n \,, \end{aligned} \tag{3.3}$$

where $\wedge_{i=1}^{n} \mathcal{B}_i$ denotes the outer product of the operators $\mathcal{B}_i$ acting on the Banach space $\wedge_{i=1}^{n} B$.

Hence for $\beta \neq \beta_k$ we have

$$det(1 \pm \mathcal{L}_\beta) = \sum_{n=0}^{\infty} (\pm)^n \, trace \, \wedge_n \mathcal{L}_\beta \,. \tag{3.4}$$

Inserting the decomposition of $\mathcal{L}_\beta$ into the operators $\mathcal{A}_\beta^{(k)}$ and $\mathcal{L}_\beta^{(k)}$ as defined in (2.7), where $\mathcal{A}_\beta^{(k)}$ is an operator of finite rank $k+1$, namely the sum of the rank 1 operators $\mathcal{N}_\beta^{(l)}$

$$\mathcal{N}_\beta^{(l)} \, f(z) := \frac{f^{(l)}(0)}{l\,!} \, \zeta(2\beta + l, z+1)\,, \quad l \in \{\,0,\,1,\,\cdots,\,k\,\}\,, \tag{3.5}$$

into expression (3.4) gives

$$det(1 \pm \mathcal{L}_\beta) = \sum_{n=0}^{\infty} (\pm)^n \, trace \, \wedge_n \; \Big[\, \sum_{l=0}^{k} \mathcal{N}_\beta^{(l)} + \mathcal{L}_\beta^{(k)} \,\Big] \,. \tag{3.6}$$

The outer product $\wedge_n$ of the operators on the right hand side can be written as the following sum

$$\sum_{n_0, n_1, \cdots, n_k \in \{0,1\}, \sum_{i=-1}^{k} n_i = n} a_{n_{-1}, n_0, n_1, \cdots, n_k} \wedge_{n_{-1}} \mathcal{L}_\beta^{(k)} \wedge_{n_0} \mathcal{N}_\beta^{(0)} \wedge_{n_1} \mathcal{N}_\beta^{(1)} \wedge_{n_2} \cdots \wedge_{n_k} \mathcal{N}_\beta^{(k)} \,, \tag{3.7}$$

where $a_{n_{-1}, n_0, n_1, \cdots, n_k} = \frac{n\,!}{n_{-1}\,!}$ is just a combinatorial factor describing the different possibilities for getting a fixed outer product. Thereby we used the facts that exchanging any two operators in an outer product doesn't change the result and that $n_l \in \{0,1\}$ for $l = 0, 1, \cdots, k$ since the $\mathcal{N}_\beta^{(l)}$ are rank-1 operators and hence $\mathcal{N}_\beta^{(l)} \wedge \mathcal{N}_\beta^{(l)} = 0$ for all $l = 0, 1, \cdots, k$. We next use formula (3.3) repeatedly to transform

$$trace\,[\,\wedge_{n_{-1}} \mathcal{L}_\beta^{(k)} \wedge_{n_0} \mathcal{N}_\beta^{(0)} \wedge_{n_1} \mathcal{N}_\beta^{(1)} \wedge_{n_2} \cdots \wedge_{n_k} \mathcal{N}_\beta^{(k)}\,]$$

into a form where no outer product of operators is present anymore. The trace can then be written as a sum of products of traces of the type:

$$trace\,[\,(\mathcal{L}_\beta^{(k)})^{p_{-1}} (\mathcal{N}_\beta^{(l_0)})^{q_0} (\mathcal{L}_\beta^{(k)})^{p_0} (\mathcal{N}_\beta^{(l_1)})^{q_1} (\mathcal{L}_\beta^{(k)})^{p_1} \cdots (\mathcal{N}_\beta^{(l_m)})^{q_m} (\mathcal{L}_\beta^{(k)})^{p_m}\,]\,, \tag{3.8}$$

where $l_i \in \{0, 1, \cdots, k\}$, $q_i \in \{0,1\}$ for $i \in \{0, 1, \cdots, m\}$ and $p_j \in I\!N_0$ for $j \in \{-1, 0, \cdots, m\}$ with $m \in I\!N_0$. It is obvious that the operators in (3.8) do not commute. In order to calculate the residue of $det(1 \pm \mathcal{L}_\beta)$ at $\beta = \beta_k$ our first task will be to see in what cases the trace in (3.8) will diverge in the limit $\beta \to \beta_k$ and to determine its residue. This answer is achieved in

Lemma 2. *The trace in (3.8) has a pole at $\beta = \beta_k$ only in the following two cases:*

(i) For $k = 0$ only trace $\mathcal{N}_\beta^{(0)}$ has a pole of first order at $\beta = \beta_0$ with residue $\frac{1}{2}$.

(ii) For $k \geq 1$ only trace $[\,\mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}\,]$ has a pole of first order at $\beta = \beta_k$ with residue $-\frac{1}{2\,k}$.

Proof. Obviously the trace in (3.8) has a singularity at $\beta = \beta_k$ only if the operator $\mathcal{N}_\beta^{(k)}$ appears in the expression under the trace, since it is the only operator singular at $\beta = \beta_k$. We have to consider three cases:

(1) only the operators $\mathcal{N}_\beta^{(k)}$ and $\mathcal{L}_\beta^{(k)}$ appear in the trace of expression (3.8): that means we have to consider

$$\begin{aligned} &trace\,[\,(\mathcal{L}_\beta^{(k)})^{i+j} \mathcal{N}_\beta^{(k)}\,] = trace\,[\,(\mathcal{L}_\beta^{(k)})^{j} \mathcal{N}_\beta^{(k)}\,(\mathcal{L}_\beta^{(k)})^{i}\,] \\ &= trace\,[\,\mathcal{N}_\beta^{(k)}\,(\mathcal{L}_\beta^{(k)})^{i+j}\,] \text{ for } i,\, j \geq 0,\; i+j\, > 0\,. \end{aligned}$$

But the operator $\mathcal{L}_\beta^{(k)} \mathcal{N}_\beta^{(k)}$ is not singular at $\beta = \beta_k$, since the singular part in the image of $f \in B(D)$ under $\mathcal{N}_\beta^{(k)}$ for $\beta \to \beta_k$ is just $\frac{f^{(k)}(0)}{\beta - \beta_k}$, which however lies in the kernel of $\mathcal{L}_\beta^{(k)}$. The operator $(\mathcal{L}_\beta^{(k)})^{i+j} \mathcal{N}_\beta^{(k)}$ is therefore regular at $\beta = \beta_k$ for $i,\, j \geq 0,\; i+j > 0$. Hence this is true for its trace also;

(2) besides the operators $\mathcal{N}_\beta^{(k)}$ and $\mathcal{L}_\beta^{(k)}$ at least one other $\mathcal{N}_\beta^{(l)}$, $l \neq k$ appears in expression (3.8): as the operator $\mathcal{L}_\beta^{(k)} \mathcal{N}_\beta^{(l)}$ converges for $\beta \to \beta_k$ to the zero operator all traces of operators containing this combination after some cyclic permutation will vanish at $\beta = \beta_k$.

If however the combination $\mathcal{L}_\beta^{(k)}\mathcal{N}_\beta^{(k)}\mathcal{N}_\beta^{(l)}$ appears in the trace after some cyclic permutation, the trace will be holomorphic at $\beta=\beta_k$ as the operator $\mathcal{L}_\beta^{(k)}\mathcal{N}_\beta^{(k)}$ is holomorphic at $\beta=\beta_k$, as argued in (1);

(3) the operator $\mathcal{L}_\beta^{(k)}$ doesn't appear in the trace (3.8):
We consider first the case $k=0$ where only the operator $\mathcal{N}_\beta^{(0)}$ can appear in trace (3.8). From its definition

$$\mathcal{N}_\beta^{(0)}\, f(z)=f(0)\zeta(2\beta,z+1)\,,$$

we see, that its only eigenvalue different from zero is $\lambda_\beta=\zeta(2\beta,1)=\zeta(2\beta)$ and hence its trace is equal to this eigenvalue. In the vicinity of $\beta=\beta_k=\frac{1}{2}$ it behaves like

$$\lambda_\beta=\zeta(2\beta)=\frac{1}{2}\,\frac{1}{\beta-\frac{1}{2}}+O(1)\,.$$

Consider next the case $k\geq 1$. With the condition $l_i\neq l_j$ for $i\neq j$, where $l_i\in\{0,1,\cdots,k\}$, we can determine the operator $\mathcal{N}_\beta^{(l_n)}\mathcal{N}_\beta^{(l_{n-1})}\cdots\mathcal{N}_\beta^{(l_0)}$ step by step as follows

$$\mathcal{N}_\beta^{(l_0)}f(z)=\frac{f^{(l_0)}(0)}{l_0\,!}\zeta(2\beta+l_0,z+1)\,,$$

$$\mathcal{N}_\beta^{(l_1)}\mathcal{N}_\beta^{(l_0)}f(z)=\frac{f^{(l_0)}(0)}{l_0\,!}\frac{\zeta^{(l_1)}(2\beta+l_0)}{l_1\,!}\zeta(2\beta+l_1,z+1)\,,$$

where

$$\zeta^{(l_1)}(2\beta+l_0):=\frac{d^{l_1}}{dz^{l_1}}\zeta(2\beta+l_0,z+1)|_{z=0}\,.$$

After n steps we get

$$\begin{aligned}\text{(3.9)}\qquad &\mathcal{N}_\beta^{(l_n)}\mathcal{N}_\beta^{(l_{n-1})}\cdots\mathcal{N}_\beta^{(l_0)}f(z)\\ =\;&\frac{f^{(l_0)}(0)}{l_0\,!}\frac{\zeta^{(l_1)}(2\beta+l_0)}{l_1\,!}\cdots\frac{\zeta^{(l_n)}(2\beta+l_{n-1})}{l_n\,!}\zeta(2\beta+l_n,z+1)\,.\end{aligned}$$

Since all the l_i, $i=0,1,\cdots$ are different we have $n\leq k$ and get for the trace of the operator in (3.9)

$$\mathit{trace}\,[\mathcal{N}_\beta^{(l_n)}\mathcal{N}_\beta^{(l_{n-1})}\cdots\mathcal{N}_\beta^{(l_0)}]=\frac{\zeta^{(l_0)}(2\beta+l_n)}{l_0\,!}\prod_{i=1}^{n}\frac{\zeta^{(l_i)}(2\beta+l_{i-1})}{l_i\,!}\,.$$
(3.10)

Since for $n\geq 1$ $\lim_{s\to 1}\frac{d^n}{dz^n}\,\zeta(s,z)\,|_{z=1}<\infty$, (3.10) can be singular at $\beta=\beta_k$ only in the following two cases:

(a) $l_0 = 0$ and $2\beta_k + l_n = 1 - k + l_n = 1$, i.e. the operator in (3.9) has the form

$$\mathcal{N}_\beta^{(k)} \mathcal{N}_\beta^{(l_{n-1})} \cdots \mathcal{N}_\beta^{(l_1)} \mathcal{N}_\beta^{(0)} := \mathcal{N}_{(a)}\,; \tag{3.11}$$

(b) $l_i = 0$ for some $i \geq 1$ and $2\beta_k + l_{i-1} = 1 - k + l_{i-1} = 1$, i.e. the operator in (3.9) has the form

$$\mathcal{N}_\beta^{(l_n)} \mathcal{N}_\beta^{(l_{n-1})} \cdots \mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)} \cdots \mathcal{N}_\beta^{(l_0)} := \mathcal{N}_{(b)}\,. \tag{3.12}$$

Up to a cyclic permutation the operators $\mathcal{N}_{(a)}$ and $\mathcal{N}_{(b)}$ are of the same form and their trace is of the form

$$trace\,[\,\underbrace{\mathcal{N}_\beta^{(l_n)} \mathcal{N}_\beta^{(l_{n-1})} \cdots \mathcal{N}_\beta^{(l_2)}}_{regular\ at\ \beta=\beta_k} \ \underbrace{\mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}}_{singular\ at\ \beta=\beta_k}\,]\,, \tag{3.13}$$

where the $l_i \neq 0$ for all i. We will show next that the trace in (3.13) is regular at $\beta = \beta_k$ if $n \geq 2$. Inserting $\mathcal{N}_\beta^{(l_0)} = \mathcal{N}_\beta^{(k)}$ and $\mathcal{N}_\beta^{(l_1)} = \mathcal{N}_\beta^{(0)}$ into expression (3.10), we see that

$$trace\,[\,\mathcal{N}_\beta^{(l_n)} \mathcal{N}_\beta^{(l_{n-1})} \cdots \mathcal{N}_\beta^{(l_2)} \mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}\,]$$

is equal to

$$\frac{\zeta^{(k)}(2\beta + l_n)}{k\,!} \zeta(2\beta + k) \frac{\zeta^{(l_2)}(2\beta)}{l_2\,!} \prod_{i=3}^{n} \frac{\zeta^{(l_i)}(2\beta + l_{i-1})}{l_i\,!}\,. \tag{3.14}$$

In the limit $\beta \to \beta_k$ the only singular term in (3.14) is $\zeta(2\beta + k)$: it behaves like $(\beta - \beta_k)^{-1}$. On the other hand the term $\zeta^{(k)}(2\beta + l_n)$ in (3.14) has the form

$$\zeta^{(k)}(2\beta + l_n) = (-1)^k\,(2\beta + l_n)_k\,\zeta(2\beta + l_n + k) \tag{3.15}$$

for $l_n \in \{1, 2, \cdots, k-1\}$, with $(s)_n := s \cdot (s+1) \cdots (s+n-1)$ the Pochhammer symbol, which for $\beta \to \beta_k$ behaves like $(\beta - \beta_k)$. This cancels just the singularity above and the trace of (3.13) tends to a constant in the limit $\beta \to \beta_k$. For $k \geq 1$ the only trace of the type (3.10) which becomes singular for $\beta \to \beta_k$ is therefore

$$\begin{aligned} trace\,[\,\mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}\,] &= \frac{1}{k\,!} \left[\frac{d^k}{d\,z^k}\,\zeta(2\beta, z+1)\right]_{z=0} \zeta(2\beta + k) \\ &= \frac{(-1)^k}{k\,!}\,(2\beta)_k\,(\zeta(2\beta + k))^2\,. \end{aligned} \tag{3.16}$$

For $\beta \to \beta_k$ the function $\zeta(2\beta + k)$ behaves like $(2\beta + k - 1)^{-1}$ and $trace\,[\,\mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}\,]$ in (3.16) hence behaves like

$$-\frac{1}{2k} \frac{1}{(\beta - \beta_k)}\,.$$

The residue of $trace\,[\,\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}\,]$ at $\beta = \beta_k$ is just the one of part (ii) of Lemma 2.

□

Knowing that only $trace\,[\,\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}\,]$ for $k \geq 1$ respectively $trace\,\mathcal{N}_\beta^{(0)}$ for $k = 0$ have a pole at $\beta = \beta_k$, we have to determine now the factors which multiply $trace\,[\,\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}\,]$ respectively $trace\,\mathcal{N}_\beta^{(0)}$ in expansion (3.6) of $det(1 \pm \mathcal{L}_\beta)$. For this we need:

Lemma 3. *For $n \geq 0$ and $l_1, l_2, \cdots, l_m \in \{1, \cdots, k-1\}$ for $m \geq 1$ and $l_i < l_j$ for $i < j$ the following formulas hold:*
(a)

$$(n+1)\,trace\,[\,\wedge_{n+1}\mathcal{L}_\beta^{(k)}\,] = \sum_{r=0}^{n}(-1)^r\,trace\,[\,\wedge_{n-r}\mathcal{L}_\beta^{(k)}\,]\,trace(\mathcal{L}_\beta^{(k)})^{r+1}\,. \tag{3.17}$$

(b)

$$trace\,[\,\wedge_n\mathcal{L}_{\beta_k}^{(k)} \wedge \mathcal{N}_{\beta_k}^{(l_1)} \wedge \mathcal{N}_{\beta_k}^{(l_2)} \wedge \cdots \wedge \mathcal{N}_{\beta_k}^{(l_m)}\,] = \frac{n!\,m!}{(n+m)!}\,trace\,[\,\wedge_n\mathcal{L}_{\beta_k}^{(k)}\,]\,trace\,[\,\mathcal{N}_{\beta_k}^{(l_1)} \wedge \mathcal{N}_{\beta_k}^{(l_2)} \wedge \cdots \wedge \mathcal{N}_{\beta_k}^{(l_m)}\,]\,. \tag{3.18}$$

Proof.
(a) Repeated application of formula (3.3) in Lemma 1 to $(n+1)\,trace\,\wedge_{n+1}\mathcal{L}_\beta^{(k)}$ gives

$$\begin{aligned}
&(n+1)\,trace\,[\,\wedge_{n+1}\mathcal{L}_\beta^{(k)}\,]\\
&= (n+1)\,trace\,[\,\wedge_n\mathcal{L}_\beta^{(k)} \wedge \mathcal{L}_\beta^{(k)}\,]\\
&= trace\,[\,\wedge_n\mathcal{L}_\beta^{(k)}\,]\,trace\mathcal{L}_\beta^{(k)} - n\,trace\,[\,\wedge_{n-1}\mathcal{L}_\beta^{(k)} \wedge (\mathcal{L}_\beta^{(k)})^2\,]\\
&= trace\,[\,\wedge_n\mathcal{L}_\beta^{(k)}\,]\,trace\mathcal{L}_\beta^{(k)} - trace\,[\,\wedge_{n-1}\mathcal{L}_\beta^{(k)}\,]\,trace(\mathcal{L}_\beta^{(k)})^2\\
&\quad + (n-1)\,trace\,[\,\wedge_{n-2}\mathcal{L}_\beta^{(k)} \wedge (\mathcal{L}_\beta^{(k)})^3\,]\,,
\end{aligned}$$

and after n steps

$$\begin{aligned}
&(n+1)\,trace\,[\,\wedge_{n+1}\mathcal{L}_\beta^{(k)}\,]\\
&= \sum_{r=0}^{n-1}(-1)^r trace\,[\,\wedge_{n-r}\mathcal{L}_\beta^{(k)}\,]\,trace(\mathcal{L}_\beta^{(k)})^{r+1}\\
&\quad + (-1)^n[n-(n-1)]\,trace\,[\,\wedge_{n-n}\mathcal{L}_\beta^{(k)} \wedge (\mathcal{L}_\beta^{(k)})^{n+1}\,]\,.
\end{aligned}$$

But the last term is just $trace(\mathcal{L}_\beta^{(k)})^{n+1}$ and statement (a) of Lemma 3 follows.

(b) To prove statement (b) we use induction on n. For $n=0$ the statement is trivially true. Assume next that the statement is true for $n=N$, that means

$$\begin{aligned} & trace\,[\wedge_N \mathcal{L}^{(k)}_{\beta_k} \wedge \mathcal{N}^{(l_1)}_{\beta_k} \wedge \mathcal{N}^{(l_2)}_{\beta_k} \wedge \cdots \wedge \mathcal{N}^{(l_m)}_{\beta_k}\,] \\ &= \frac{N!\,m!}{(N+m)!}\, trace\,[\wedge_N \mathcal{L}^{(k)}_{\beta_k}\,]\, trace\,[\mathcal{N}^{(l_1)}_{\beta_k} \wedge \mathcal{N}^{(l_2)}_{\beta_k} \wedge \cdots \wedge \mathcal{N}^{(l_m)}_{\beta_k}\,]\,. \end{aligned} \tag{3.19}$$

We will show that it is true also for $n=N+1$. For this we set

$$\mathcal{N} := \mathcal{N}^{(l_1)}_{\beta_k} \wedge \mathcal{N}^{(l_2)}_{\beta_k} \wedge \cdots \wedge \mathcal{N}^{(l_m)}_{\beta_k}\,. \tag{3.20}$$

Using formula (3.3) we get

$$\begin{aligned} & trace\,[\wedge_{N+1} \mathcal{L}^{(k)}_{\beta_k} \wedge \mathcal{N}^{(l_1)}_{\beta_k} \wedge \mathcal{N}^{(l_2)}_{\beta_k} \wedge \cdots \wedge \mathcal{N}^{(l_m)}_{\beta_k}\,] \\ &= \frac{1}{(N+m+1)} \Big\{ trace\,[\wedge_N \mathcal{L}^{(k)}_{\beta_k} \wedge \mathcal{N}\,]\, trace \mathcal{L}^{(k)}_{\beta_k} \\ &\quad - N\, trace\,[\wedge_{N-1} \mathcal{L}^{(k)}_{\beta_k} \wedge \mathcal{N} \wedge (\mathcal{L}^{(k)}_{\beta_k})^2\,] \\ &\quad - \sum_{j=1}^{m} trace\,[\wedge_{N-1} \mathcal{L}^{(k)}_{\beta_k} \wedge \mathcal{N}^{(l_1)}_{\beta_k} \wedge \cdots \wedge \mathcal{L}^{(k)}_{\beta_k} \mathcal{N}^{(l_j)}_{\beta_k} \wedge \cdots \wedge \mathcal{N}^{(l_m)}_{\beta_k}\,] \Big\}\,. \end{aligned}$$

The traces in the last sum vanish since $\mathcal{L}^{(k)}_{\beta_k} \mathcal{N}^{(l_i)}_{\beta_k}$ is the zero operator for $l_i \neq k$. Thus we get

$$\begin{aligned} & trace\,[\wedge_{N+1} \mathcal{L}^{(k)}_{\beta_k} \wedge \mathcal{N}^{(l_1)}_{\beta_k} \wedge \mathcal{N}^{(l_2)}_{\beta_k} \wedge \cdots \wedge \mathcal{N}^{(l_m)}_{\beta_k}\,] \\ &= \frac{trace\,[\wedge_N \mathcal{L}^{(k)}_{\beta_k} \wedge \mathcal{N}\,]}{(N+m+1)}\, trace \mathcal{L}^{(k)}_{\beta_k} \\ &\quad - \frac{N}{(N+m+1)}\, trace\,[\wedge_{N-1} \mathcal{L}^{(k)}_{\beta_k} \wedge \mathcal{N} \wedge (\mathcal{L}^{(k)}_{\beta_k})^2\,]\,. \end{aligned}$$

Applying formula (3.3) N times finally leads to

$$\begin{aligned} & trace\,[\wedge_{N+1} \mathcal{L}^{(k)}_{\beta_k} \wedge \mathcal{N}^{(l_1)}_{\beta_k} \wedge \mathcal{N}^{(l_2)}_{\beta_k} \wedge \cdots \wedge \mathcal{N}^{(l_m)}_{\beta_k}\,] \\ &= \sum_{r=0}^{N-1} \Big\{ (-1)^r \frac{(N+m-r)!}{(N+m+1)!} \frac{N!}{(N-r)!} \\ &\qquad \times\, trace\,[\wedge_{N-r} \mathcal{L}^{(k)}_{\beta_k} \wedge \mathcal{N}\,]\, trace (\mathcal{L}^{(k)}_{\beta_k})^{(r+1)} \Big\} \\ &\quad + (-1)^N \frac{(N+m-(N-1))!}{(N+m+1)!} \frac{N!}{(N-(N-1)-1)!} \\ &\qquad \times\, trace\,[\wedge_{N-N} \mathcal{L}^{(k)}_{\beta_k} \wedge \mathcal{N} \wedge (\mathcal{L}^{(k)}_{\beta_k})^{(N+1)}\,]\,. \end{aligned}$$

Using next (3.19) we get

$$
\begin{aligned}
& trace\,[\wedge_{N+1}\mathcal{L}^{(k)}_{\beta_k}\wedge\mathcal{N}^{(l_1)}_{\beta_k}\wedge\mathcal{N}^{(l_2)}_{\beta_k}\wedge\cdots\wedge\mathcal{N}^{(l_m)}_{\beta_k}\,] \\
&= \sum_{r=0}^{N-1}\Big\{(-1)^r\frac{(N+m-r)!}{(N+m+1)!}\frac{N!}{(N-r)!} \\
&\quad\times\frac{(N-r)!\,m!}{(N+m-r)!}trace\,[\wedge_{N-r}\mathcal{L}^{(k)}_{\beta_k}\,]\,trace\mathcal{N}trace(\mathcal{L}^{(k)}_{\beta_k})^{(r+1)}\Big\} \\
&\quad+(-1)^N\frac{(m+1)\,!}{(N+m+1)!}\,\frac{N!}{0!}\,\frac{m\,!\,1\,!}{(m+1)\,!}trace\mathcal{N}\,trace(\mathcal{L}^{(k)}_{\beta_k})^{(N+1)} \\
&= \frac{N!\,m!}{(N+m+1)!}trace\mathcal{N}\,\sum_{r=0}^{N-1}(-1)^r trace\,[\wedge_{N-r}\mathcal{L}^{(k)}_{\beta_k}\,]\,trace(\mathcal{L}^{(k)}_{\beta_k})^{(r+1)} \\
&\quad+\frac{N!\,m!}{(N+m+1)!}(-1)^N trace\mathcal{N}\,trace(\mathcal{L}^{(k)}_{\beta_k})^{(N+1)} \\
&= \frac{N!\,m!}{(N+m+1)!}trace\mathcal{N}\,\sum_{r=0}^{N}(-1)^r trace\,[\wedge_{N-r}\mathcal{L}^{(k)}_{\beta_k}\,]\,trace(\mathcal{L}^{(k)}_{\beta_k})^{(r+1)} \\
&= \frac{(N+1)!\,m!}{(N+1+m)!}trace\,[\wedge_{N+1}\mathcal{L}^{(k)}_{\beta_k}\,trace\mathcal{N}\,]\,,
\end{aligned}
$$

where we used also part (a) of the Lemma. This shows that also part (b) of Lemma 3 holds. ∎

Now we are finally prepared to prove Proposition 1.

Proof of Proposition 1.
(i) Since $\mathcal{A}^{(0)}_\beta=\mathcal{N}^{(0)}_\beta$, its Fredholm determinants $det(1\pm\mathcal{A}^{(0)}_\beta)$ can be simply expanded by using formula (3.2) as

$$
\begin{aligned}
det(1\pm\mathcal{A}^{(0)}_\beta) &= \sum_{n=0}^{\infty}(\pm)^n trace\wedge_n\mathcal{A}^{(0)}_\beta \\
&= \sum_{n=0}^{\infty}(\pm)^n trace\wedge_n\mathcal{N}^{(0)}_\beta \\
&= 1\pm trace\mathcal{N}^{(0)}_\beta\,,
\end{aligned}
$$

since $trace\wedge_n\mathcal{N}^{(0)}_\beta=0$ for $n\geq 2$. Its residue at $\beta=\beta_k$ is equal to $\pm$ the residue of $trace\mathcal{N}^{(0)}_\beta$ which according to Lemma 2 is $\pm\frac{1}{2}$.

(ii) We first calculate $trace\wedge_n\mathcal{L}_\beta$. As before we set

$$trace\wedge_0\mathcal{L}_\beta=1\,.$$

For $n \geq 1$ we have

$$\begin{aligned} trace \wedge_n \mathcal{L}_\beta &= trace \wedge_n (\mathcal{L}_\beta^{(0)} + \mathcal{N}_\beta^{(0)}) \\ &= n\, trace\, [\wedge_{n-1} \mathcal{L}_\beta^{(0)} \wedge \mathcal{N}_\beta^{(0)}] + trace \wedge_n \mathcal{L}_\beta^{(0)} \\ &= trace \wedge_{n-1} \mathcal{L}_\beta^{(0)}\, trace \mathcal{N}_\beta^{(0)} + rest\,, \end{aligned}$$

where *rest* denotes here and in the following formulas always terms which are regular at $\beta = \beta_0$. The Fredholm determinants $det(1 \pm \mathcal{L}_\beta)$ hence can be written as

$$\begin{aligned} det(1 \pm \mathcal{L}_\beta) &= \sum_{n=0}^{\infty} (\pm)^n trace \wedge_n \mathcal{L}_\beta \\ &= 1 + \sum_{n=1}^{\infty} (\pm)^n trace \wedge_n \mathcal{L}_\beta \\ &= trace \mathcal{N}_\beta^{(0)} \sum_{n=1}^{\infty} (\pm)^n trace \wedge_{n-1} \mathcal{L}_\beta^{(0)} + rest \\ &= \pm trace \mathcal{N}_\beta^{(0)} \sum_{n=0}^{\infty} (\pm)^n trace \wedge_n \mathcal{L}_\beta^{(0)} + rest \\ &= \pm trace \mathcal{N}_\beta^{(0)}\, det(1 \pm \mathcal{L}_\beta^{(0)}) + rest\,. \end{aligned}$$

The residue of $det(1 \pm \mathcal{L}_\beta)$ at $\beta = \beta_0 = \frac{1}{2}$ is therefore equal to $\pm\frac{1}{2}\, det(1 \pm \mathcal{L}_\beta^{(0)})$.

(iii) Let us first look for possible poles of $trace \wedge_n \mathcal{A}_\beta^{(k)}$ at $\beta = \beta_k$ for $n \geq 0$: Since $trace \wedge_0 \mathcal{A}_\beta^{(k)} = 1$ and $trace \wedge_1 \mathcal{A}_\beta^{(k)} = trace\, [\sum_{l=0}^{k} \mathcal{N}_\beta^{(l)}]$ there is no pole at $\beta = \beta_k$, since $trace \mathcal{N}_\beta^{(k)}$ is regular there [M2]. For $2 \leq n \leq k+1$ we have

$$trace \wedge_n \mathcal{A}_\beta^{(k)} = trace\ \wedge_n \left(\mathcal{N}_\beta^{(0)} + \mathcal{N}_\beta^{(l_1)} + \cdots + \mathcal{N}_\beta^{(k)} \right)\,. \tag{3.21}$$

As we have seen in Lemma 2, only $trace\, [\mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}]$ is for $k \neq 0$ singular at $\beta = \beta_k$, so we try to isolate it in $trace \wedge_n \mathcal{A}_\beta^{(k)}$ in the following way: first, due to $\mathcal{A} \wedge \mathcal{B} = \mathcal{B} \wedge \mathcal{A}$, we expand expression (3.21) as follows

$$\begin{aligned} & trace \wedge_n \mathcal{A}_\beta^{(k)} \\ &= \sum\nolimits^{*1} n\,(n-1)\,(n-2)!\, trace[\mathcal{N}_\beta^{(l_1)} \wedge \cdots \wedge \mathcal{N}_\beta^{(l_{n-2})} \wedge \mathcal{N}_\beta^{(0)} \wedge \mathcal{N}_\beta^{(k)}]\,, \end{aligned}$$

where $\sum^{*1} := \sum_{1 \leq l_1 < l_2 < \cdots < l_{n-2} \leq k-1}$. Next we apply formula (3.3) twice to get

$$trace \wedge_n \mathcal{A}_\beta^{(k)} \tag{3.22}$$

$$= \sum\nolimits^{*1} (-1)(n-2)!\, trace\, [\mathcal{N}_\beta^{(l_1)} \wedge \cdots \wedge \mathcal{N}_\beta^{(l_{n-2})}]\, trace\, [\mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}] + rest$$
$$= -trace\, [\mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}] \sum\nolimits^{*1} (n-2)\,!\, trace\, [\mathcal{N}_\beta^{(l_1)} \wedge \cdots \wedge \mathcal{N}_\beta^{(l_{n-2})}] + rest$$
$$= -trace\, [\mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}\,]\, S_\beta^{n-2}(k) + rest\,,$$

where $rest$ denotes again terms regular at $\beta = \beta_k$ and $S_\beta^{n-2}(k)$ denotes the sum in $\sum^{*1}$. Obviously one finds $\sum_{n=0}^{k-1} (\pm 1)^n S_\beta^n(k) = det(1 \pm \bar{\mathcal{A}}_\beta^{(k)})$ with $\bar{\mathcal{A}}_\beta^{(k)}$ defined in (2.8).

For $n \geq k+2$ one has $trace \wedge_n \mathcal{A}_\beta^{(k)} = 0$, because in this case at least one $\mathcal{N}_\beta^{(l)}$ must appear twice in the outer product and hence makes it vanish.

Summarizing the discussions above about $trace \wedge_n \mathcal{A}_\beta^{(k)}$ for $n \in I\!N_0$, the singularity of the Fredholm determinant $det(1 \pm \mathcal{A}_\beta^{(k)})$ at $\beta = \beta_k$ can be determined:

$$\begin{aligned}
& det(1 \pm \mathcal{A}_\beta^{(k)}) \\
&= \sum_{n=0}^{\infty} (\pm)^n trace \wedge_n \mathcal{A}_\beta^{(k)} \\
&= trace \wedge_0 \mathcal{A}_\beta^{(k)} \pm trace \wedge_1 \mathcal{A}_\beta^{(k)} + \sum_{n=2}^{k+1} (\pm)^n trace \wedge_n \mathcal{A}_\beta^{(k)} \\
&= \sum_{n=2}^{k+1} (\pm)^n \left(-\, trace\, [\mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}\,] \right) S_\beta^{n-2}(k) + rest \\
&= -trace\, [\,\mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}\,] \sum_{n=2}^{k+1} (\pm)^n\, S_\beta^{n-2}(k) + rest \\
&= -trace\, [\,\mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}\,] \sum_{n=0}^{k-1} (\pm)^n S_\beta^n(k) + rest \\
&= -trace\, [\,\mathcal{N}_\beta^{(0)} \mathcal{N}_\beta^{(k)}\,]\, det(1 \pm \bar{\mathcal{A}}_\beta^{(k)}) + rest\,,
\end{aligned}$$

where $rest$ denotes again terms regular at $\beta = \beta_k$. According to Lemma 2, $det(1 \pm \mathcal{A}_\beta^{(k)})$ hence has residue $\frac{1}{2\,k}\, det(1 \pm \bar{\mathcal{A}}_\beta^{(k)})$ at the point $\beta = \beta_k$.

(iv) We proceed as in the proof of statement (iii) of Proposition 1. Again, since $trace \wedge_0 \mathcal{L}_\beta = 1$ and $trace \wedge_1 \mathcal{L}_\beta = trace\, [\,\mathcal{L}_\beta^{(k)} + \sum_{l=0}^{k} \mathcal{N}_\beta^{(l)}\,]$ they are regular for $\beta = \beta_k$. For $n \geq 2$ one finds

$$\begin{aligned}
& trace \wedge_n \mathcal{L}_\beta = trace \wedge_n \left(\mathcal{L}_\beta^{(k)} + \sum_{l=0}^{k} \mathcal{N}_\beta^{(l)} \right) \\
&= \sum\nolimits^{*2} \sum\nolimits^{*1} \frac{n\,!}{(n-r-2)\,!}
\end{aligned}$$

$$\times\, trace\,[\,\wedge_{n-r-2}\mathcal{L}_\beta^{(k)} \wedge \mathcal{N}_\beta^{(l_1)} \wedge \cdots \wedge \mathcal{N}_\beta^{(l_r)} \wedge \mathcal{N}_\beta^{(0)} \wedge \mathcal{N}_\beta^{(k)}\,] + rest\,,$$

where $rest$ again denotes terms regular at $\beta = \beta_k$ and $\sum^{*2} := \sum_{r=0}^{min\{n-2,k-1\}}$. Using formula (3.3) we find

$$trace \wedge_n \mathcal{L}_\beta = \sum^{*2}\sum^{*1} \frac{n\,!}{(n-r-2)\,!}\,\frac{-trace\,[\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}\,]}{n\,(n-1)}$$
$$\times\, trace \wedge_{n-r-2} \mathcal{L}_\beta^{(k)} \wedge\, [\mathcal{N}_\beta^{(l_1)} \wedge \cdots \wedge \mathcal{N}_\beta^{(l_r)}\,] + rest\,.$$

The last trace in this expression can be split by means of (3.18) in Lemma 3, if $\beta = \beta_k$. For β values near β_k formula (3.18) is only correct up to a term of order $o(1)$ which vanishes for $\beta = \beta_k$. Since this term does not contribute to the residue at $\beta = \beta_k$, we can put it into the $rest$ term. Hence we get

$$\begin{aligned}
&trace \wedge_n \mathcal{L}_\beta \\
&= \sum^{*2}\Big\{\sum^{*1} \frac{(n-2)\,!}{(n-r-2)\,!}(-trace\,[\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}\,])\frac{(n-r-2)\,!\,r\,!}{(n-2)\,!} \\
&\quad \times trace\,[\,\wedge_{n-r-2}\mathcal{L}_\beta^{(k)}\,]\,trace\,[\mathcal{N}_\beta^{(l_1)} \wedge \cdots \wedge \mathcal{N}_\beta^{(l_r)}\,]\Big\} + rest \\
&= -trace\,[\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}\,]\sum^{*2}\Big\{trace\,[\,\wedge_{n-r-2}\mathcal{L}_\beta^{(k)}\,] \\
&\quad \times \sum^{*1} r!\,trace\,[\mathcal{N}_\beta^{(l_1)} \wedge \cdots \wedge \mathcal{N}_\beta^{(l_r)}\,]\Big\} + rest \\
&= -trace\,[\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}\,]\sum^{*2} trace\,[\,\wedge_{n-r-2}\mathcal{L}_\beta^{(k)}\,]\,S_\beta^r(k) + rest\,,
\end{aligned}$$

with $S_\beta^r(k)$ as defined in (3.22). The Fredholm determinant $det(1 \pm \mathcal{L}_\beta)$ can be written for β near β_k as

$$\begin{aligned}
&det(1 \pm \mathcal{L}_\beta) \\
&= \sum_{n=0}^{\infty} (\pm)^n trace \wedge_n \mathcal{L}_\beta \\
&= \sum_{n=2}^{\infty} (\pm)^n trace \wedge_n \mathcal{L}_\beta + rest \\
&= \sum_{n=2}^{\infty} (\pm)^n \Big(-trace\,[\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}]\Big)\sum^{*2} trace\,[\wedge_{n-r-2}\mathcal{L}_\beta^{(k)}]S_\beta^r(k) + rest \\
&= -trace\,[\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}\,]\sum_{n=2}^{\infty}\sum^{*2}(\pm)^n trace\,[\,\wedge_{n-r-2}\mathcal{L}_\beta^{(k)}\,]\,S_\beta^r(k) + rest \\
&= -trace\,[\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}\,]\sum_{r=0}^{k-1}\sum_{n=2+r}^{\infty}(\pm)^n trace\,[\,\wedge_{n-r-2}\mathcal{L}_\beta^{(k)}\,]\,S_\beta^r(k) + rest
\end{aligned}$$

$$= -trace\,[\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}\,]\sum_{r=0}^{k-1}(\pm)^r det(1\pm\mathcal{L}_\beta^{(k)})S_\beta^r(k)+rest$$

$$= -trace\,[\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}\,]\,det(1\pm\mathcal{L}_\beta^{(k)})\sum_{r=0}^{k-1}(\pm)^r S_\beta^r(k)+rest$$

$$= -trace\,[\mathcal{N}_\beta^{(0)}\mathcal{N}_\beta^{(k)}\,]\,det(1\pm\mathcal{L}_\beta^{(k)})det(1\pm\bar{\mathcal{A}}_\beta^{(k)})+rest\,.$$

Hence due to Lemma 2 the Fredholm determinant $det(1\pm\mathcal{L}_\beta)$ has residue $\frac{1}{2\,k}\,det(1\pm\mathcal{L}_\beta^{(k)})det(1\pm\bar{\mathcal{A}}_\beta^{(k)})$ at the point $\beta=\beta_k$. □

Remark Obviously the poles in the Fredholm determinants are absent in case the corresponding residues vanish. From (iii) and (iv) it follows that in case the residue of $det(1\pm\mathcal{A}_\beta^{(k)})$ vanishes the residue of $det(1\pm\mathcal{L}_\beta)$ also will vanish. One can also expect some kind of factorization of $det(1\pm\mathcal{L}_\beta)$ into the two Fredholm determinants $det(1\pm\mathcal{L}_\beta^{(k)})$ and $det(1\pm\mathcal{A}_\beta^{(k)})$ for $\beta=\beta_k$. We will come back to this point when discussing the properties of $\mathcal{A}_\beta^{(k)}$ and $\mathcal{L}_\beta^{(k)}$ in more detail.

3.2. Spectral properties of the operator $\mathcal{A}_\beta^{(k)}$. Next we discuss some properties of the operator $\mathcal{A}_\beta^{(k)}$ which obviously is of rank $k+1$.

Proposition 2.

(i) *The operator $\mathcal{A}_\beta^{(k)}$ has a pole of first order at $\beta=\beta_k$ with residue the rank 1 operator*

$$\mathcal{N}^{(k)}f(z)=\frac{1}{2}\frac{f^{(k)}(0)}{k\,!}\,.$$

(ii) *Besides the eigenvalue $\lambda_\beta=\zeta(2\beta)$ corresponding to the eigenfunction $f_\beta(z)=\zeta(2\beta,z+1)$ all the eigenvalues of the operator $\mathcal{A}_\beta^{(0)}$ vanish for $\beta\to\beta_0=1/2$. The eigenvalue λ_β and the eigenfunction f_β are singular at $\beta=\beta_0$ and behave asymptotically as follows*

$$\begin{aligned}\lambda_\beta=\zeta(2\beta) &\overset{\beta\to\beta_0}{\sim} \frac{1}{2(\beta-\beta_0)}+O(1)\,,\\ f_\beta(z)=\zeta(2\beta,z+1) &\overset{\beta\to\beta_0}{\sim} \frac{1}{2(\beta-\beta_0)}-\psi(z+1)+o(1)\,,\\ &\qquad with\;\;\psi(z):=\frac{\Gamma'(z)}{\Gamma(z)}\,.\end{aligned}$$

(iii) *The operator $\mathcal{A}_\beta^{(k)}$ has for $k\geq 2$ exactly $k-1$ regular nonvanishing eigenvalues at $\beta=\beta_k$. The corresponding eigenfunctions are* *polynomials*

$$f(z) = \sum_{l=0}^{k-1} a_l\, z^l = \lim_{\beta\to\beta_k} f_\beta(z)\,,$$

where the eigenfunctions f_β *behave in the limit* $\beta \to \beta_k$ *asymptotically as follows*

$$f_\beta(z) = \sum_{l=0}^{\infty} a_l(\beta)\, z^l$$

$$\text{(3.23)}\quad \textit{with} \begin{cases} a_0(\beta) = o(1)\,, & \\ a_l(\beta) = a_l + o(1)\,, & 1 \le l \le k-1\,, \\ a_k(\beta) = c_k\,(\beta-\beta_k) + o(\beta-\beta_k)\,, & \\ a_l(\beta) = o(1)\,, & k+1 \le l\,. \end{cases}$$

The coefficients a_l, $(1 \le l \le k-1)$ *and the eigenvalues* λ *are determined by the system of linear equations*

$$\text{(3.24)}\qquad \lambda\, a_r = \sum_{l=1}^{k-1} \frac{\zeta^{(r)}(1-k+l)}{r!}\, a_l\,, \qquad 1 \le r \le k-1,$$

with $\zeta^{(r)}(1-k+l)$ *the* r*-th derivative of Hurwitz's zeta function* $\zeta(1-k+l, z+1)$ *at* $z=0$*; the coefficients* c_k *are defined for* $k \ge 2$ *as*

$$\text{(3.25)}\qquad c_k = -2\sum_{l=1}^{k-1} \zeta(1-k+l)\, a_l\,,$$

with ζ *Riemann's zeta function, i.e. the analytic extension of* $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $(\Re s > 1)$.

(*iv*) *The operator* $\mathcal{A}_\beta^{(k)}$ *has for* $k \ge 1$ *two eigenvalues which in the limit* $\beta \to \beta_k$ *are singular:*

$$\text{(3.26)}\qquad \lambda_\beta^{\pm} = \pm\sqrt{\frac{1}{2k}}\, i\, \frac{1}{\sqrt{\beta-\beta_k}} + \frac{(S_k + \zeta(1-k))}{2} + o(1)\,.$$

The asymptotic behaviour of their eigenfunctions

$$\text{(3.27)}\qquad f_\beta^{\pm}(z) = \sum_{l=0}^{k} a_l(\beta)\, \zeta(2\beta + l, z+1)$$

is given in the limit $\beta \to \beta_k$ *by*

$$\text{(3.28)}\qquad a_0(\beta) = \mp\sqrt{\frac{k}{2}}\, i\, \frac{1}{\sqrt{\beta-\beta_k}} + \frac{(S_k - \zeta(1-k))k}{2} + o(1)\,,$$

$$\text{(3.29)}\qquad a_l(\beta) = -k\binom{k-1}{l}\zeta(1-k+l) + o(1),\ l = 1, \cdots, k-1,$$

$$\text{(3.30)}\qquad a_k(\beta) = 1\,,$$

with $S_k := \sum_{r=1}^{k} \frac{a_r(\beta_k)\zeta^{(k)}(2\beta+r)}{k!}$.

Proof.
(i) is clear.

(ii) The rank-1 operator $\mathcal{A}_\beta^{(0)}$ is defined as

$$\mathcal{A}_\beta^{(0)} f_\beta(z) = f_\beta(0)\zeta(2\beta, z+1)\,. \tag{3.31}$$

The eigenfunction f_β with non-vanishing eigenvalue can be chosen as $f_\beta(z) = \zeta(2\beta, z+1)$. Its eigenvalue λ_β is obviously equal to $f_\beta(0) = \zeta(2\beta)$. For $\beta \to \beta_0 = 1/2$ one gets the asymptotic behaviour

$$\begin{aligned} \zeta(2\beta) \quad &\sim \quad \frac{1}{2(\beta-\beta_0)} + O(1)\,, \\ \text{respectively} \quad \zeta(2\beta, z+1) \quad &\sim \quad \frac{1}{2(\beta-\beta_0)} - \psi(z+1) + o(1)\,, \end{aligned}$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ with Γ Eulers gamma function. From this property (ii) follows immediately.

(iii) To prove this statement we proceed as follows. Suppose the operator $\mathcal{A}_\beta^{(k)}$, defined as

$$\mathcal{A}_\beta^{(k)} f_\beta(z) = \sum_{l=0}^{k} \frac{f_\beta^{(l)}(0)}{l\,!}\zeta(2\beta+l, z+1)\,, \tag{3.32}$$

has a regular eigenvalue λ at $\beta = \beta_k$ with eigenfunction f, that means $\lambda = \lim_{\beta\to\beta_k} \lambda_\beta$, $f(z) = \lim_{\beta\to\beta_k} f_\beta(z)$ and $\mathcal{A}_\beta^{(k)}\, f_\beta(z) = \lambda_\beta\, f_\beta(z)$, $\beta \neq \beta_k$. Since $f_\beta \in B(D)$ we have for small absolut values of z

$$f_\beta(z) = \sum_{r=0}^{\infty} a_r(\beta) z^r\,, \tag{3.33}$$

where for $\beta \to \beta_k$

$$a_r(\beta) \sim c_r(\beta - \beta_k)^{p_r} \quad \text{for some } c_r \neq 0 \text{ and } p_r \geq 0\,, \tag{3.34}$$

since we demand all the coefficients of $f_\beta(z)$ to be regular at $\beta = \beta_k$. Since f_β is an eigenfunction of $\mathcal{A}_\beta^{(k)}$ we find

$$\mathcal{A}_\beta^{(k)} f_\beta(z) \quad = \quad \sum_{l=0}^{k} \frac{1}{l!} \left[\frac{d^l}{dz^l} f_\beta(z) \right]_{z=0} \zeta(2\beta+l, z+1)$$

$$
\begin{aligned}
&= \sum_{l=0}^{k-1} a_l(\beta)\zeta(2\beta+l,z+1) + a_k(\beta)\zeta(2\beta+k,z+1) \\
&= \lambda_\beta \sum_{r=0}^{\infty} a_r(\beta) z^r
\end{aligned} \tag{3.35}
$$

The only singular term at $\beta = \beta_k$ in the left hand side of expression (3.35) is $\zeta(2\beta + k, z + 1)$, which for $\beta \to \beta_k$ behaves as

$$
\zeta(2\beta+k,z+1) \sim \frac{1}{2(\beta-\beta_k)} + O(1)\,.
$$

To get a regular right hand side, this singularity must be cancelled by $a_k(\beta)$ for $\beta \to \beta_k$, i.e.

$$
a_k(\beta) \sim c_k(\beta-\beta_k)^p, \quad p \geq 1\,, \quad c_k \neq 0\,. \tag{3.36}
$$

The number p must be chosen such that all other a_r, $r \in \mathbb{N}_0$ have regular limits for $\beta = \beta_k$. We show now that choosing $p = 1$ will guarantee this: for $p = 1$ we get

$$
\lim_{\beta\to\beta_k} a_k(\beta)\zeta(2\beta+k,z+1) = \frac{c_k}{2} < \infty\,,
$$

and the eigenfunction equation (3.35) at $\beta = \beta_k$ reads as

$$
\lim_{\beta\to\beta_k} \mathcal{A}_\beta^{(k)} f_\beta(z) = \sum_{l=0}^{k-1} a_l(\beta_k)\zeta(2\beta_k+l,z+1) + \frac{c_k}{2} = \lambda \sum_{r=0}^{\infty} a_r(\beta_k) z^r\,. \tag{3.37}
$$

Since $\zeta(2\beta_k + l, z + 1)$ for $l = 0, \cdots, k - 1$ is a polynomial [1] of degree $\leq k$ and $a_k(\beta_k) = 0$ because of (3.36), all the $a_r(\beta_k)$ must vanish for $r \geq k$. On the other hand, the only polynomial of degree k on the left hand side of expression (3.37) is $\zeta(2\beta_k, z + 1)$, corresponding to $l = 0$. Hence it follows that $a_0(\beta_k) = 0$. Expression (3.37) can be simplified therefore as:

$$
\sum_{l=1}^{k-1} a_l(\beta_k)\zeta(1-k+l,z+1) + \frac{c_k}{2} = \lambda \sum_{r=1}^{k-1} a_r(\beta_k) z^r\,. \tag{3.38}
$$

Expanding the left hand side into a Taylor series and comparing the coefficients with the right hand side one gets

$$
\lambda a_r(\beta_k) = \sum_{l=1}^{k-1} \frac{\zeta^{(r)}(1-k+l)}{r!} a_l(\beta_k)\,, \quad 1 \leq r \leq k-1\,, \tag{3.39}
$$

[1] Let $n \in \mathbb{N}_0$, $z \in \mathbb{C}$, then

$$
\zeta(-n,z) = -\frac{\varphi_{n+1}(z)}{n+1}\,,
$$

where $\varphi_{n+1}(z)$ is the $(n+1)$-th Bernoulli polynomial, which has degree $n+1$.

and

$$c_k = -2\sum_{l=1}^{k-1}\zeta(1-k+l)a_l(\beta_k)\,,$$

where $\zeta^{(r)}(s)$ denotes the r-th derivative of $\zeta(s,z+1)$ at $z=0$. We claim the eigenvalue λ doesn't vanish. For this consider expression (3.39). Since the function $\zeta(1-k+l,z+1)$ is a polynomial of degree $k-l$, $\zeta^{(r)}(1-k+l)=0$ for $r>k-l$. Suppose now $\lambda=0$. Consider equation (3.39) for $r=k-1$. The only non-zero term in the sum is the one with $l=1$. Hence a_1 has to vanish. Consider next $r=k-2$. Now there are two non-zero terms on the right hand side of (3.39), namely $l=1,\,2$. As $a_1=0$ it follows that $a_2=0$. Repeating this procedure we get $a_r=0$ for all r, i.e. the eigenfunction must vanish identically. The assumption $\lambda=0$ is therefore wrong.

(iv) Let λ_β be an eigenvalue of $\mathcal{A}_\beta^{(k)}$ with eigenfunction

$$f_\beta(z) = \sum_{r=0}^{k} a_r(\beta)\zeta(2\beta+r,z+1)\,,$$

and assume $\lim_{\beta\to\beta_k}\lambda_\beta=\infty$. The eigenfunction equation $\mathcal{A}_\beta^{(k)}f_\beta(z) = \lambda_\beta f_\beta(z)$ reads again

$$\begin{aligned}
&\sum_{r=0}^{k} a_r(\beta)\zeta(2\beta+r)\zeta(2\beta,z+1) \qquad (3.40)\\
&\quad+\sum_{l=1}^{k-1}\frac{1}{l!}\sum_{r=0}^{k} a_r(\beta)\zeta^{(l)}(2\beta+r)\zeta(2\beta+l,z+1)\\
&\quad+\frac{1}{k!}\sum_{r=0}^{k} a_r(\beta)\zeta^{(k)}(2\beta+r)\zeta(2\beta+k,z+1)\\
&=\lambda_\beta\Big[a_0(\beta)\zeta(2\beta,z+1)+\sum_{l=1}^{k-1}a_l(\beta)\zeta(2\beta+l,z+1) \qquad (3.41)\\
&\quad+a_k(\beta)\zeta(2\beta+k,z+1)\Big]\,.
\end{aligned}$$

Comparing the coefficients of the zeta functions in (3.40) and (3.41) we get

$$\lambda_\beta a_0(\beta) = \sum_{r=0}^{k} a_r(\beta)\zeta(2\beta+r)\,, \tag{3.42}$$

$$\lambda_\beta a_l(\beta) = \frac{1}{l!}\sum_{r=0}^{k} a_r(\beta)\zeta^{(l)}(2\beta+r)\,,\quad 1\le l\le k-1\,, \tag{3.43}$$

$$\lambda_\beta a_k(\beta) = \frac{1}{k!}\sum_{r=0}^{k} a_r(\beta)\zeta^{(k)}(2\beta+r)\,. \tag{3.44}$$

Without loss of generality we can assume $a_k(\beta_k) = 1$. Then at least one of the coefficients $a_l(\beta)$, $l = 0, \cdots, k$ must be singular at $\beta = \beta_k$, as one sees from equation (3.44). Indeed $a_0(\beta)$ must be the term most singular for $\beta \to \beta_k$, all other $a_l(\beta)$, $1 \leq l \leq k-1$ are less singular as can be seen from equation (3.43). Moreover, from the leading terms of equations (3.42) and (3.44) for $\beta \to \beta_k$ we find

$$a_0(\beta) \overset{\beta\to\beta_k}{\sim} \frac{1}{\sqrt{\beta-\beta_k}} \quad \text{and} \quad \lambda_\beta \overset{\beta\to\beta_k}{\sim} \frac{1}{\sqrt{\beta-\beta_k}}\,.$$

Inserting this into equations (3.43) shows that the coefficients $a_l(\beta)$, $l = 1, \cdots, k-1$ must be regular at $\beta = \beta_k$. This allows us to calculate the exact form of the leading terms in relations (3.44) and (3.42) for $\beta \to \beta_k$: since

$$\lambda_\beta a_k(\beta) = \frac{1}{k!} a_0(\beta)\zeta^{(k)}(2\beta) + O(1)$$

we get

$$\frac{a_0(\beta)}{\lambda_\beta} = \frac{a_k(\beta)k!}{(-1)^k(2\beta)_k\zeta(2\beta+k)} + o(1)$$

and therefore

$$\lim_{\beta\to\beta_k} \frac{a_0(\beta)}{\lambda_\beta} = \frac{k!}{-(k-1)!} = -k\,. \tag{3.45}$$

From relations (3.43) we find in leading order

$$a_l(\beta) = \frac{a_0(\beta)}{\lambda_\beta}\frac{(-1)^l}{l!}(2\beta)_l\zeta(2\beta+l) \qquad l = 1, \cdots, k-1$$

and hence

$$\lim_{\beta\to\beta_k} a_l(\beta) = -k\frac{(k-1)!}{l!(k-l-1)!}\zeta(1-k+l)$$

and therefore

$$a_l(\beta_k) = -k\binom{k-1}{l}\zeta(1-k+l) \qquad l = 1, \cdots, k-1\,.$$

Define for $l = 0, \cdots, k$ the numbers S_l as follows:

$$S_0 := \sum_{r=1}^{k-1} a_r(\beta_k)\zeta(2\beta_k + r)\,, \tag{3.46}$$

$$S_l := \sum_{r=1}^{k} \frac{a_r(\beta_k)\zeta^{(l)}(2\beta_k+r)}{l!}\,, \quad l = 1, 2, \cdots, k\,, \tag{3.47}$$

and rewrite equations (3.42) to (3.44) in terms of S_0 and S_l as

$$\lambda_\beta\, a_0(\beta) = a_0(\beta)\zeta(2\beta) + S_0 + \zeta(2\beta + k) + o(1)\,, \tag{3.48}$$

$$\lambda_\beta\, a_l(\beta) = \frac{a_0(\beta)\zeta^{(l)}(2\beta)}{l!} + S_l + o(1)\,,\;\; 1 \le l \le k-1\,, \tag{3.49}$$

$$\lambda_\beta = \frac{a_0(\beta)\zeta^{(k)}(2\beta)}{k!} + S_k + o(1)\,. \tag{3.50}$$

Dividing expression (3.48) by (3.50) yields

$$a_0(\beta) = \frac{a_0(\beta)\zeta(2\beta) + S_0 + \zeta(2\beta+k) + o(1)}{\frac{a_0(\beta)\zeta^{(k)}(2\beta)}{k!} + S_k + o(1)}\,.$$

This is a quadratic equation for $a_0(\beta)$ of the form

$$A(\beta)a_0(\beta)^2 + B(\beta)a_0(\beta) - C(\beta) + o(1) = 0\,, \tag{3.51}$$

with

$$A(\beta) := \left[\frac{\zeta^{(k)}(2\beta)}{k!}\right]\,,\;\; B(\beta) := [S_k - \zeta(2\beta)]$$
$$\text{and}\quad C(\beta) := [S_0 + \zeta(2\beta+k)]\;.$$

Solving for $a_0(\beta)$ gives

$$a_0(\beta) = -\frac{B(\beta)}{2A(\beta)} \pm \frac{\sqrt{B(\beta)^2 + 4A(\beta)C(\beta)}}{2A(\beta)} + o(1)\,.$$

The only singularity for $\beta \to \beta_k$ is in $C(\beta)$. Expanding the square root gives

$$\sqrt{B(\beta)^2 + 4A(\beta)C(\beta)} = 2\sqrt{A(\beta)}\sqrt{\zeta(2\beta+k)} + o(1)\,.$$

Since $\lim_{\beta\to\beta_k} A(\beta) = -1/k$, $\lim_{\beta\to\beta_k} B(\beta) = (S_k - \zeta(1-k))$ and $\zeta(2\beta + k) \overset{\beta\to\beta_k}{\sim} 1/(2(\beta-\beta_k))$, we finally get in the limit $\beta \to \beta_k$, $k \ge 1$:

$$a_0(\beta) = \mp\sqrt{\frac{k}{2}}\, i\frac{1}{\sqrt{\beta-\beta_k}} + \frac{(S_k - \zeta(1-k))k}{2} + o(1)\,,$$

and with (3.50)

$$\lambda_\beta = \pm\sqrt{\frac{1}{2k}}\, i\frac{1}{\sqrt{\beta-\beta_k}} + \frac{(S_k + \zeta(1-k))}{2} + o(1)\,.$$

□

Since the operator $\mathcal{A}_\beta^{(k)}$ is of finite rank $k+1$, its eigenspace for the eigenvalues different from zero is of dimension $k+1$. A natural basis for this space is given by the Hurwitz functions $\zeta(2\beta+l,z+1)$, $l=0,\,1,\,\cdots,\,k$. In the limit $\beta\to\beta_k=\frac{1-k}{2}$ the functions $\zeta(1-k+l,z+1)$ are for $l<k$ proportional to the Bernoulli polynomials whereas $\zeta(2\beta+k,z+1)$ develops a singularity at $\beta=\beta_k$. Obviously one can also use the powers $\{z^l\}$, $l\in\mathbb{N}_0$ as a basis of the eigenspace of $\mathcal{A}_\beta^{(k)}$. One only has to be careful with convergence properties of the corresponding series expansions, since the functions are only known to belong to the space $B(D)$. In the limit $\beta\to\beta_k$ however one gets

Proposition 3.

(i) *Let λ be a regular eigenvalue of $\mathcal{A}_\beta^{(k)}$ for $\beta\to\beta_k$ with regular eigenfunction $f(z)$ with $\lim_{\beta\to\beta_k}\lambda_\beta=\lambda\neq 0$ and $\lim_{\beta\to\beta_k}f_\beta(z)=f(z)$. If the eigenfunctions f_β in the basis $\{z^l\}$, $l\in\mathbb{N}_0$ and the basis $\{\zeta(2\beta+l,z+1)\}$, $l=0,\,1,\,\cdots,\,k$ have the representations*

$$f_\beta(z)=\sum_{l=0}^{\infty}a_l(\beta)\,z^l=\sum_{l=0}^{\infty}b_l(\beta)\zeta(2\beta+l,z+1)\,,\quad b_l(\beta)=0\ for\ l>k\,,$$

then for $\beta=\beta_k$ the coefficients $a_l(\beta_k)$ and $b_l(\beta_k)$ coincide up to a common constant factor.

(ii) *Let λ_β be an eigenvalue of $\mathcal{A}_\beta^{(k)}$ with eigenfunction $f_\beta(z)$ which for $\beta\to\beta_k$ becomes singular. If the f_β in the basis $\{z^l\}$, $l\in\mathbb{N}_0$ and the basis $\{\zeta(2\beta+l,z+1)\}$, $l=0,\,1,\,\cdots,\,k$ have the representations*

$$f_\beta(z)=\sum_{l=0}^{\infty}a_l(\beta)\,z^l=\sum_{l=0}^{\infty}b_l(\beta)\zeta(2\beta+l,z+1)\,,\quad b_l(\beta)=0\ for\ l>k\,,\tag{3.52}$$

then for $\beta\to\beta_k$ the coefficients $a_l(\beta)$ and $b_l(\beta)$ can be chosen such that for all l

$$\lim_{\beta\to\beta_k}|\,\lambda_\beta^{-1}a_l(\beta)\;-\;b_l(\beta)\,|\;=\;0\,.\tag{3.53}$$

Proof.
(i) To prove statement (i) of this Proposition, assume the eigenfunction f_β of $\mathcal{A}_\beta^{(k)}$ has the two representations

$$f_\beta(z)=\sum_{l=0}^{\infty}a_l(\beta)\,z^l=\sum_{l=0}^{\infty}b_l(\beta)\zeta(2\beta+l,z+1)\,,$$

with $b_l(\beta)\equiv 0$ for $l\geq k+1$. We know from Proposition 2 that

$$a_0(\beta_k)\;=\;0,$$

$$\begin{aligned} a_l(\beta_k) &= O(1), \quad 1 \le l \le k-1, \\ a_k(\beta_k) &= 0, \\ a_l(\beta_k) &= 0, \quad l \ge k+1. \end{aligned}$$

We want to calculate $b_l(\beta_k)$ for $0 \le l \le k$. Obviously $b_k(\beta_k) = 0$, else $b_k(\beta)\zeta(2\beta + l, z+1)$ would diverge for $\beta = \beta_k$. Moreover $b_0(\beta_k) = 0$, otherwise $f_{\beta_k}(z)$ would be a polynomial of degree k because $\zeta(2\beta_k, z+1)$ is a polynomial of degree k. This would contradict Proposition 2 which shows that every regular eigenfunction of $\mathcal{A}^{(k)}_{\beta_k}$ is a polynomial of degree $k-1$. To prove finally that $\lambda\, b_l(\beta_k) = a_l(\beta_k)$ for $1 \le l \le k-1$ consider the equation

$$\begin{aligned} f_\beta(z) &= \frac{1}{\lambda_\beta}\mathcal{A}^{(k)}_\beta f_\beta(z) = \sum_{l=0}^{k} \frac{1}{\lambda_\beta}\frac{f^{(l)}_\beta(0)}{l\,!}\zeta(2\beta + l, z+1) \\ &= \sum_{l=0}^{k} b_l(\beta)\,\zeta(2\beta + l, z+1)\,. \end{aligned}$$

Since $\lambda \neq 0$ we can find a neighbourhood of $\beta = \beta_k$ such that $\lambda_\beta \neq 0$. But $\frac{f^{(l)}_\beta(0)}{l\,!}$ is nothing but $a_l(\beta)$. Taking the limit $\beta \to \beta_k$ proves (i) of Proposition 3.

(ii) To prove this statement recall the asymptotic form of the divergent eigenvalues of the operator $\mathcal{A}^{(k)}_{\beta_k}$ given in Proposition 2:

$$\lambda_\beta = \frac{X}{\sqrt{\beta - \beta_k}}\left[1 + \frac{Y}{X}\sqrt{\beta - \beta_k} + o(\sqrt{\beta - \beta_k})\right],$$

with $X := \pm\sqrt{\frac{1}{2k}}i$ and $Y := \frac{(S_k + \zeta(1-k))}{2}$. Their inverses are

$$\begin{aligned} \lambda_\beta^{-1} &= \frac{\sqrt{\beta - \beta_k}}{X}\left[1 + \frac{Y}{X}\sqrt{\beta - \beta_k} + o(\sqrt{\beta - \beta_k})\right]^{-1} \\ &= \pm\frac{\sqrt{2k}}{i}\sqrt{\beta - \beta_k} + (S_k + \zeta(1-k))\,k\,(\beta - \beta_k) + o(\beta - \beta_k)\,. \end{aligned}$$

The eigenfunction $f_\beta(z)$ can be written as

$$\begin{aligned} f_\beta(z) &= b_0(\beta)\zeta(2\beta, z+1) + \sum_{l=1}^{k-1} b_l(\beta)\zeta(2\beta + l, z+1) \\ &\quad + b_k(\beta)\zeta(2\beta + k, z+1)\,. \end{aligned}$$

The asymptotic behavior of the different terms are respectively

$$\begin{aligned} &\pm\sqrt{\frac{k}{2}}\frac{1}{i}\frac{1}{\sqrt{\beta - \beta_k}}\zeta(2\beta_k, z+1) + o((\beta - \beta_k)^{-1/2})\,, \quad O(1) \\ &\text{and} \quad \frac{1}{2(\beta - \beta_k)} + o((\beta - \beta_k)^{-1/2})\,. \end{aligned}$$

Therefore we get asymptotically

$$\begin{aligned}
\lambda_\beta^{-1} f_\beta(z) &= \left[\pm\frac{\sqrt{2k}}{i}\sqrt{\beta-\beta_k} + (S_k+\zeta(1-k))k(\beta-\beta_k) + o(\beta-\beta_k)\right] \\
&\quad\times\left[\pm\sqrt{\frac{k}{2}}\frac{1}{i}\frac{1}{\sqrt{\beta-\beta_k}}\zeta(2\beta_k, z+1) + \frac{1/2}{\beta-\beta_k} + o((\beta-\beta_k)^{-\frac{1}{2}})\right] \\
&= -k\,\zeta(2\beta_k, z+1) \pm \sqrt{\frac{k}{2}}\frac{1}{i}\frac{1}{\sqrt{\beta-\beta_k}} + \frac{(S_k+\zeta(1-k))k}{2} + o(1) \\
&= \left[\pm\sqrt{\frac{k}{2}}\frac{1}{i}\frac{1}{\sqrt{\beta-\beta_k}} + \frac{(S_k-\zeta(1-k))k}{2}\right] \\
&\quad+\sum_{l=1}^{k-1}\left[-k\binom{k-1}{l}\zeta(1-k+l)\right]z^l + z^k + o(1) \\
&= \sum_{l=0}^{k} b_l(\beta) z^l + o(1)\,. \qquad (3.54)
\end{aligned}$$

On the other hand we have

$$\lambda_\beta^{-1} f_\beta(z) = \sum_{l=0}^{\infty} \lambda_\beta^{-1} a_l(\beta) z^l\,. \qquad (3.55)$$

Comparing these two representations for $\lambda_\beta^{-1} f_\beta(z)$ shows

$$\sum_{l=0}^{k}[\lambda_\beta^{-1}a_l(\beta) - b_l(\beta)]z^l + \sum_{l=k+1}^{\infty}\frac{a_l(\beta)}{\lambda_\beta}z^l + o(1) = 0\,.$$

Hence

$$\lim_{\beta\to\beta_k} |\lambda_\beta^{-1}a_l(\beta) - b_l(\beta)| = 0\,.$$

This just says that the coefficients $a_l(\beta)$ can be chosen to coincide with the $b_l(\beta)$ for $\beta\to\beta_k$ up to order $(\beta-\beta_k)^0$. □

It turns out that the regular eigenvalues of $\mathcal{A}_\beta^{(k)}$ for $\beta\to\beta_k$ are just the eigenvalues of the operator $\bar{\mathcal{A}}_{\beta_k}^{(k)}$, as shows the next result.

Proposition 4. *Let* $\bar{\mathcal{A}}_\beta^{(k)} : B(D)\to B(D)$ $(k\geq 2)$ *be the operator defined in (2.8):*

$$\bar{\mathcal{A}}_\beta^{(k)} f_\beta(z) = \sum_{l=1}^{k-1}\frac{f_\beta^{(l)}(0)}{l!}\zeta(2\beta+l, z+1)\,.$$

Then for $\beta=\beta_k$ *one has:*

(i) $\bar{\mathcal{A}}^{(k)}_{\beta_k}$ *has real spectrum.*

(ii) The regular eigenvalues of $\mathcal{A}^{(k)}_{\beta_k}$ *coincide with the eigenvalues of* $\bar{\mathcal{A}}^{(k)}_{\beta_k}$.

(iii) Let λ *be a regular eigenvalue of* $\mathcal{A}^{(k)}_{\beta_k}$ *with eigenfunction* $f(z)$. *Let* $\bar{f}(z)$ *be the eigenfunction of* $\bar{\mathcal{A}}^{(k)}_{\beta_k}$ *with the same eigenvalue. Then* $f(z)$ *and* $\bar{f}(z)$ *can be chosen to coincide up to a constant:*

$$f(z) = \bar{f}(z) - \frac{1}{\lambda}\sum_{l=1}^{k-1}\frac{f^{(l)}(0)}{l\,!}\zeta(1-k+l)\,,$$

with ζ *Riemann's zeta function.*

Proof.
(i) To show $\bar{\mathcal{A}}^{(k)}_{\beta_k}$ has real spectrum we rewrite $\zeta(2\beta_k+l, z+1)$ for $\beta_k = \frac{1-k}{2}$ and $l = 1, \cdots, k-1$ as follows:

$$\zeta(2\beta_k + l, z+1) = -\frac{B_{k-l}(z+1)}{k-l} = \sum_{r=0}^{k}\frac{-\binom{k-l}{r}}{(k-l)}B_{k-l-r}(1)\,z^r\,.$$

The operator $\bar{\mathcal{A}}^{(k)}_{\beta_k}$ can then be written as

$$\bar{\mathcal{A}}^{(k)}_{\beta_k} f(z) = \sum_{l=1}^{k-1}\frac{f^{(l)}(0)}{l\,!}\sum_{r=0}^{k}\frac{-\binom{k-l}{r}}{(k-l)}B_{k-l-r}(1)\,z^r\,.$$

Consider next the operator

$$\hat{\mathcal{A}}^{(k)}_{\beta_k} f(z) = \bar{\mathcal{A}}^{(k)}_{\beta_k} f(z) - \sum_{l=1}^{k-1}\frac{f^{(l)}(0)}{l\,!}\zeta(2\beta_k + l)\,.$$

It is clear that the operators $\hat{\mathcal{A}}^{(k)}_{\beta_k}$ and $\bar{\mathcal{A}}^{(k)}_{\beta_k}$ have the same spectra: if $\bar{\mathcal{A}}^{(k)}_{\beta_k} f(z) = \lambda f(z)$ and $\lambda \neq 0$ then

$$g(z) = f(z) - \frac{1}{\lambda}\sum_{l=1}^{k-1}\frac{f^{(l)}(0)}{l\,!}\zeta(2\beta_k + l)$$

is an eigenfunction of $\hat{\mathcal{A}}^{(k)}_{\beta_k}$ to the same eigenvalue λ:

$$\hat{\mathcal{A}}^{(k)}_{\beta_k} g(z) = \lambda f(z) - \sum_{l=1}^{k-1}\frac{f^{(l)}(0)}{l\,!}\zeta(2\beta_k + l) = \lambda g(z)\,.$$

And vice versa if $\hat{\mathcal{A}}^{(k)}_{\beta_k} g(z) = \lambda g(z)$ and $\lambda \neq 0$ then the function

$$f(z) = g(z) + \frac{1}{\lambda}\sum_{l=1}^{k-1}\frac{g^{(l)}(0)}{l\,!}\zeta(2\beta_k + l)$$

is an eigenfunction of $\bar{\mathcal{A}}^{(k)}_{\beta_k}$ with the same eigenvalue λ. Obviously every function $f \in B(D)$ with $f(z) = z^r\, h(z)$ for $r \geq k$ and $h \in B(D)$ is both in the kernel of $\bar{\mathcal{A}}^{(k)}_{\beta_k}$ and $\hat{\mathcal{A}}^{(k)}_{\beta_k}$. In the basis $\{z^{\,i}\}$, $i = 1, 2, \cdots, k-1$ of the space $\mathcal{P}_0^{\leq(k-1)}$ of polynomials of degree $\leq k-1$ vanishing at $z = 0$ the operator $\hat{\mathcal{A}}^{(k)}_{\beta_k}$ has the matrix representation

$$\hat{A}_{i\,j} = \begin{cases} -\binom{k-j}{i}\,\frac{B_{k-j-i}(1)}{(k-j)}\,, & for\ i+j \leq k\,, \\ 0\,, & else\,. \end{cases}$$

Define the invertible operator $\mathcal{U} : \mathcal{P}_0^{\leq(k-1)} \to \mathcal{P}_0^{\leq(k-1)}$ by

$$Uf(z) = \sum_{l=1}^{k-1} \sqrt{\frac{k-1}{\binom{k-1}{l}}}\;\frac{f^{(l)}(0)}{l\,!}\; z^l\,,$$

which in matrix form reads

$$U_{i\,j} = \sqrt{\frac{k-1}{\binom{k-1}{i}}}\;\delta_{i\,j}\,,\ \text{for } i,j \in \{1, \cdots, k-1\}$$

and has the property $[U^{-1}]_{i\,j} = 1/U_{i\,j}$. We will prove that $[U\hat{A}U^{-1}]_{i\,j}$ is a symmetric matrix. For the case (i,j) with $i+j > k$ we get due to $\hat{A}_{i\,j} = 0$

$$[U\hat{A}U^{-1}]_{i\,j} = U_{i\,r}\hat{A}_{r\,l}U^{-1}_{l\,j} = 0\,.$$

In the remaining cases we have

$$\begin{aligned} [U\hat{A}U^{-1}]_{i\,j} &= U_{i\,r}\hat{A}_{r\,l}U^{-1}_{l\,j} \\ &= \sqrt{\frac{k-1}{\binom{k-1}{i}}}\;\delta_{i\,r}\;\frac{-\binom{k-l}{r}}{(k-l)}\;B_{k-l-r}(1)\;\sqrt{\frac{\binom{k-1}{l}}{k-1}}\;\delta_{l\,j} \\ &= -\sqrt{\frac{(k-i-1)\,!\,(k-j-1)\,!}{i\,!\,j\,!}}\;\frac{B_{k-i-j}(1)}{(k-i-j)\,!}\,. \end{aligned}$$

Hence $[U\hat{A}U^{-1}]_{i\,j}$ is symmetric and has therefore real spectrum. Since $\mathcal{U}\,\hat{\mathcal{A}}^{(k)}_{\beta_k}\,\mathcal{U}^{-1}$ is isomorphic to $\hat{\mathcal{A}}^{(k)}_{\beta_k}$ and $\bar{\mathcal{A}}^{(k)}_{\beta_k}$ in the space $\mathcal{P}_0^{\leq(k-1)}$, also the spectrum of the operator $\bar{\mathcal{A}}^{(k)}_{\beta_k}$ is real.

(ii), (iii) Since

$$\bar{\mathcal{A}}^{(k)}_{\beta} f_\beta(z) = \mathcal{A}^{(k)}_{\beta} f_\beta(z) - f(0)\zeta(2\beta, z+1) - \frac{f^{(k)}(0)}{k\,!}\zeta(2\beta+k, z+1)$$

we get for a regular eigenfunction f of $\mathcal{A}^{(k)}_{\beta_k}$ with eigenvalue $\lambda \neq 0$

$$\begin{aligned} &\lim_{\beta\to\beta_k} \mathcal{A}^{(k)}_{\beta} f_\beta(z) = \lambda f(z) \\ &= \bar{\mathcal{A}}^{(k)}_{\beta_k} f(z) + f(0)\zeta(2\beta_k, z+1) + \lim_{\beta\to\beta_k} \frac{f^{(k)}(0)}{k\,!}\zeta(2\beta+k, z+1)\,. \end{aligned}$$

For $g(z) = f(z) + a$ we then find

$$\begin{aligned}\bar{\mathcal{A}}^{(k)}_{\beta_k} g(z) &= \bar{\mathcal{A}}^{(k)}_{\beta_k} f(z) \\ &= \lambda f(z) - f(0)\zeta(2\beta_k, z+1) - \lim_{\beta\to\beta_k} \frac{f^{(k)}(0)}{k\,!}\zeta(2\beta + k, z+1)\,.\end{aligned}$$

Since according to Proposition 2 every regular eigenfunction f of $\mathcal{A}^{(k)}_{\beta_k}$ vanishes at $z = 0$ we have

$$\bar{\mathcal{A}}^{(k)}_{\beta_k} g(z) = \lambda f(z) - \lim_{\beta\to\beta_k} \frac{f^{(k)}(0)}{k\,!}\zeta(2\beta + k, z+1)\,.$$

The above limit however is a constant c_f which according to (3.25) is just $-\sum_{l=1}^{k-1} \frac{f^{(l)}(0)}{l\,!}\zeta(1-k+l)$ and therefore choosing $a = -\frac{c_f}{\lambda}$ we find

$$\bar{\mathcal{A}}^{(k)}_{\beta_k} g(z) = \lambda(f(z) + a) = \lambda g(z)\,.$$

If on the other hand $g(z)$ is an eigenfunction of $\bar{\mathcal{A}}^{(k)}_{\beta_k}$ with $\bar{\mathcal{A}}^{(k)}_{\beta_k} g(z) = \lambda g(z)$, $\lambda \neq 0$, we find for the function $f_\beta(z) = g(z) + a + (\beta - \beta_k)\,2c\,z^k + o(\beta - \beta_k)$

$$\begin{aligned}\lim_{\beta\to\beta_k} \mathcal{A}^{(k)}_{\beta} f_\beta(z) &= \bar{\mathcal{A}}^{(k)}_{\beta_k} g(z) + (g(0) + a)\zeta(2\beta_k, z+1) + c \\ &= \lambda g(z) + (g(0) + a)\zeta(2\beta_k, z+1) + c\,.\end{aligned}$$

Choosing $a = -g(0)$ and $c = a\lambda$ we find

$$\lim_{\beta\to\beta_k} \mathcal{A}^{(k)}_{\beta} f_\beta(z) = \lambda(g(z) + a) = \lambda f(z)$$

and therefore f is a regular eigenfunction of $\mathcal{A}^{(k)}_{\beta_k}$ to the same eigenvalue λ. □

So far some spectral properties of the operator $\mathcal{A}^{(k)}_{\beta}$ at $\beta = \beta_k$. The next results are related to the operator $\mathcal{L}^{(k)}_{\beta_k}$.

3.3. Spectral properties of the operator $\mathcal{L}^{(k)}_{\beta}$.

Proposition 5.

(*i*) *For $k \in \mathbb{N}_0$ the polynomials of degree $\leq k$ belong to the kernel of $\mathcal{L}^{(k)}_{\beta_k}$.*

(*ii*) *For $k \in \mathbb{N}_0$ the operator $\mathcal{L}^{(k)}_{\beta_k}$ has $\lambda = (-1)^{k+1}$ as an eigenvalue with eigenfunction*

$$f(z) = \frac{1}{z+1}\,.$$

(*iii*) *For $k \in \mathbb{N}$ the operator $\mathcal{L}^{(k)}_{\beta_k}$ has real spectrum.*

Remark The eigenfunction in (ii) doesn't depend on k and is just the density of the Gauss measure.

Proof.
(i) This follows immediately from the definition of the operator $\mathcal{L}_{\beta_k}^{(k)}$.

(ii) To show that $f(z) = \frac{1}{z+1}$ is an eigenfunction of the operator $\mathcal{L}_{\beta_k}^{(k)}$ with eigenvalue $\lambda = (-1)^{k+1}$ we insert this function into the definition of $\mathcal{L}_{\beta_k}^{(k)}$ and get

$$\begin{aligned}
\mathcal{L}_{\beta_k}^{(k)} f(z) &= \sum_{n=1}^{\infty} (\frac{1}{z+n})^{1-k} \left[\frac{1}{1+\frac{1}{z+n}} - \sum_{l=0}^{k} \frac{f^{(l)}(0)}{l\,!} (\frac{1}{z+n})^l \right] \\
&= \sum_{n=1}^{\infty} (\frac{1}{z+n})^{1-k} \left[\frac{z+n}{z+n+1} - \sum_{l=0}^{k} (-1)^l (\frac{1}{z+n})^l \right] .
\end{aligned}$$

The last expression can be written also as

$$\begin{aligned}
&(-1)^k \sum_{n=1}^{\infty} \left[\frac{1}{z+n+1} - \frac{1}{z+n} \right] \\
&+ \sum_{n=1}^{\infty} \left[\frac{(z+n)^k - (-1)^k}{z+n+1} - \sum_{l=0}^{k-1} (-1)^l (z+n)^{k-l-1} \right] .
\end{aligned}$$

It is a simple calculation that the term in the second sum vanishes whereas the first sum just gives $(-1)^{k+1} \frac{1}{z+1}$.

(iii) The proof will be done in two steps. In the first step we show that the operator $\mathcal{L}_{\beta_k}^{(k)}$ is isomorphic to the operator $(-1)^k (\mathcal{L}_{-\beta_k+1} - \mathcal{N}_{-\beta_k+1}^{(0)})$ where $\mathcal{N}_\beta^{(0)} f_\beta(z) = f_\beta(0) \zeta(2\beta, z+1)$. Indeed denote by $\mathcal{D}_z^k$ the differential operator

$$\mathcal{D}_z^k := \frac{d^k}{dz^k} .$$

Then we have for $f \in B(D)$ the following

Lemma 4.

$$\mathcal{D}_z^k \mathcal{L}_{\beta_k}^{(k)} f(z) = (-1)^k (\mathcal{L}_{-\beta_k+1} - \mathcal{N}_{-\beta_k+1}^{(0)}) \mathcal{D}_z^k f(z) , \tag{3.56}$$

Proof. Every function $f \in B(D)$, $D := \{ z : |z-1| < \frac{3}{2} \}$ has a series expansion

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(1)}{j\,!} (z-1)^j$$

which converges uniformly and absolutely in every compactum in D and absolutely on the boundary ∂D [H]. Since

$$\mathcal{L}_\beta^{(k)}(z-1)^j = 0 \quad \text{for} \quad j \le k$$

and

$$\mathcal{L}_\beta^{(k)}(z-1)^j = \sum_{l=k+1}^{j} (-1)^{j-l}\binom{j}{l}\zeta(2\beta+l, z+1) \quad \text{for} \quad j > k\,,$$

we get for $f \in B(D)$

$$\mathcal{L}_\beta^{(k)} f(z) = \sum_{j=k+1}^{\infty}\sum_{l=k+1}^{j} (-1)^{j-l}\binom{j}{l}\frac{f^{(j)}(1)}{j!}\zeta(l+2\beta, z+1)\,.$$

Differentiating k times, one gets for the left hand side of (3.56):

$$\begin{aligned} &D_z^k\,\mathcal{L}_\beta^{(k)}\, f(z) \\ &= \sum_{j=k+1}^{\infty}\sum_{l=k+1}^{j} (-1)^{j-l}\binom{j}{l}\frac{f^{(j)}(1)}{j!}(-1)^k(l+2\beta)_k\zeta(l+2\beta+k, z+1)\,, \end{aligned}$$

which at $\beta = \beta_k$ gives:

$$(-1)^k \sum_{j=k+1}^{\infty}\sum_{l=k+1}^{j} (-1)^{j-l}\frac{f^{(j)}(1)}{(j-l)!(l-k)!}\zeta(l+1, z+1)\,. \tag{3.57}$$

On the other hand, the right hand side of expression (3.56) is, up to the factor $(-1)^k$,

$$\begin{aligned} &(\mathcal{L}_{-\beta+1} - \mathcal{N}_{-\beta+1}^{(0)})\,\mathcal{D}_z^k\, f(z) = (\mathcal{L}_{-\beta+1} - \mathcal{N}_{-\beta+1}^{(0)})\, f^{(k)}(z) \\ &= \sum_{n=1}^{\infty}(\frac{1}{z+n})^{-2\beta+2}\left\{f^{(k)}(\frac{1}{z+n}) - f^{(k)}(0)\right\} \end{aligned}$$

Expanding the curly bracket around $z = 1$ leads to

$$\begin{aligned} &(\mathcal{L}_{-\beta+1} - \mathcal{N}_{-\beta+1}^{(0)})\,\mathcal{D}_z^k\, f(z) \\ &= \sum_{n=1}^{\infty}(\frac{1}{z+n})^{-2\beta+2}\left\{\sum_{j=k+1}^{\infty}\frac{f^{(j)}(1)}{(j-k)!}\left[(\frac{1}{z+n}-1)^{j-k} - (-1)^{j-k}\right]\right\} \\ &= \sum_{n=1}^{\infty}(\frac{1}{z+n})^{-2\beta+2} \end{aligned}$$

$$\times \left\{ \sum_{j=k+1}^{\infty} \frac{f^{(j)}(1)}{(j-k)!} \left[\sum_{l=0}^{j-k} \binom{j-k}{l} (\frac{1}{z+n})^l (-1)^{j-k-l} - (-1)^{j-k} \right] \right\}$$

$$= \sum_{j=k+1}^{\infty} \frac{f^{(j)}(1)}{(j-k)!} \sum_{l=1}^{j-k} \binom{j-k}{l} (-1)^{j-k-l} \zeta(-2\beta+2+l, z+1)$$

$$= \sum_{j=k+1}^{\infty} \sum_{l=1}^{j-k} (-1)^{j-k-l} \frac{f^{(j)}(1)}{(j-k-l)!l!} \zeta(-2\beta+2+l, z+1)$$

$$= \sum_{j=k+1}^{\infty} \sum_{l=k+1}^{j} (-1)^{j-l} \frac{f^{(j)}(1)}{(j-l)!(l-k)!} \zeta(-2\beta+l-k+2, z+1)\,.$$

For $\beta = \beta_k = \frac{1-k}{2}$ this gives

$$\sum_{j=k+1}^{\infty} \sum_{l=k+1}^{j} (-1)^{j-l} \frac{f^{(j)}(1)}{(j-l)!(l-k)!} \zeta(l+1, z+1)\,,$$

which coincides with expression (3.57) for the left hand side of (3.56) up to the factor $(-1)^k$. ◻

Let us go back to the proof of statement (iii) of Proposition 5. The space of polynomials of degree $\le k-1$, denoted by $\mathcal{P}^{\le k-1}$, lies obviously in the kernel of $\mathcal{L}_\beta^{(k)}$. Thus, up to the point zero, $\mathcal{L}_\beta^{(k)}$ has the same spectrum on $B(D)$ as on the quotient space $B(D)/\mathcal{P}^{\le k-1}$. Furthermore, the operator $\mathcal{D}_z^k$ is invertible on $B(D)/\mathcal{P}^{\le k-1}$. That means, on the space $B(D)/\mathcal{P}^{\le k-1}$ we have

$$\mathcal{L}_{\beta_k}^{(k)} = (-1)^k (\mathcal{D}_z^k)^{-1} (\mathcal{L}_{-\beta_k+1} - \mathcal{N}_{-\beta_k+1}^{(0)}) \mathcal{D}_z^k \tag{3.58}$$

and hence the operators $\mathcal{L}_{\beta_k}^{(k)}$ and the operator

$$(-1)^k\,(\mathcal{D}_z^k)^{-1}\,(\mathcal{L}_{-\beta_k+1} - \mathcal{N}_{-\beta_k+1}^{(0)})\,\mathcal{D}_z^k$$

have the same spectrum on this space. But the last operator has besides zero the same spectrum as the operator $(-1)^k\,(\mathcal{L}_{-\beta_k+1} - \mathcal{N}_{-\beta_k+1}^{(0)})$ in the space $B(D)$.

In [M2] it was shown that the operator $\mathcal{L}_\beta : B(D) \to B(D)$ for $\Re\beta > \frac{1}{2}$ has the same spectrum as the operator $\mathcal{L}_\beta$ when acting in the Hilbert space $\mathcal{H}_{\Re\beta}^{(2)}(H_{-1/2})$, which is some generalized Hardy space:

$$\begin{aligned}
&\mathcal{H}_{\Re\beta}^{(2)}(H_{-1/2}) \\
&= \Big\{ f \textit{ holomorphic in } H_{-1/2},\ \textit{bounded in } H_{-1/2+\varepsilon}\,\forall\,\varepsilon>0 \textit{ and} \\
&\quad \int_0^\infty x^{2\Re\beta-2} dx \int_{-\infty}^{+\infty} dy (\mid f(x-\frac{1}{2}+i\,y)\mid^2 - \mid f(x+i\,y)\mid^2) < \infty \Big\}\,,
\end{aligned}$$

where H_δ denotes the half plane $\Re z > \delta$. It was furthermore shown in [M2] that $f \in \mathcal{H}^{(2)}_{\Re\beta}(H_{-1/2})$ iff there exists $\varphi \in L_2(dm, I\!R_+)$ such that

$$f(z) = \int_0^\infty dm(s) e^{-sz} s^{\Re\beta - 1/2} \varphi(s) \, ,$$

with $dm(s) = \frac{ds}{e^s - 1}$. Consider then the operator

$$\mathcal{N}^{(0)}_\beta \; : \; \mathcal{H}^{(2)}_{\Re\beta}(H_{-1/2}) \to \mathcal{H}^{(2)}_{\Re\beta}(H_{-1/2})$$

with

$$\mathcal{N}^{(0)}_\beta f(z) = f(0) \zeta(2\beta, z+1) \, . \tag{3.59}$$

Since [MOS]

$$\zeta(2\beta, z+1) = \frac{1}{\Gamma(2\beta)} \int_0^\infty s^{2\beta - 1} e^{-zs} \, dm(s) \tag{3.60}$$

this function obviously belongs to $\mathcal{H}^{(2)}_{\Re\beta}(H_{-1/2})$. Denote next by $\mathcal{M}^{(0)}_\beta$: $L_2(dm, I\!R_+) \to L_2(dm, I\!R_+)$ the operator

$$\mathcal{M}^{(0)}_\beta \, \varphi(t) := \int_0^\infty dm(s) \frac{t^{\beta - 1/2}}{\Gamma(2\beta)} s^{\beta - 1/2} \, \varphi(s) \, . \tag{3.61}$$

Then we get

Lemma 5. *For $\beta \in I\!\!\!C$ with $\Re\beta > 1/2$ the operators $\mathcal{N}^{(0)}_\beta$ and $\mathcal{M}^{(0)}_\beta$ are isomorphic via the isomorphism $\mathcal{J}_\beta : \mathcal{L}_2(dm, I\!R_+) \to \mathcal{H}^{(2)}_{\Re\beta}(H_{-1/2})$ given by*

$$\mathcal{J}_\beta \, \varphi(z) := \int_0^\infty dm(s) e^{-sz} s^{\beta - 1/2} \varphi(s) \, .$$

Proof. Since for $\beta \in I\!\!\!C$ with $\Re\beta > \frac{1}{2}$ every $f \in \mathcal{H}^{(2)}_{\Re\beta}(H_{-1/2})$ has the unique representation

$$f(z) = \int_0^\infty dm(s) e^{-sz} s^{\beta - 1/2} \varphi(s) \, ,$$

with $\varphi \in L_2(dm, I\!R_+)$ we get

$$f(0) = \int_0^\infty dm(s) s^{\beta - 1/2} \varphi(s) \, .$$

Using representation (3.60) for the Hurwitz zeta function the operator $\mathcal{N}^{(0)}_\beta$ can be written as

$$\mathcal{N}^{(0)}_\beta f(z) = \int_0^\infty dm(t) e^{-tz} t^{\beta - 1/2} \left(\mathcal{M}^{(0)}_\beta \varphi \right)(t) \, , \tag{3.62}$$

with $\mathcal{M}_\beta^{(0)}$ defined on $L_2(dm, I\!R_+)$ as in (3.61). Introducing hence the isomorphism

$$\mathcal{J}_\beta : L_2(dm, I\!R_+) \to \mathcal{H}^{(2)}_{\Re\beta}(H_{-1/2})$$

as

$$\mathcal{J}_\beta\, \varphi(z) := \int_0^\infty dm(s) e^{-sz} s^{\beta-1/2} \varphi(s)\,, \tag{3.63}$$

relation (3.62) can be rewritten as

$$\mathcal{N}_\beta^{(0)} \circ \mathcal{J}_\beta = \mathcal{J}_\beta \circ \mathcal{M}_\beta^{(0)}\,.$$

□

That $\mathcal{J}_\beta$ is indeed an isomorphism can be seen easily as in [M2]. In [M2] it was shown also that for $\Re\beta > \frac{1}{2}$ the operator $\mathcal{L}_\beta$ when acting on $\mathcal{H}^{(2)}_{\Re\beta}(H_{-1/2})$ is isomorphic via the same isomorphism $\mathcal{J}_\beta$ in (3.63) to the integral operator $\mathcal{K}_\beta : L_2(dm, I\!R_+) \to L_2(dm, I\!R_+)$ whose kernel is $\mathcal{K}_\beta(s,t) = \mathcal{J}_{2\beta-1}(2\sqrt{st})$ with $\mathcal{J}_r$ the Bessel function of order r.

From this and Lemma 5 we then get

Lemma 6. *For $\beta \in C\!\!\!\!I$ with $\Re\beta > 1/2$ the operator $\mathcal{L}_\beta - \mathcal{N}_\beta^{(0)}$ acting in $\mathcal{H}^{(2)}_{\Re\beta}(H_{-1/2})$ is isomorphic to the operator $\mathcal{K}_\beta - \mathcal{M}_\beta^{(0)}$ acting in $L_2(dm, I\!R_+)$.*

This shows that indeed for $\Re\beta > \frac{1}{2}$ the eigenvalues of the operator $\mathcal{L}_\beta - \mathcal{N}_\beta^{(0)}$ acting in the Banach space $B(D)$ are real. The same holds then true for the operator $\mathcal{L}_{\beta_k}^{(k)}$ which has the same eigenvalues as $(-1)^k\,(\mathcal{L}_{-\beta_k+1} - \mathcal{N}^{(0)}_{-\beta_k+1})$ for $\beta_k = \frac{1-k}{2}$, $k \in I\!N$ as was shown before. □

Remark In the case $k = 0$ the operator $\mathcal{K}_\beta - \mathcal{M}_\beta^{(0)}$ has still a symmetric kernel but the isomorphism $\mathcal{J}_\beta$ does not exist anymore for $\beta = \frac{1}{2}$, so that our argument does not work. But we conjecture that $\mathcal{L}_{\beta_0}^{(0)}$ has real spectrum also.

We are now ready to discuss the relation of the spectra of $\mathcal{A}_\beta^{(k)}$ and $\mathcal{L}_\beta^{(k)}$ and the spectrum of the generalized Perron-Frobenius operator $\mathcal{L}_\beta$ for $\beta \to \beta_k$.

3.4. Spectral properties of the operator $\mathcal{L}_\beta$.

Theorem 1.

(i) In the limit $\beta \to \beta_k$ the regular eigenvalues of $\mathcal{L}_\beta$ belong either to the regular spectrum of $\mathcal{A}_\beta^{(k)}$ or to the spectrum of $\mathcal{L}_{\beta_k}^{(k)}$.

(ii) Let f be a regular eigenfunction with regular eigenvalue λ of $\mathcal{A}_{\beta_k}^{(k)}$. Then f is a regular eigenfunction with regular eigenvalue λ of the operator $\mathcal{L}_{\beta_k}$.

(iii) Let $\lambda_\beta^\pm$ be the two eigenvalues of the operator $\mathcal{A}_\beta^{(k)}$ with eigenfunctions $f_\beta^\pm(z)$ diverging for $\beta \to \beta_k$. Then the operator $\mathcal{L}_\beta$ has also exactly two eigenvalues $\bar{\lambda}_\beta^\pm$ divergent for $\beta \to \beta_k$ such that

$$\lim_{\beta \to \beta_k} |\, \bar{\lambda}_\beta^\pm(z) - \lambda_\beta^\pm(z)\, | \;=\; 0\,.$$

The eigenfunctions $\bar{f}_\beta^\pm(z)$ can be chosen such that

$$\lim_{\beta \to \beta_k} |\, \bar{f}_\beta^\pm(z) - f_\beta^\pm(z)\, | \;=\; 0\,.$$

(iv) Let λ be an eigenvalue of the operator $\mathcal{L}_{\beta_k}^{(k)}$ with eigenfunction f. If λ is not a regular eigenvalue of $\mathcal{A}_{\beta_k}^{(k)}$, then λ is a regular eigenvalue of $\mathcal{L}_{\beta_k}$.

Since $\mathcal{A}_{\beta_k}^{(k)}$ has only finitely many eigenvalues different from zero the spectrum of $\mathcal{L}_{\beta_k}$ differs from the combined spectra of the operators $\mathcal{L}_{\beta_k}^{(k)}$ and $\mathcal{A}_{\beta_k}^{(k)}$ only in finitely many points. As we will see later, it turns out however that for odd k the number $\lambda = -1$ which is both an eigenvalue of $\mathcal{L}_{\beta_k}^{(k)}$ and $\mathcal{A}_{\beta_k}^{(k)}$ is also an eigenvalue of $\mathcal{L}_{\beta_k}$. For even $k \geq 2$ however $\lambda = -1$ is both an eigenvalue of $\mathcal{L}_{\beta_k}^{(k)}$ and $\mathcal{A}_{\beta_k}^{(k)}$ but in general it does not belong to the regular spectrum of $\mathcal{L}_{\beta_k}$. In the case $k = 0$ the number $\lambda = -1$ is not an eigenvalue of $\mathcal{A}_{\beta_k}^{(k)}$ and Theorem 1 can be applied. As a corollary of Theorem 1 we hence get

Corollary 1. *The Fredholm determinant $det(1 \pm \mathcal{L}_\beta)$ can be written for $\beta \to \beta_k$ as*

$$det(1 \pm \mathcal{L}_\beta) = det(1 \pm \mathcal{L}_\beta^{(k)})\, det(1 \pm \mathcal{A}_\beta^{(k)})\, \phi_\beta^{(k)}\,.$$

For $\beta \to \beta_k$ the function $\phi_\beta^{(k)}$ is simply given as $\prod_j (1 \pm \lambda_j)^{-1}$, where the product is over the finite set of eigenvalues λ_j of $\mathcal{L}_\beta^{(k)}$ which are not eigenvalues of $\mathcal{L}_{\beta_k}$. For k odd the $\lambda_j \neq \pm 1$ for all j.

Remark We conjecture that $\phi_{\beta_k}^{(k)} \equiv 1$ for k odd.

From Proposition 4 (i) and Proposition 5 (i) (iii) also follows

Corollary 2. *Besides the two divergent eigenvalues the spectrum of* $\mathcal{L}_\beta$ *at* $\beta = \beta_k$ *for* $k \in I\!N$ *is real.*

Proof of Theorem 1.
(i) Let λ_β be an eigenvalue of $\mathcal{L}_\beta$ with eigenfunction f_β such that $\lim_{\beta\to\beta_k} \lambda_\beta = \lambda < \infty$ and $\lim_{\beta\to\beta_k} f_\beta = f$ exist. The eigenfunction equation in the limit $\beta \to \beta_k$ can be written as

$$\begin{aligned} \lim_{\beta\to\beta_k} \lambda_\beta f_\beta &= \lambda f = \lim_{\beta\to\beta_k} \mathcal{L}_\beta f_\beta = \lim_{\beta\to\beta_k} \mathcal{A}_\beta^{(k)} f_\beta + \lim_{\beta\to\beta_k} \mathcal{L}_\beta^{(k)} f_\beta \\ &= P_k + \mathcal{L}_{\beta_k}^{(k)} f\,, \end{aligned} \tag{3.64}$$

where P_k is a polylomial of degree $\leq k$. Define $p_k(z) = P_k(z)/\lambda$ and $g(z) = f(z) - p_k(z)$. Then either (1) $g(z) \equiv 0$ or (2) $g(z) \not\equiv 0$.
(1) If $g(z) = f(z) - p_k(z) \equiv 0$, then $f(z) = p_k(z)$ is a polynomial. Therefore

$$\lambda f = \lim_{\beta\to\beta_k} \mathcal{A}_\beta^{(k)} f_\beta + \mathcal{L}_{\beta_k}^{(k)} f = \lim_{\beta\to\beta_k} \mathcal{A}_\beta^{(k)} f_\beta + \mathcal{L}_{\beta_k}^{(k)} p_k\,.$$

Since $p_k(z)$ lies in the kernel of the operator $\mathcal{L}_{\beta_k}^{(k)}$, $\mathcal{L}_{\beta_k}^{(k)} p_k(z)$ vanishes. Thus

$$\lim_{\beta\to\beta_k} \mathcal{A}_\beta^{(k)} f_\beta = \lambda f$$

and λ and $f(z)$ are eigenvalue and eigenfunction of the operator $\mathcal{A}_\beta^{(k)}$ in the limit $\beta \to \beta_k$.
(2) If $g(z) \not\equiv 0$, then

$$\begin{aligned} \mathcal{L}_{\beta_k}^{(k)} g &= \mathcal{L}_{\beta_k}^{(k)} (f - p_k) = \mathcal{L}_{\beta_k}^{(k)} f = \lim_{\beta\to\beta_k} (\mathcal{L}_\beta f_\beta - \mathcal{A}_\beta^{(k)} f_\beta) \\ &= \lambda f - P_k = \lambda f - \lambda p_k = \lambda (f - p_k) = \lambda g\,. \end{aligned}$$

Hence λ is an eigenvalue of the operator $\mathcal{L}_{\beta_k}^{(k)}$ with eigenfunction g.

(ii) We want to show that for $\beta \to \beta_k$ the regular eigenvalues and eigenfunctions of $\mathcal{A}_\beta^{(k)}$ are also eigenvalues and eigenfunctions of $\mathcal{L}_\beta$.

Let λ_β be an eigenvalue of $\mathcal{A}_\beta^{(k)}$ with eigenfunction f_β with

$$\lim_{\beta\to\beta_k} \lambda_\beta = \lambda < \infty \text{ and } \lim_{\beta\to\beta_k} f_\beta = f\,.$$

Then, since f according to Proposition 2 is a polynomial

$$\lim_{\beta\to\beta_k} \mathcal{L}_\beta f_\beta = \lim_{\beta\to\beta_k} \mathcal{A}_\beta^{(k)} f_\beta + \lim_{\beta\to\beta_k} \mathcal{L}_\beta^{(k)} f_\beta = \lim_{\beta\to\beta_k} \mathcal{A}_\beta^{(k)} f_\beta + \mathcal{L}_{\beta_k}^{(k)} f = \lambda f$$

implies that λ is a regular eigenvalue with eigenfunction f of the operator $\mathcal{L}_\beta$ for $\beta \to \beta_k$.

(iii) We will show the two divergent eigenvalues $\bar\lambda^{\pm}_\beta$ of the operator $\mathcal{L}_\beta$ and the divergent eigenvalues $\lambda^{\pm}_\beta$ of the operator $\mathcal{A}^{(k)}_\beta$ for $\beta \to \beta_k$ are up to terms of order $o(1)$ the same as well as their eigenfunctions: indeed we can show

$$\lim_{\beta\to\beta_k} |\,\bar\lambda_\beta - \lambda_\beta\,| = 0 \quad \text{and} \quad \lim_{\beta\to\beta_k} |\,\bar f_\beta(z) - f_\beta(z)\,| = 0\,,$$

if the functions $\bar f^{\pm}_\beta(z)$ and $f^{\pm}_\beta(z)$ are chosen in an appropriate way.

Let $\bar\lambda_\beta$ be a singular eigenvalue of $\mathcal{L}_\beta$ with eigenfunction $\bar f_\beta(z)$. We proceed as in the discussion of the operator $\mathcal{A}^{(k)}_\beta$ in Proposition 2 (iv): inserting the ansatz

$$\bar f_\beta(z) = \sum_{l=0}^{\infty} \bar a_l(\beta)\,\zeta(2\beta + l, z+1)$$

into the eigenfunction equation for $\mathcal{L}_\beta$, we get the equations

$$\begin{aligned}
\bar\lambda_\beta \bar a_0(\beta) &= \sum_{r=0}^{\infty} \bar a_r(\beta)\zeta(2\beta + r)\,,\\
\bar\lambda_\beta \bar a_l(\beta) &= \frac{1}{l!}\sum_{r=0}^{\infty} \bar a_r(\beta)\zeta^{(l)}(2\beta + r)\,,\quad 1 \le l \le k-1\,,\\
\bar\lambda_\beta \bar a_k(\beta) &= \frac{1}{k!}\sum_{r=0}^{\infty} \bar a_r(\beta)\zeta^{(k)}(2\beta + r)\,,\\
\bar\lambda_\beta \bar a_l(\beta) &= \frac{1}{l!}\sum_{r=0}^{\infty} \bar a_r(\beta)\zeta^{(l)}(2\beta + r)\,,\quad l \ge k+1\,.
\end{aligned} \tag{3.65}$$

This system of linear equations is very similar to the one for the operator $\mathcal{A}^{(k)}_\beta$ from (3.42) to (3.44). We can assume again that $\bar a_k(\beta_k) = 1$ and conclude that the coefficient $\bar a_0(\beta)$ must be the most singular term for $\beta \to \beta_k$. Indeed one finds again:

$$\bar a_0(\beta),\, \bar\lambda_\beta \overset{\beta\to\beta_k}{\sim} \frac{1}{\sqrt{\beta - \beta_k}} \tag{3.66}$$

and therefore

$$\bar a_l(\beta) = O(1) \; for \; l \ge 1\,. \tag{3.67}$$

For $l \ge k+1$ the result (3.67) can be sharpended: $\bar a_l(\beta_k) \overset{\beta\to\beta_k}{\sim} \sqrt{\beta - \beta_k}$, $l \ge k+1$. Indeed from this follows, that the right side of (3.65) is regular at

$\beta = \beta_k$, since the only singularity in $\bar{a}_0(\beta) \sim (\beta - \beta_k)^{-1/2}$ will be cancelled by $\zeta^{(l)}(2\beta) \sim (\beta - \beta_k)$ for all $l \geq k+1$ and even tends to zero. Therefore $\bar{a}_l(\beta)$ for $l \geq k+1$ on the left side of (3.65) has to vanish for $\beta \to \beta_k$ at least as $(\beta - \beta_k)^{1/2}$ to cancel the singularity in λ_β. At $\beta = \beta_k$ the term $\bar{a}_k(\beta_k)\zeta^{(l)}(2\beta_k + k)$ is then the only non-zero term on the right hand side in (3.65), because $\zeta^{(l)}(2\beta_k + r) = 0$ for $r < k$, $\bar{a}_r(\beta_k) = 0$ for $r > k$, $\zeta^{(l)}(2\beta_k + k) \neq 0$ and $\bar{a}_k(\beta) = 1$. Hence $\bar{\lambda}_\beta \bar{a}_l(\beta)$ on the left hand side of (3.65) cannot vanish for $\beta \to \beta_k$ and $\bar{a}_l(\beta)$ must behave exactly like $\sqrt{\beta - \beta_k}$ for all $l > k$.

It is then clear that λ_β and $\bar{\lambda}_\beta$, respectively $\bar{a}_l(\beta)$ and $a_l(\beta)$, are up to terms of order $o(1)$ identical and (iii) of Theorem 1 is true.

(iv) Let λ be an eigenvalue of $\mathcal{L}_{\beta_k}^{(k)}$ with eigenfunction $f(z)$. Define the function $g_\beta(z) := f(z) + P_\beta(z)$ with $P_\beta(z) = P_k(z) + (\beta - \beta_k)cz^k + o(\beta - \beta_k)$, where $P_k(z)$ is a polynomial of degree $\leq k$. Then we have

$$\begin{aligned} \mathcal{L}_\beta g_\beta &= \mathcal{L}_\beta(f + P_\beta) = \mathcal{L}_\beta^{(k)}(f + P_\beta) + \mathcal{A}_\beta^{(k)}(f + P_\beta) \\ &= \lambda f + \mathcal{A}_\beta^{(k)}(f + P_\beta) + o((\beta - \beta_k)) \,. \end{aligned}$$

Hence if $\lim_{\beta \to \beta_k} \mathcal{A}_\beta^{(k)}(f + P_\beta) = \lambda P_k$ holds then λ will be a regular eigenvalue of $\mathcal{L}_\beta$ for $\beta \to \beta_k$ with eigenfunction $f + P_k$. But

$$\begin{aligned} &\mathcal{A}_\beta^{(k)}(f(z) + P_\beta(z)) \\ &= \sum_{l=0}^{k} \frac{f^{(l)}(0) + P_\beta^{(l)}(0)}{l!} \zeta(2\beta + l, z+1) \\ &= (f(0) + P_\beta(0))\zeta(2\beta, z+1) + \sum_{l=1}^{k-1} \frac{f^{(l)}(0) + P_\beta^{(l)}(0)}{l!} \zeta(2\beta + l, z+1) \\ &\quad + (\frac{f^{(k)}(0) + P_\beta^{(k)}(0)}{k!}) \zeta(2\beta + k, z+1) \,. \end{aligned} \tag{3.68}$$

Since the right hand side must be regular at $\beta = \beta_k$, $P_\beta^{(k)}(0) + f^{(k)}(0)$ must behave as $2\,k\,!\,c\,(\beta - \beta_k)$ for $\beta \to \beta_k$, where c is some constant. Thus we get

$$P_k^{(k)}(0) = -f^{(k)}(0) \,. \tag{3.69}$$

The equation $\lim_{\beta \to \beta_k} \mathcal{A}_\beta^{(k)}(f + P_\beta) = \lambda P_k$ then has the form

$$\begin{aligned} &\lim_{\beta \to \beta_k} \mathcal{A}_\beta^{(k)}(f(z) + P_\beta(z)) \\ &= (f(0) + P_k(0))\zeta(2\beta_k, z+1) \\ &+ \sum_{l=1}^{k-1} \frac{f^{(l)}(0) + P_k^{(l)}(0)}{l!} \zeta(2\beta_k + l, z+1) + c = \lambda P_k \,. \end{aligned} \tag{3.70}$$

Compare next the coefficients of z^k in (3.70). On the left hand side it is the coefficient of z^k in $\zeta(2\beta_k, z+1)$ which is $-1/k$. The coefficient of z^k on the right hand side is $\frac{\lambda P_k^{(k)}(0)}{k!}$ and therefore

$$-\frac{(f(0)+P_k(0))}{k} = \frac{\lambda P_k^{(k)}(0)}{k!}\,.$$

Due to (3.69), $P_k(0)$ can then be expressed as

$$P_k(0) = \frac{-\lambda P_k^{(k)}(0)}{(k-1)\,!} - f(0) = \frac{\lambda f^{(k)}(0)}{(k-1)\,!} - f(0)\,. \tag{3.71}$$

Inserting this into expression (3.70) gives

$$\frac{\lambda f^{(k)}(0)}{k-1!}\zeta(2\beta_k, z+1) + \bar{\mathcal{A}}_{\beta_k}^{(k)}(f(z)+P_k(z)) + c = \lambda P_k\,, \tag{3.72}$$

with $\bar{\mathcal{A}}_{\beta_k}^{(k)}$ as defined in (2.8). Denote again by

$$\mathcal{P}_0^{\leq(k-1)} := \{p:\, p\in\mathcal{P}^{\leq(k-1)},\, p(0)=0\}$$

the space of polynomials of degree $\leq k-1$ vanishing at $z=0$. Since

$$\zeta(2\beta_k, z+1) = -\frac{1}{k}z^k + P + \zeta(2\beta_k) \quad \text{for some} \quad P\in\mathcal{P}_0^{\leq(k-1)}$$

and

$$\begin{aligned} P_k &= P_k(0) + Q + \frac{P_k^{(k)}(0)}{k!}z^k \\ &= P_k(0) + Q - \frac{f^{(k)}(0)}{k!}z^k \text{ for some } Q\in\mathcal{P}_0^{\leq(k-1)}\,, \end{aligned}$$

we can rewrite (3.72) as

$$\begin{aligned} &\Big[-\frac{\lambda f^{(k)}(0)}{k!}z^k + \frac{\lambda f^{(k)}(0)}{(k-1)\,!}P + \frac{\lambda f^{(k)}(0)}{(k-1)\,!}\zeta(2\beta_k)\Big] \\ &+\Big[\bar{\mathcal{A}}_{\beta_k}^{(k)}f(z) + \bar{\mathcal{A}}_{\beta_k}^{(k)}P_k(0) + \bar{\mathcal{A}}_{\beta_k}^{(k)}Q(z) + \bar{\mathcal{A}}_{\beta_k}^{(k)}\frac{P_k^{(k)}(0)}{k!}z^k\Big] + c \\ &= \lambda P_k(0) + \lambda Q - \frac{\lambda f^{(k)}(0)}{k!}z^k\,. \end{aligned} \tag{3.73}$$

Since $P_k(0)$ and $\frac{P_k^{(k)}(0)}{k!}z^k$ lie in the kernel of $\bar{\mathcal{A}}_{\beta_k}^{(k)}$ we get

$$\begin{aligned} (\bar{\mathcal{A}}_{\beta_k}^{(k)} - \lambda)\,Q &= \left[-\frac{\lambda f^{(k)}(0)}{(k-1)\,!}P - \bar{\mathcal{A}}_{\beta_k}^{(k)}f(z) + \bar{\mathcal{A}}_{\beta_k}^{(k)}f(z)|_{z=0}\right] \\ &+ \left[-\bar{\mathcal{A}}_{\beta_k}^{(k)}f(z)|_{z=0} + \lambda P_k(0) - \frac{\lambda f^{(k)}(0)}{(k-1)\,!}\zeta(2\beta_k) - c\right]\,. \end{aligned} \tag{3.74}$$

The first bracket is a well known polynomial in $\mathcal{P}_0^{\leq(k-1)}$ which we call R. The second bracket is some constant which we call c'. Hence we arrive at the following equation:

$$(\bar{\mathcal{A}}^{(k)}_{\beta_k} - \lambda)\, Q = R + c' \in \mathcal{P}_0^{\leq(k-1)}\,. \tag{3.75}$$

The problem now is to find a polynomial $Q \in \mathcal{P}_0^{\leq(k-1)}$ and a constant c' such that equation (3.75) holds. In case λ is in the resolvent set of $\bar{\mathcal{A}}^{(k)}_{\beta_k}$, this problem can be solved: define

$$Q' = (\bar{\mathcal{A}}^{(k)}_{\beta_k} - \lambda)^{-1}(R + c')\,. \tag{3.76}$$

Then Q' will in general be a polynomial in $\mathcal{P}_0^{\leq(k-1)}$. But chosing the constant c' appropriately we can make Q' vanish at $z = 0$. But every λ which is not a regular eigenvalue of $\mathcal{A}^{(k)}_{\beta}$ at $\beta = \beta_k$ obviously belongs to the resolvent set of $\bar{\mathcal{A}}^{(k)}_{\beta_k}$. $\square$

4. Poles and trivial zeros of Selberg's zeta function. From the approach to Selberg's zeta function via the trace formula it is known that $Z_S(s)$ has poles respectively 'trivial' zeros at the points $s = \beta_k$ for k even respectively k odd. From Theorem 1 (iii) it is clear that a singularity in $det(1 \pm \mathcal{L}_\beta)$ at the point $\beta = \beta_k$ can arize only from a singularity of $det(1 \pm \mathcal{A}^{(k)}_{\beta})$ at this point, which comes from the two singular eigenvalues of the operator $\mathcal{A}^{(k)}_{\beta_k}$. From Proposition 1 we know that this singularity is at most of order 1. This singularity can be cancelled however by the presence of eigenvalues $\lambda = \pm 1$ in the spectra of $\mathcal{A}^{(k)}_{\beta_k}$ respectively $\mathcal{L}^{(k)}_{\beta_k}$. Obviously also the manner these values are approached when β tends to β_k determines the behaviour of $det(1 \pm \mathcal{L}_\beta)$ at $\beta = \beta_k$. To these questions relate the following results:

Proposition 6.

(i) The function $h_k^+(z) = (z+1)^{k-1} - 1$ is an eigenfunction of both the operators $\mathcal{A}^{(k)}_{\beta_k}$ and $\mathcal{L}_{\beta_k}$ with eigenvalue $\lambda = -1$ for all $k \geq 2$.

(ii) Let

$$p_k(z) = \sum_{-1 \leq n \leq k,\, n\ odd} \frac{B_{n+1}}{(n+1)!}\frac{B_{k-n}}{(k-n)!} z^n \tag{4.1}$$

denote the odd part of the period function of the holomorphic Eisenstein series of weight $k+1$ for odd $k \geq 3$. Then the function $h_k^-(z) = p_k(z+1)$ is an eigenfunction of the operator $\mathcal{L}_{\beta_k}$ with eigenvalue $\lambda = 1$.

(iii) The function $f_0(z) = \frac{1}{z+1} - 1$ is a regular eigenfunction of $\mathcal{L}_\beta$ at $\beta = \beta_0 = \frac{1}{2}$ with eigenvalue $\lambda = -1$. The function $f_1(z) = \frac{1}{z+1} - 2 + z$

is a regular eigenfunction of $\mathcal{L}_\beta$ *at* $\beta = \beta_1 = 0$ *with eigenvalue* $\lambda = 1$.

Remark For more details on period functions and polynomials see [Z2].

Before we prove this Proposition we need a relation of the operator $\mathcal{L}_\beta$ to a certain functional equation introduced quite recently by J. Lewis [L1] in connection with his work on the Maass cusp forms for the group $PSL(2,\mathbb{Z})$.

Proposition 7. *If* $f_\beta \in B(D)$ *is an eigenfunction of the operator* $\mathcal{L}_\beta$ *with eigenvalue* λ_β *then* f_β *fulfills the functional equation*

$$\lambda_\beta f_\beta(z) - \lambda_\beta f_\beta(z+1) = (\frac{1}{z+1})^{2\beta} f_\beta(\frac{1}{z+1})\,. \tag{4.2}$$

On the other hand every solution of this equation which is holomorphic in the complex z*-plane cut along* $(-\infty,-1]$ *defines an eigenfunction of* $\mathcal{L}_\beta$ *for* $\Re\beta > -\frac{k}{2}$, $k \in \mathbb{N}_0$ *with* $\beta \neq \beta_l$, $l = 0, 1, \cdots, k$ *iff*

$$\lim_{N\to\infty}\left[\lambda_\beta\, f_\beta(z+N) - \sum_{l=0}^{k}\frac{f_\beta^{(l)}(0)}{l\,!}\zeta(2\beta+l, z+1+N)\right] = 0 \tag{4.3}$$

uniformly in D. *At the points* $\beta = \beta_k$, $k \in \mathbb{N}_0$ *a holomorphic solution* f *in the cut* z*-plane of the equation*

$$\lambda f(z) - \lambda f(z+1) = (\frac{1}{z+1})^{2\beta_k} f(\frac{1}{z+1}) \tag{4.4}$$

determines a regular eigenfunction of $\mathcal{L}_{\beta_k}$ *iff* $f^{(k)}(0) = 0$ *and*

$$\lim_{N\to\infty}\left[\lambda\, f(z+N) - \sum_{l=0}^{k-1}\frac{f^{(l)}(0)}{l\,!}\zeta(1-k+l, z+1+N)\right] = c \tag{4.5}$$

uniformly in D *for some constant* c.

Proof. Consider an eigenfunction f_β of $\mathcal{L}_\beta$ for $\beta \neq \beta_k$, respectively a regular eigenfunction for $\beta = \beta_k$. Then

$$\mathcal{L}_\beta f_\beta(z) = \lambda f_\beta(z) \;\text{ respectively }\; \lim_{\beta\to\beta_k}\mathcal{L}_\beta f_\beta(z) = \lambda_\beta f(z)\,,$$

with $f(z) = \lim_{\beta\to\beta_k} f_\beta(z)$. Since for $\Re\beta > -\frac{k}{2}$

$$\begin{aligned}\mathcal{L}_\beta f_\beta(z) \;&=\; \sum_{n=1}^{\infty}(\frac{1}{n+z})^{2\beta}\Big[f_\beta(\frac{1}{n+z}) - \sum_{l=0}^{k}\frac{f_\beta^{(l)}(0)}{l\,!}(\frac{1}{n+z})^{l}\Big]\\ &\quad+\sum_{l=0}^{k}\frac{f_\beta^{(l)}(0)}{l\,!}\zeta(2\beta+l, z+1)\end{aligned}$$

we find

$$
\begin{aligned}
&\mathcal{L}_\beta f_\beta(z) - \mathcal{L}_\beta f_\beta(z+1) \\
&= (\frac{1}{1+z})^{2\beta}\Big[f_\beta(\frac{1}{1+z}) - \sum_{l=0}^{k} \frac{f_\beta^{(l)}(0)}{l\,!}(\frac{1}{1+z})^l\Big] \\
&\quad + \sum_{l=0}^{k}\Big[\frac{f_\beta^{(l)}(0)}{l\,!}\zeta(2\beta+l, z+1) - \frac{f_\beta^{(l)}(0)}{l\,!}\zeta(2\beta+l, z+2)\Big] .
\end{aligned}
$$

But the Hurwitz zeta function obeys for all $\beta \in \mathbb{C}$ the functional equation [MOS]

$$\zeta(s,z) - \zeta(s, z+1) = z^{-s}$$

and hence for f_β an eigenfunction of $\mathcal{L}_\beta$ we get

$$\lambda_\beta f_\beta(z) - \lambda_\beta f_\beta(z+1) = (\frac{1}{z+1})^{2\beta} f_\beta(\frac{1}{z+1})$$

which is just Lewis' functional equation. The same result holds in case of a regular eigenfunction for $\beta = \beta_k$.

Assume on the other hand f_β is a solution of this functional equation holomorphic in the z-plane cut along $(-\infty, -1]$. Obviously we have

$$\lambda_\beta f_\beta(z) - \lambda_\beta f_\beta(z+1) = \mathcal{L}_\beta f_\beta(z) - \mathcal{L}_\beta f_\beta(z+1)$$

and hence

$$\lambda_\beta f_\beta(z) - \mathcal{L}_\beta f_\beta(z) = \lambda_\beta f_\beta(z+N) - \mathcal{L}_\beta f_\beta(z+N)$$

for all $N \in \mathbb{N}$. But for $N \to \infty$ one gets

$$
\begin{aligned}
&\lim_{N\to\infty}\Big[\lambda_\beta f_\beta(z+N) - \mathcal{L}_\beta f_\beta(z+N)\Big] \\
&= \lim_{N\to\infty}\Big[\lambda_\beta f_\beta(z+N) - \sum_{l=0}^{k}\frac{f_\beta^{(l)}(0)}{l\,!}\zeta(2\beta+l, z+1+N)\Big]
\end{aligned}
$$

uniformly in $z \in D$ and hence if

$$\lim_{\beta\to\beta_k}\Big[\lambda_\beta f_\beta(z+N) - \sum_{l=0}^{k}\frac{f_\beta^{(l)}(0)}{l\,!}\zeta(2\beta+l, z+1+N)\Big] = 0$$

we find

$$\lambda_\beta f_\beta(z) - \mathcal{L}_\beta f_\beta(z) = 0$$

and f_β is an eigenfunction of $\mathcal{L}_\beta$. The argument goes through also in the case $\beta \to \beta_k$ for regular λ and f:

$$\lambda f(z) - \lim_{\beta\to\beta_k}\mathcal{L}_\beta f_\beta(z) = \lambda f(z+N) - \lim_{\beta\to\beta_k}\mathcal{L}_\beta f_\beta(z+N) .$$

If therefore $f^{(k)}(0) = 0$

$$
\begin{aligned}
&\lim_{N\to\infty}\Big[\lambda f(z+N) - \lim_{\beta\to\beta_k}\mathcal{L}_\beta f_\beta(z+N)\Big]\\
&= \lim_{N\to\infty}\Big[\lambda f(z+N) - \sum_{l=0}^{k-1}\frac{f_\beta^{(l)}(0)}{l\,!}\zeta(1-k+l,z+1+N)\\
&\quad - \lim_{\beta\to\beta_k}\frac{f_\beta^{(k)}(0)}{k\,!}\zeta(2\beta+k,z+1+N)\Big]\\
&= c - \lim_{\beta\to\beta_k}\frac{f_\beta^{(k)}(0)}{k\,!}\,\frac{1}{2\,(\beta-\beta_k)}\,.
\end{aligned}
$$

Hence chosing $\frac{f_\beta^{(k)}(0)}{k\,!} = 2\,c\,(\beta-\beta_k)$ we see that $f = \lim_{\beta\to\beta_k} f_\beta(z)$ is a regular eigenfunction for $\mathcal{L}_\beta$ at $\beta = \beta_k$ with eigenvalue λ.

For β-values with $\Re\beta > \frac{1}{2}$ condition (4.3) reduces to $\lim_{N\to\infty} f_\beta(z+N) = 0$, since $\lim_{N\to\infty}\zeta(2\beta, z+1+N) = 0$ if $\Re\beta > \frac{1}{2}$. In the case $\Re\beta = \frac{1}{2}$ on the other hand a solution f_β of Lewis' equation (4.2) vanishing at infinity determines an eigenfunction of the operator $\mathcal{L}_\beta$ iff $f_\beta(0) = 0$. This follows again from condition (4.3) since for $\Re\beta = \frac{1}{2}$ the Hurwitz function $\zeta(2\beta, z+N)$ does not vanish for $N\to\infty$. ◻

Proof of Proposition 6.
(i) Consider first the function $h_k^+(z) = (z+1)^{k-1} - 1$ for $k \ge 2$. For $\beta = \beta_k = \frac{1-k}{2}$ and $\lambda = -1$ we get

$$
\begin{aligned}
&-h_k^+(z+1) + \Big(\frac{1}{z+1}\Big)^{\beta_k} h_k^+\Big(\frac{1}{z+1}\Big)\\
&= -(z+2)^{k-1} + 1 + \Big(\frac{1}{z+1}\Big)^{1-k}\Big(\Big(1+\frac{1}{z+1}\Big)^{k-1} - 1\Big)\\
&= -(z+2)^{k-1} + 1 + (z+1)^{k-1}\,\frac{(z+2)^{k-1}}{(z+1)^{k-1}} - (z+1)^{k-1}\\
&= -((z+1)^{k-1} - 1) = -h_k^+(z)
\end{aligned}
$$

and $h_k^+(z)$ is a solution of Lewis equation with $\lambda = -1$. Obviously $h_k^{+(k)}(0) = 0$ and $h_k^+(0) = 0$. Next consider condition (4.5) of Proposition 7:

$$
\begin{aligned}
(4.6)\qquad &-h_k^+(z) - \sum_{l=0}^{k-1}\frac{h_k^{+(l)}(0)}{l\,!}\zeta(2\beta_k+l,z+1)\\
&= -(z+1)^{k-1} + 1 - \sum_{l=1}^{k-1}\binom{k-1}{l}\,\zeta(1-k+l,z+1)\,.
\end{aligned}
$$

Since $\zeta(1-k+l, z+1) = -\frac{B_{k-l}(z+1)}{k-l}$ and

$$\binom{k-1}{l}\frac{1}{k-l} = \frac{(k-1)\,!}{l\,!\,(k-l-1)\,!}\frac{1}{k-l} = \frac{1}{k}\frac{k\,!}{l\,!\,(k-l)\,!} = \frac{1}{k}\binom{k}{k-l}$$

we see with the identity [MOS]

$$z^{k-1} = \frac{1}{k-1}\sum_{l=0}^{k-1}\binom{k}{l}\,B_l(z)$$

that (4.6) is just $\frac{1}{k-1}B_0(z)+1 = -\frac{1}{k-1}+1$. This shows that

$$-h_k^+(z) - \sum_{l=0}^{k-1}\frac{h_k^{+(l)}(0)}{l\,!}\zeta(2\beta_k+l, z+1) - \lim_{\beta\to\beta_k}\frac{h_k^{+(k)}(0)}{k\,!}\zeta(2\beta_k+k, z+1) = 0$$

if $h_\beta(z) = h_k^+(z) + c\,(2\beta-(1-k))\,z^k$ and $c = \frac{1}{k-1}-1$. Hence $h_k^+(z)$ is a regular eigenfunction of $\mathcal{L}_\beta$ at $\beta=\beta_k$ with eigenvalue $\lambda=-1$.

(ii) Consider next for odd $k\geq 3$ the functions

$$h_k^-(z) = \sum_{-1\leq n\leq k,\, n\ odd}\frac{B_{n+1}}{(n+1)!}\frac{B_{k-n}}{(k-n)!}(z+1)^n \tag{4.7}$$

To prove, that these functions are eigenfunctions of the operator $\mathcal{L}_\beta$ at $\beta=\beta_k$ with eigenvalue $\lambda=1$ we recall a result by Zagier, who showed in [Z1] that the function h_β, defined for $\Re\beta>1$ as

$$h_\beta(z) = \sum_{n,m\geq 1}\left(\frac{1}{m(z+1)+n}\right)^{2\beta} + \frac{1}{2}\,\zeta(2\beta)\,\left(1+\left(\frac{1}{z+1}\right)^{2\beta}\right).$$

has an analytic extension into the entire β-plane with a simple pole at $\beta=1$. Its analytic extension is a solution of Lewis equation for all $\beta\neq 1$. We show next

Lemma 7. *The analytic extension of the function h_β at $\beta=\beta_k$, $k=3,5,7,\cdots$ is up to a factor $(-1)^{k-1}\,(k-1)!$ the function $h_k^-(z)$ in (4.7).*

Proof. The term of the function $h_\beta(z)$ at $\beta=\beta_k$ proportional to $\zeta(2\beta)$ vanishes for $\beta=-1,-2,-3,\cdots$, this means, since $\beta_k=\frac{1-k}{2}$, for $k=(2n+1)$ and $n\in I\!N$. Hence it remains to determine the analytic extension of the function

$$g_\beta(z) = \sum_{n,m=1}^{\infty}\left(\frac{1}{m(z+1)+n}\right)^{2\beta}$$

to these β-values. To achieve this extension we use the Mellin transform method [C], which gives for $\Re\beta > 1$

$$\Gamma(2\beta)\; g_\beta(z) = \int_0^\infty \sum_{n,m=1}^{\infty} e^{-t\,(m(z+1)+n)}\, t^{2\beta-1}\, dt\,.$$

Consider next the function $F_z(t)$ defined for $t > 0$ as

$$F_z(t) = \sum_{n,m=1}^{\infty} e^{-t\,(m(z+1)+n)} = \frac{e^{-t}}{1-e^{-t}}\; \frac{e^{-t\,(z+1)}}{1-e^{-t\,(z+1)}}\,.$$

The asymptotic expansion of $F_z(t)$ for $t \to 0$ follows from the expansion [C]

$$\frac{1}{e^t - 1} \sim \sum_{r=-1}^{\infty} \frac{B_{r+1}}{(r+1)\,!}\, t^r \tag{4.8}$$

as

$$F_z(t) \sim \sum_{r=-1}^{\infty} \frac{B_{r+1}}{(r+1)\,!}\, t^r \sum_{l=-1}^{\infty} \frac{B_{l+1}}{(l+1)\,!}\,(z+1)^l\, t^l$$

respectively

$$\begin{aligned} F_z(t) \quad &\sim \quad \frac{1}{z+1}\, t^{-2} + B_0 B_1 (1 + \frac{1}{z+1})\, t^{-1} \\ &\quad + \sum_{l=0}^{\infty} \left(\sum_{n=-1}^{l+1} \frac{B_{n+1}\, B_{l-n+1}}{(n+1)!\,(l-n+1)!} (z+1)^n \right)\, t^l\,. \end{aligned}$$

From this asymptotic expansion one gets for the analytic extension of $g_\beta(z)$ to the half plane $\Re z > \frac{-k-1}{2}$

$$\begin{aligned} g_\beta(z) \quad &= \frac{1}{\Gamma(2\beta)} \Big\{ \sum_{n=-2}^{k} \frac{c_n}{2\beta + n} + \int_0^1 \Big[F_z(t) - \sum_{n=-2}^{k} c_n(z)\, t^n \Big]\, t^{2\beta-1}\, dt \\ &\quad + \int_1^\infty F_z(t)\, t^{2\beta-1}\, dt \Big\} \end{aligned}$$

with $c_{-2} = \frac{1}{z+1}$; $c_{-1} = B_0 B_1\, (1 + \frac{1}{z+1})$ and

$$c_l = \sum_{n=-1}^{l+1} \frac{B_{n+1}\, B_{l-n+1}}{(n+1)!\,(l-n+1)!} (z+1)^n\,, \quad l \in \mathbb{N}_0\,.$$

The Γ function has poles at the points $2\beta = 0, -1, -2, \cdots$ with residue $\frac{(-1)^n}{n\,!}$ for $2\beta = -n$. This shows that $g_\beta(z)$ has a pole only at the point $\beta = 1$

with residue $\frac{1}{2}\frac{1}{z+1}\frac{1}{\Gamma(2)}$ and at the point $\beta = \frac{1}{2}$ with residue $\frac{B_0\,B_1}{2\,\Gamma(1)}(1+\frac{1}{z+1})$. At the points $2\beta = -n$ we get

$$\begin{aligned} &g_{-n/2}(z) \\ &= \lim_{\beta\to -n/2} c_n \frac{1}{2\beta+n}\frac{1}{\Gamma(2\beta)} = c_n(-1)^n\, n\,! \lim_{\beta\to -n/2}\frac{2\beta+n}{2\beta+n} \\ &= (-1)^n\, n\,! \sum_{r=-1}^{n+1} \frac{B_{r+1}\,B_{n-r+1}}{(r+1)!\,(n-r+1)!}(z+1)^r\,. \end{aligned}$$

This shows that the function $g_\beta(z)$ takes for $\beta = \beta_k = \frac{1-k}{2}$ the following form

$$g_{\beta_k}(z) = (-1)^{k-1}\,(k-1)\,! \sum_{r=-1}^{k} \frac{B_{r+1}\,B_{k-r}}{(r+1)!\,(k-r)!}(z+1)^r$$

which up to the factor $(-1)^{k-1}(k-1)\,!$ is just the function $h_k^-(z)$ in (4.7). This proves Lemma 7. □

As a corollary we get from this Lemma that the functions $h_k^-(z)$ fulfill Lewis equation (4.4) for $\beta = \beta_k$, $k = 3, 5, 7, \cdots$. To make sure that these functions are indeed eigenfunctions of the operator $\mathcal{L}_\beta$ at $\beta = \beta_k$ we still have to check if the following relations hold

(i)

$$\sum_{r=1}^{k} \frac{B_{r+1}\,B_{k-r}}{(r+1)!\,(k-r)!} z^r - \sum_{l=0}^{k-1} \frac{h_k^{-(l)}(0)}{l\,!}\frac{B_{k-l}(z+1)}{k-l} = c \tag{4.9}$$

for some constant $c \in \mathbb{C}$, where $B_n(z)$ denotes the Bernoulli polynomial of degree n,

(ii)

$$h_k^{-(k)}(0) = 0\,. \tag{4.10}$$

Property (ii), is easy to verify: since

$$\frac{h_k^{-(l)}(0)}{l\,!} = \frac{B_0\,B_{k+1}}{(k+1)\,!}(-1)^l + \sum_{r=l}^{k} \frac{B_{r+1}\,B_{k-r}}{(r+1)!\,(k-r)!}\binom{r}{l} \tag{4.11}$$

we find $h_k^{-(k)}(0) = \frac{B_0\,B_{k+1}}{(k+1)\,!}[(-1)^k + 1] = 0$ since k is odd, respectively $B_{k+1} = 0$ for $k \geq 2$ even.

Property (i) is more subtle to prove. We know that $h_k^-(z)$ fulfills Lewis functional equation for $\beta = \beta_k$ and $\lambda = 1$. A straightforward but rather tedious calculation shows that from this the following relations for the

Bernoulli numbers follow for $l = 0, 1, 2, \cdots, k-1$

$$\sum_{n=l+1}^{k-2} \binom{n}{l} \frac{B_{n+1}\, B_{k-n}}{(n+1)!\,(k-n)!} + \sum_{n=1}^{l} \binom{k-1-n}{l-n} \frac{B_{k-n}\, B_{n+1}}{(k-n)!\,(n+1)!} + \frac{B_0\, B_{k+1}}{(k+1)!}\left[(-1)^l + \binom{k+1}{l+1}\right] = 0\,. \tag{4.12}$$

For $l = 0$ the last sum in (4.12) is absent. To derive relations (4.12) the following formula was used for odd k,

$$\frac{(z+1)^k + 1}{z+2} = \sum_{n=2}^{k-1} (-1)^n (z+1)^n - z\,,$$

which can be easily verified. On the other hand, expressing $(z+1)^n$ in terms of the Bernoulli polynomials $B_l(z+1)$ as

$$(z+1)^n = \sum_{l=k-n}^{k-1} \binom{n}{k-l-1} \frac{B_{k-l}(z+1)}{k-l} + \frac{1}{n+1}$$

which follows immediately from the identity [MOS]

$$(n+1) z^n = \sum_{m=0}^{n} \binom{n+1}{m} B_m(z)\,, \tag{4.13}$$

the first term on the left hand side of relation (4.9) has therefore the form

$$\sum_{l=0}^{k-1} \sum_{n=-1}^{l-1} \binom{k-n-1}{l-n} b_{k-n-1} \frac{B_{k-l}(z+1)}{k-l} + \sum_{n=0}^{k} \frac{b_n}{n+1} \tag{4.14}$$

with $b_n = \frac{B_{k-n}\, B_{n+1}}{(k-n)!\,(n+1)!} = b_{k-1-n}$ for $0 \le l \le k$. The second term on the left hand side of (4.9) on the other hand can be written when using expression (4.11) for $h_k^{-(l)}(0)$:

$$-\sum_{l=0}^{k-1} \left[\sum_{n=l}^{k} \binom{n}{l} b_n + (-1)^l b_{-1}\right] \frac{B_{k-l}(z+1)}{k-l} \tag{4.15}$$

Hence identity (4.9) is fulfilled iff

$$-\sum_{l=0}^{k-1} \left[\sum_{n=l}^{k} \binom{n}{l} b_n + (-1)^l b_{-1} + \sum_{n=-1}^{l-1} \binom{k-n-1}{l-n} b_n\right] \times \frac{B_{k-l}(z+1)}{k-l} - \sum_{n=0}^{k} \frac{b_n}{n+1} = c\,.$$

But it is easy to check that the expression under the brackets coincides with the left hand side of relation (4.12) and hence vanishes for all $0 \leq l \leq k-1$. Chosing hence $c = -\sum_{n=0}^{k} \frac{b_n}{n+1}$ shows that also Property (ii) in (4.10) is fulfilled for the functions $h_k^-(z)$ and they are regular eigenfunctions for $\mathcal{L}_{\beta_k}$ as we claimed.

(iii) We apply Proposition 7. For $\beta = \beta_0 = \frac{1}{2}$ the Lewis equation with $\lambda_0 = -1$ reads

$$-f(z) = -f(z+1) + \frac{1}{z+1} f(\frac{1}{z+1}).$$

A trivial calculation shows that $f_0(z) = \frac{1}{z+1} - 1$ fulfills this equation. Obviously $f_0(0) = 0$ and

$$\lim_{N\to\infty} \Big[-\frac{1}{z+1+N} + 1 \Big] = const.$$

Hence $f_0(z)$ determines a regular eigenfunction of $\mathcal{L}_\beta$ at $\beta = \frac{1}{2}$ with eigenvalue $\lambda = -1$. Indeed, we will see later that the corresponding λ_β behaves for $\beta \to \frac{1}{2}$ as $\lambda_\beta = -1 - 4\,(\beta - \frac{1}{2}) + o(\beta - \frac{1}{2})$.

For $\beta = \beta_1 = 0$ Lewis functional equation for $\lambda = 1$ reads

$$f(z) = f(z+1) + f(\frac{1}{z+1}).$$

A trivial calculation shows that the function $f_1(z) = \frac{1}{z+1} - 2 + z$ solves this equation. Obviously $f_1'(0) = 0$ and

$$\begin{aligned} &\lim_{N\to\infty} \Big[f_1(z+N) - f_1(0)\,\zeta(0, z+1+N) \Big] \\ &= -2 + \lim_{N\to\infty} \Big[z + N - B_1(z+1+N) \Big] = const. \end{aligned}$$

Hence also $f_1(z)$ is a regular eigenfunction of $\mathcal{L}_\beta$ for $\beta = 0$ with eigenvalue $\lambda = 1$. $\square$

Remark The functions $r_k^-(z) = h_k^-(z-1)$ are just the odd parts of the period functions of the holomorphic Eisenstein series of weight $k+1$, $k = 3, 5, 7, \cdots$ for the group $PSL(2, \mathbb{Z})$ determined some time ago by Zagier in [Z2]. The even parts of the period functions to these non cusp forms are the functions [Z2]

$$r_k^+(z) = h_k^+(z-1) = z^{k-1} - 1,$$

which were discussed in (i) Proposition 6.

In a next step we discuss the behaviour of the eigenvalue λ_β of $\mathcal{L}_\beta$ which for $\beta \to \beta_k$ tends to the eigenvalue $\lambda = 1$ corresponding to the eigenfunction $h_k^-(z)$. Instead of working with the operator $\mathcal{L}_\beta$ for $\beta \to \beta_k$ we can also investigate the behaviour of an eigenvalue λ_β when β tends to β_k by using Lewis functional equation (4.2). Let

$$\lambda_\beta = \lambda + \bar{\lambda}\,(\beta - \beta_k) + \tilde{\lambda}\,(\beta - \beta_k)^2 + O(\,(\beta - \beta_k)^3\,) \tag{4.16}$$

and

$$f_\beta(z) = f(z) + \bar{f}(z)\,(\beta - \beta_k) + \tilde{f}(z)\,(\beta - \beta_k)^2 + O(\,(\beta - \beta_k)^3\,)\,. \tag{4.17}$$

Then one shows

Proposition 8. *The eigenvalue λ_β of $\mathcal{L}_\beta$ approaching the regular eigenvalue $\lambda = 1$ belonging to the eigenfunction $f(z) := h_k^-(z) = p_k(z+1)$ of $\mathcal{L}_{\beta_k}$ has in its $\beta - \beta_k$ expansion the linear coefficient*

$$\bar{\lambda} = \left[\,\frac{\bar{f}^{(k)}(0)}{k\,!} + \sum_{i=0}^{k-1} \frac{f^{(i)}(0)}{i\,!}\,\frac{(-1)^i}{(k-i)}\,\right] \;\Big/\; \left[\frac{f^{(k)}(1)}{k\,!}\right]\,. \tag{4.18}$$

For $k = 1$ we find $\bar{\lambda} = -8$ whereas $\bar{\lambda} = 0$ for $k = 3, 5, 7, \cdots$.

Before we prove Proposition 8, we state two Lemmas. In the first several properties of the function $h_k^-(z) := p_k(z+1)$ and $\bar{f}(z)$ are collected.

Lemma 8. *Let $k \geq 3$ be an odd integer. Then the function*

$$h_k^-(z) = \sum_{-1 \leq n \leq k,\; n\ odd} \frac{B_{n+1}}{(n+1)!}\,\frac{B_{k-n}}{(k-n)!}(z+1)^n = \sum_{n=-1}^{k} b_n (z+1)^n$$

has the following properties:
(i)

$$b_n \;=\; b_{k-1-n} \;\textit{for}\; -1 \leq n \leq k \;\textit{and}\; b_n = 0 \;\textit{for}\; n \;\textit{even}, \tag{4.19}$$

$$h_k^{-(k)}(0) \;=\; 0\,, \tag{4.20}$$

$$\frac{h_k^{-(k)}(1)}{k\,!} \;=\; \frac{B_{k+1}}{(k+1)\,!}\,[\,1 - 2^{-k-1}\,] \neq 0\,, \tag{4.21}$$

$$h_k^-(0) \;=\; -k\,\frac{B_{k+1}}{(k+1)\,!}\,, \tag{4.22}$$

$$\frac{h_k^{-(k-1)}(0)}{(k-1)\,!} \;=\; (k+1)\,\frac{B_{k+1}}{(k+1)\,!}\,, \tag{4.23}$$

$$\frac{(\bar{f}^{k-})^{(k)}(0)}{k\,!} \;=\; -2\sum_{l=0}^{k-1} \frac{f^{(l)}(0)}{l\,!}\zeta(1-k+l) + 2\,f(0) - 2\frac{B_{k+1}}{(k+1)\,!}\,. \tag{4.24}$$

(ii) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ denotes the Taylor expansion of $h_k^-(z)$ around $z = 0$ then the coefficients a_n are related to the coefficients b_n as follows:

$$a_l = \sum_{i=\frac{l}{2}+1}^{\frac{k+1}{2}} b_{2i-1} \binom{2i-1}{l} + b_{-1}\,, \qquad \text{for } l \text{ even}\,, \tag{4.25}$$

$$a_l = \sum_{i=\frac{l+1}{2}}^{\frac{k+1}{2}} b_{2i-1} \binom{2i-1}{l} - b_{-1}\,, \qquad \text{for } l \text{ odd}\,. \tag{4.26}$$

(These relations are also valid for $k = 1$).

Proof. The only nontrivial statements are (4.22) and (4.24) so that we can restrict ourselves to proving those.
For statement (4.22), since $B_{2n+1} = 0$, $\forall n \in I\!N$ we see that

$$\begin{aligned} h_k^-(0) &= \sum_{-1 \le n \le k,\, n\ odd} \frac{B_{n+1}}{(n+1)!} \frac{B_{k-n}}{(k-n)!} \\ &= \sum_{n=0}^{k+1} \frac{B_n}{n\,!} \frac{B_{k-n+1}}{(k-n+1)!} \quad \text{for } k = 3,\, 5,\, 7,\, \cdots \qquad (4.27) \\ &= \sum_{n=0}^{k'} \frac{B_n}{n\,!} \frac{B_{k'-n}}{(k'-n)!} \quad \text{for } k' = k+1 = 4,\, 6,\, 8,\, \cdots\,. \qquad (4.28) \end{aligned}$$

The last sum is a Cauchy product, indeed it is the coefficient $c_{k'}$ of $z^{k'}$ in the Taylor series of the function $g^2(z)$, where $g(z)$ is defined as

$$g(z) := \left(\frac{z}{e^z - 1}\right) = \sum_{k'=0}^{\infty} \frac{B_{k'}}{k'\,!}\, z^{k'}\,.$$

But then

$$g^2(z) = \left(\frac{z}{e^z - 1}\right)^2 = \sum_{k'=0}^{\infty} c_{k'}\, z^{k'}\,,$$

with $c_{k'} = h_{k'-1}^-(0)$ for $k' = 4,\, 6,\, \cdots$, respectively

$$g^2(z) = \sum_{k'=0}^{\infty} \frac{B_{k'}^{(2)}}{k'\,!}\, z^{k'}$$

is the generating function for the Bernoulli numbers $B_{k'}^{(2)}$ of order 2. Hence $c_{k'} = \frac{B_{k'}^{(2)}}{k'\,!}$. Differentiating the function $g(z)$ leads to

$$\frac{d}{d\,z}\, g(z) = -\frac{z}{e^z - 1} + \frac{1}{z}\left(\left(\frac{z}{e^z - 1}\right) - \left(\frac{z}{e^z - 1}\right)^2\right),$$

and hence to the following Taylor series expansions

$$\sum_{k'=1}^{\infty} \frac{k'\,B_{k'}}{k'\,!}\, z^{k'-1} = -\sum_{k'=0}^{\infty} \frac{B_{k'}}{k'\,!}\, z^{k'} + \frac{1}{z} \sum_{k'=0}^{\infty} \frac{B_{k'}}{k'\,!}\, z^{k'} - \frac{1}{z} \sum_{k'=0}^{\infty} \frac{B^{(2)}_{k'}}{k'\,!}\, z^{k'} .$$

Comparing then the coefficients of $z^{k'-1}$ we get

$$\frac{k'\,B_{k'}}{k'\,!} = -\frac{B_{k'-1}}{(k'-1)\,!} + \frac{B_{k'}}{k'\,!} - \frac{B^{(2)}_{k'}}{k'\,!}\,, \quad k' = 0,\, 1,\, 2,\, \cdots .$$

This implies

$$c_{k'} = \frac{B^{(2)}_{k'}}{k'\,!} = -\frac{(k'-1)\,B_{k'}}{k'\,!} - \frac{B_{k'-1}}{(k'-1)\,!}\,, \quad k' = 0,\, 1,\, 2,\, \cdots .$$

Restricting k' to the values $k' = k+1 = 4,\, 6,\, 8,\, \cdots$ for which $B_{k'-1} = 0$, we get finally

$$h^-_{k'-1}(0) = c_{k'} = \frac{B^{(2)}_{k'}}{k'\,!} = -\frac{(k'-1)\,B_{k'}}{k'\,!}\,, \quad k' = 4,\, 6,\, 8,\, \cdots .$$

To prove statement (4.24) we proceed as follows:
Denote by $f_\beta(z)$ the eigenfunction of $\mathcal{L}_\beta$ which for $\beta \to \beta_k$ tends to $h^-_k(z)$ with

$$\mathcal{L}_\beta f_\beta(z) - \lambda_\beta\, f_\beta(z) = 0\,.$$

It has the general form:

$$f_\beta(z) = f(z) + \bar{f}(z)(\beta-\beta_k) + O(\,(\beta-\beta_k)^2\,) = \frac{b_{-1}}{z+1} + P_k + O(\,(\beta-\beta_k)\,)\,,$$

where $b_{-1} = \frac{B_{k+1}}{(k+1)\,!}$ and P_k is a polynomial of degree $\leq k$. Since P_k is in the kernel of $\mathcal{L}^{(k)}_\beta$, we get the eigenvalue equation

$$\sum_{l=0}^{k-1} \frac{f^{(l)}_\beta(0)}{l\,!}\,\zeta(2\beta+l, z+1) + \frac{f^{(k)}_\beta(0)}{k\,!}\,\zeta(2\beta+k, z+1) + \mathcal{L}^{(k)}_\beta\, f_\beta(z) - \lambda_\beta\, f_\beta(z) = 0\,.$$

Taking next the limit $\beta \to \beta_k = \frac{1-k}{2}$ and setting $z = 0$, the first term is equal to

$$\sum_{l=0}^{k-1} \frac{h_k^{-\,(l)}(0)}{l\,!}\,\zeta(1-k+l)\,.$$

The second term, due to $h_k^{-\,(k)}(0) = 0$ is equal to

$$\lim_{\beta\to\beta_k} \Big[\,\frac{\bar{f}^{(k)}(0)}{k\,!}(\beta-\beta_k) \,+ O((\beta-\beta_k)^2)\Big]\Big[\frac{1}{2(\beta-\beta_k)} + O(1)\Big] = \frac{\bar{f}^{(k)}(0)}{2\,k\,!}\,.$$

The third term is in the limit $\beta \to \beta_k$ equal to

$$\left[\mathcal{L}^{(k)}_{\beta_k}\Big(\frac{b_{-1}}{z+1} + P_k\Big)\right]_{z=0} = b_{-1}(-1)^{k+1}\,,$$

since $\mathcal{L}^{(k)}_{\beta_k}\,\frac{1}{z+1} = (-1)^{k+1}\,\frac{1}{z+1}$. Summing up the three terms, we get statement (4.24). □

In the next Lemma we have collected two identities for Bernoulli numbers, which we need for the proof of Proposition 8, and which we did not find in the literature. The proof of statement (ii) was communicated to us by S. Johansson (Göteborg).

Lemma 9.
(i) *Let $k \geq 3$ be an odd integer. Then*

$$\varphi(k) := \sum_{l=0}^{k-2} \frac{2\,B_{k-l}\,\left[\binom{k}{l} - 1\right] + \left[1 + (-1)^l\binom{k}{l}\right]}{(k-l)} = 0\,.$$

(ii) *For $k \geq 1$ let r be an integer with $r \leq 2k$. Then*

$$\varphi(k,r) := c(2k+1,r) + c(2k+1,2k-r) = 0\,, \tag{4.29}$$

where

$$c(k,r) := \sum_{l=0}^{r} \frac{2\,B_{k-l} + (-1)^l}{k-l}\binom{r}{l}\,. \tag{4.30}$$

Proof.
(i) This identity is proved by induction on $k = 3, 5, 7, \cdots$. The case $k = 3$ is trivial since $\varphi(3) = 2B_2 - \frac{1}{3} = 0$ as $B_2 = \frac{1}{6}$. We will show that $\varphi(k+2) - \varphi(k) = 0$. In a first step, we decompose $\varphi(k)$ into four parts:

$$\begin{aligned}
\varphi(k) \quad &= \quad 2\sum_{l=0}^{k-2} \frac{B_{k-l}}{k-l}\binom{k}{l} - 2\sum_{l=0}^{k-2}\frac{B_{k-l}}{k-l} + \sum_{l=0}^{k-2}\frac{1}{k-l} \\
&\quad + \sum_{l=0}^{k-2}\frac{(-1)^l}{k-l}\binom{k}{l} \quad \text{for} \quad k = 3,\,5,\,7,\,\cdots \\
&:= \quad c_1(k) + c_2(k) + c_3(k) + c_4(k)\,,
\end{aligned}$$

where the $c_i(k)\,, i = 1,\,2,\,3,\,4$ denote the four sums in $\varphi(k)$. Consider first

$$c_1(k) = 2\sum_{l=0}^{k-2}\frac{B_{k-l}}{k-l}\binom{k}{l}\,,$$

respectively

$$\begin{aligned} c_1(k+2) &= 2\sum_{l=0}^{k} \frac{B_{k+2-l}}{k+2-l}\binom{k+2}{l} \\ &= 2\,\frac{k+2}{k+1}B_{k+1} + 2\sum_{l=0}^{k-2} \frac{B_{k-l}}{k-l}\binom{k+2}{l+2}\,. \end{aligned} \tag{4.31}$$

Using the two formulas

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{4.32}$$

and

$$\frac{1}{n+1-m}\binom{n}{m} = \frac{1}{n+1}\binom{n+1}{m} \tag{4.33}$$

we can write

$$\begin{aligned} \frac{\binom{k+2}{l+2}}{k-l} &= \frac{\binom{k+1}{l+2}+\binom{k+1}{l+1}}{k-l} = \frac{\binom{k+2}{l+2}}{k+2} + \frac{\binom{k}{l+1}+\binom{k}{l}}{k-l} \\ &= \frac{\binom{k+2}{l+2}}{k+2} + \frac{\binom{k+1}{l+1}}{k+1} + \frac{\binom{k}{l}}{k-l}\,. \end{aligned}$$

Inserting this into (4.31) yields

$$\begin{aligned} c_1(k+2) &= 2\,\frac{k+2}{k+1}B_{k+1} + \frac{2}{k+2}\sum_{l=0}^{k-2}\binom{k+2}{l+2}B_{k-l} \\ &\quad + \frac{2}{k+1}\sum_{l=0}^{k-2}\binom{k+1}{l+1}B_{k-l} + 2\sum_{l=0}^{k-2}\binom{k}{l}\frac{B_{k-l}}{k-l}\,. \end{aligned} \tag{4.34}$$

The first and second sum of expression (4.34) can be simplified by means of (4.32) and [MOS]

$$\sum_{m=0}^{n}\binom{n}{m}B_{n-m} = \sum_{m=0}^{n}\binom{n}{m}B_m = B_n$$

as follows

$$\begin{aligned} \frac{2}{k+2}\sum_{l=0}^{k-2}\binom{k+2}{l+2}B_{k-l} &= \frac{2}{k+2}\sum_{l=2}^{k}\binom{k+2}{l}B_{k+2-l} = \frac{k}{k+2} - 2B_{k+1}\,. \\ \frac{2}{k+1}\sum_{l=0}^{k-2}\binom{k+1}{l+1}B_{k-l} &= \frac{2}{k+1}\sum_{l=1}^{k-1}\binom{k+1}{l}B_{k+1-l} = \frac{k-1}{k+1}\,. \end{aligned}$$

The third sum in (4.34) is $c_1(k)$. Hence we have

$$c_1(k+2) - c_1(k) = 2\,[\frac{B_{k+1}}{k+1} + \frac{k^2+k-1}{(k+1)(k+2)}]\,. \tag{4.35}$$

In complete analogy one shows also

$$c_4(k+2) - c_4(k) = \frac{1}{k+2} - \frac{2\,k+1}{k+1}\,. \tag{4.36}$$

respectively

$$c_2(k+2) - c_2(k) = -2\,\frac{B_{k+1}}{k+1}\,, \tag{4.37}$$

$$c_3(k+2) - c_3(k) = \frac{1}{k+2} - \frac{1}{k+1}\,. \tag{4.38}$$

Summing up the relations (4.35) to (4.38), we obtain $\varphi(k+2) - \varphi(k) = 0$.

(ii) This statement will be proved by induction on k by using the following two properties:

Property 1: Let $k \geq 1$ be an integer. Then $\varphi(k, 2k) = 0$.

Proof. Using that $B_1 = -1/2$ and $B_n = 0$ for n an odd integer $n > 1$, we get

$$\begin{aligned} c(2k+1, 2k) &= \sum_{i=0}^{2k-1} \frac{(-1)^i}{2k+1-i}\binom{2k}{i} \\ &\quad + \sum_{i=0}^{k-1} \frac{2B_{2(k-i)}}{2k-2i}\binom{2k}{2i+1}\,. \end{aligned} \tag{4.39}$$

Applying again equality (4.33) the first sum in relation (4.39) can be simplified to

$$\sum_{i=0}^{2k-1} \frac{(-1)^i}{2k+1-i}\binom{2k}{i} = -\frac{2k}{2k+1}\,,$$

The second sum can be written as

$$\sum_{i=0}^{k-1} \frac{2B_{2(k-i)}}{2k-2i}\binom{2k}{2i+1} = \frac{2}{2k+1}\sum_{i=0}^{k-1} B_{2(k-i)}\binom{2k+1}{2i+1}.$$

The last sum however is well known [N], and is given by

$$\sum_{i=0}^{k-1} B_{2(k-i)}\binom{2k+1}{2i+1} = k - \frac{1}{2}.$$

This leads to

$$c(2k+1,2k) = -\frac{1}{2k+1}.$$

From the definition of $c(k,r)$ in (4.30) we get on the other hand

$$c(2k+1,0) = \frac{1}{2k+1}.$$

and hence $\varphi(k,2k) = c(2k+1,2k) + c(2k+1,0) = 0$.

Property 2: For $k \geq 1$ let r be an integer with $k+1 \leq r \leq 2k$. Then

$$\varphi(k,r) - \varphi(k,r-1) = \sum_{j=k}^{r-2} \varphi(k-1,j) + \frac{1}{2}\varphi(k-1,k-1).$$

Proof. Using the abbreviation

$$g(n) = \frac{2B_n + (-1)^{n+1}}{n},$$

we have

$$c(2k+1,r) = \sum_{i=0}^{r} g(2k+1-i)\binom{r}{i}.$$

Using identity (4.32) and the formula

$$\binom{n}{i} = \sum_{j=0}^{n-1}\binom{j}{i-1}, \quad i,\, n,\, k \in \mathbb{N}\,,$$

we get after some simple calculation

$$\begin{aligned}
&\varphi(k,r) - \varphi(k,r-1)\\
&= \sum_{i=0}^{r}\left[\binom{r}{i} + \binom{2k-r}{i} - \binom{r-1}{i} - \binom{2k-r+1}{i}\right] g(2k+1-i)\\
&= \sum_{j=2k-r}^{r-2}\sum_{m=0}^{j}\binom{j}{m} g(2k-1-m)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\varphi(k,r) - \varphi(k,r-1) &= \sum_{j=2k-r}^{r-2} c(2k-1,i)\\
&= \sum_{j=k}^{r-2}\big[c(2k-1,j) + c(2k-1,2k-2-j)\big] + c(2k-1,k-1)\\
&= \sum_{j=k}^{r-2}\varphi(k-1,j) + \frac{1}{2}\varphi(k-1,k-1)\,,
\end{aligned}$$

which is just Property 2.

Now we can prove statement (ii) of Lemma 9 by induction on k. It is trivially true for $k=1$. Assume that it is true for $k-1$. According to Property 1, it is true for k in the special case $r=2k$. By Property 2, we have

$$\varphi(k,r)-\varphi(k,r-1)=\sum_{i=k}^{r-2}\varphi(k-1,i)+\frac{1}{2}\varphi(k-1,k-1)=0\,, \tag{4.40}$$

where the last equality follows from the induction hypothesis. Hence, starting with $r=2k$ we see from (4.40)

$$\varphi(k,r)=\varphi(k,2k)=0, \qquad \text{for all } 0\le r\le 2k$$

and statement (ii) of Lemma 9 is proved ◻

Now we are prepared to prove Proposition 8

Proof of Proposition 8.
To determine $\bar\lambda$ for the eigenvalue $\lambda=1$ belonging to the eigenfunction $f=h_k^-(z)$ of $\mathcal{L}_\beta$ at $\beta=\beta_k$, we insert the $(\beta-\beta_k)$-expansion of λ_β and $f_\beta(z)$ in (4.16) and (4.17) into Lewis equation at $\beta=\beta_k$ and get

$$\begin{aligned}
&[\,(-1)^{k+1}+\bar\lambda(\beta-\beta_k)+O(\,(\beta-\beta_k)^2\,)]\\
&\quad\times[(f(z)-f(z+1))+(\bar f(z)-\bar f(z+1))(\beta-\beta_k)+O(\,(\beta-\beta_k)^2\,)\,)\,]\\
&=[(\frac{1}{z+1})^{2\beta_k}-2\,ln(z+1)\,(\frac{1}{z+1})^{2\beta_k}(\beta-\beta_k)+O(\,(\beta-\beta_k)^2\,)]\\
&\quad\times[f(\frac{1}{z+1})+\bar f(\frac{1}{z+1})(\beta-\beta_k)+O(\,(\beta-\beta_k)^2\,)\,]\,.
\end{aligned}$$

Comparing the coefficients of $(\beta-\beta_k)^i,\ i=0,1$ we get the equations

$$(\beta-\beta_k)^0:(-1)^{k+1}(f(z)-f(z+1))-(z+1)^{k-1}f(\frac{1}{z+1})=0 \tag{4.41}$$

$$\begin{aligned}(\beta-\beta_k)^1:&(-1)^{k+1}(\bar f(z)-\bar f(z+1))-\bar\lambda(f(z)-f(z+1))\\
&-(z+1)^{k-1}\bar f(\frac{1}{z+1})+2\ln(z+1)(z+1)^{k-1}f(\frac{1}{z+1})=0\end{aligned} \tag{4.42}$$

Denoting the left hand sides of expressions (4.41) and (4.42) by $G_1(k,z)$ respectively $G_2(k,z)$ and defining $G_3(k,z)$ as

$$G_3(k,z)=G_2(k,z-1)-G_1(k,\frac{1}{z}-1)\,z^{k-1}+2\,ln(z)\,G_1(k,z-1)\,,$$

we see that

$$\begin{aligned}G_3(k,z)\quad&=\quad[\,f(z-1)-f(z)-z^{k-1}f(\frac{1}{z}-1)+z^{k-1}f(\frac{1}{z})\,]\,\bar\lambda\\
&\quad+[\,\bar f(z-1)-z^{k-1}\bar f(\frac{1}{z}-1)+2\,ln(z)f(z-1)\,]=0\,.\end{aligned}$$

Expanding $G_3(k,z)$ in a Taylor series around $z=1$ one finds

$$\frac{G_3^{(k)}(k,z)}{2\,k!}|_{z=1}$$
$$= \frac{1}{k!}\left\{ \left[f^{(k)}(0) - f^{(k)}(1) \right] \bar{\lambda} + \bar{f}^{(k)}(0) + \sum_{i=0}^{k-1} \frac{f^{(i)}(0)}{i\,!} \frac{(-1)^{k-1-i}}{(k-i)} \right\} .$$

Obviously this expression must vanish. As

$$[f^{(k)}(0) - f^{(k)}(1)] = -f^{(k)}(1)$$

according to (4.20) and (4.21) is different from zero, we have

$$\bar{\lambda} = \left[\bar{f}^{(k)}(0) + \sum_{i=0}^{k-1} \frac{f^{(i)}(0)}{i\,!} \frac{(-1)^{k-1-i}}{(k-i)} \right] / \frac{f^{(k)}(1)}{k!} . \tag{4.43}$$

It remains to show that the numerator Λ of this expression vanishes for $k = 3, 5, 7, \cdots$, where

$$\Lambda := \left[\bar{f}^{(k)}(0) + \sum_{i=0}^{k-1} \frac{f^{(i)}(0)}{i\,!} \frac{(-1)^{k-1-i}}{(k-i)} \right] .$$

Using relations (4.22), (4.23) and (4.24), one can rewrite Λ as

$$\Lambda = \sum_{l=0}^{k-2} \frac{f^{(l)}(0)}{l\,!} [-2\zeta(l-k+1) + \frac{(-1)^l}{(k-l)}] .$$

But $f^{(l)}(0)/l\,!$ is just the coefficient a_l in statement (ii) of Lemma 8. Using relations (4.25) and (4.26), Λ can be expressed also as

$$\Lambda = \sum_{l=-1}^{k} b_l\, c(k,l), \qquad \text{with} \quad c(k,l) = \sum_{i=0}^{l} \frac{2\,B_{k-i} + (-1)^i}{k-i} \binom{l}{i} .$$

Since from (4.19) $b_l = b_{k-1-l}$ and from (4.29) $c(k,l) = -c(k,k-1-l)$, we get $\Lambda = 0$ and hence $\bar{\lambda} = 0$.

In the special case $k=1$ we have

$$f(z) = \frac{B_2}{2} \left(\frac{1}{z+1} - 2 + z \right) .$$

Inserting $f(0) = -\frac{B_2}{2}$ into formula (4.24) we get $\bar{f}^{(1)}(0) = -\frac{5B_2}{2}$ and hence from (4.18)

$$\bar{\lambda} = \frac{\bar{f}^{(1)}(0) + f(0)}{f(1)} = -8 ,$$

since $f(1) = \frac{3}{8}B_2$. □

There are further eigenfunctions of the operator $\mathcal{L}_\beta$ at $\beta = \beta_k$ with eigenvalue $\lambda = \pm 1$ which are closely related to the holomorphic cusp forms for the group $PSL(2,\mathbb{Z})$. Indeed a function $\varphi_{k+1}(z)$ holomorphic in the upper half plane with the properties:
(i)

$$\varphi_{k+1}(\gamma z) = (cz+d)^{k+1}\varphi_{k+1}(z)\,, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})\,,$$

(ii) $\varphi_{k+1}(z)$ is vanishing at the cusp of the modular surface,
is called a modular cusp form of weight $k+1$ with odd integer k. Consider then such a cusp form $\varphi_{k+1}(z)$. In the Shimura-Eichler-Manin theory of periods [Z2] there is associated to this modular form a period polynomial $r_{\varphi_{k+1}}(z)$ of degree $\leq k-1$ defined as

$$r_{\varphi_{k+1}}(z) = \int_0^{i\infty} \varphi_{k+1}(z')\,(z-z')^{k-1}\,dz'\,.$$

Denote by $r_{\varphi_{k+1}}(z) = r^+_{\varphi_{k+1}}(z) + r^-_{\varphi_{k+1}}(z)$ its decomposition into the even and odd parts, that means $r^+_{\varphi_{k+1}}(z)$ is an even polynomial and $r^-_{\varphi_{k+1}}(z)$ is an odd polynomial. From the transformation property

$$\varphi_{k+1}(\gamma z) = (cz+d)^{k+1}\varphi_{k+1}(z)\,, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$$

for the modular form $\varphi_{k+1}(z)$ of weight $k+1$ one shows very easily [Z2] that the polynomial $r_{\varphi_{k+1}}(z)$ fulfills the following two relations

$$r_{\varphi_{k+1}}(z) + z^{k-1} r_{\varphi_{k+1}}(-\frac{1}{z}) = 0\,, \tag{4.44}$$

$$r_{\varphi_{k+1}}(z) + z^{k-1} r_{\varphi_{k+1}}(1-\frac{1}{z}) + (z-1)^{k-1} r_{\varphi_{k+1}}(-\frac{1}{z-1}) = 0 \tag{4.45}$$

and conversely the Eichler-Shimura-Manin theory tells us that the space of polynomials W_{k-1} of degree $\leq k-1$ obeying relations (4.44) and (4.45) modulo the subspace spanned by $z^{k-1}-1$ is isomorphic to the direct sum of two copies of S_{k+1}, the space of holomorphic cusp forms of weight $k+1$ with the isomorphism given just by the maps

$$\varphi_{k+1} \to r^+_{\varphi_{k+1}}(z) \quad \text{respectively} \quad \varphi_{k+1} \to r^-_{\varphi_{k+1}}(z)\,.$$

Extending a result by Zagier in [Z1] one shows

Proposition 9. *If $r_{\varphi_{k+1}}(z)$ is the period polynomial of a holomorphic cusp form φ_{k+1} of weight $k+1$ then $r^-_{\varphi_{k+1}}(z+1)$ is a solution of Lewis functional equation in (4.2) at $\beta = \beta_k$ with $\lambda = 1$ and $r^+_{\varphi_{k+1}}(z+1)$ is* *a solution of this equation at $\beta = \beta_k$ with $\lambda = -1$ and vice verso, every polynomial solution of Lewis equation for $\beta = \beta_k$ with $\lambda = \pm 1$ belongs to W_{k-1}. The functions $p^{\pm}_{k+1}(z) = r^{\pm}_{\varphi_{k+1}}(z+1)$ are eigenfunctions of $\mathcal{L}_\beta$ at $\beta = \beta_k$ with eigenvalue $\lambda = \mp 1$.*

Proof. The case of the odd period polynomial was discussed in [Z1]. So we can restrict ourselves to the even case and just follow Zagiers arguments more or less word by word. Denote by E, S and U the following matrices in $GL(2, \mathbb{Z})$

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

They act on z as usually:

$$\gamma z = \frac{az+b}{cz+d} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The action γ^* of γ on the space of polynomials of degree $\leq k-1$ is defined as

$$\gamma^* \psi(z) = (cz+d)^{k-1} \psi(\gamma z).$$

It is clear that the action of E^* commutes with that of $U^* + {U^*}^2$ and S^* and hence leaves the space W_{k-1} invariant. Since E^* is symmetric it has a complete set of eigenfunctions in W_{k-1}, and because of ${E^*}^2 = id$ its eigenvalues are just $\rho = \pm 1$. Let $\psi \in W_{k-1}$ be an eigenfunction of E^* with eigenvalue -1. Then ψ must be an even polynomial: $\psi \in W_{k-1}$ satisfies $-\psi = S^*\psi = E^*\psi$ and therefore

$$\psi(z) = E^* S^* \psi(z) = E^*(z^{k-1}\psi(-\frac{1}{z})) = \psi(-z).$$

Because

$$(1+S^*)\psi = (1 + U^* + {U^*}^2)\psi = 0$$

we find $S^*\psi = (U^* + {U^*}^2)\psi$ and therefore

$$\psi = S^*(U^* + {U^*}^2)\psi = T^*\psi + E^*T^*E^*\psi = T^*\psi - E^*T^*\psi,$$

which just says

$$\psi(z) = \psi(z+1) - z^{k-1}\psi(\frac{1}{z}+1).$$

Thereby we used $T^*\psi(z) = \psi(z+1)$ and ${S^*}^2 = id$. Defining $f(z) = \psi(z+1)$ with $\psi \in W_{k-1}$ we find finally for f:

$$f(z) = f(z+1) - (z+1)^{k-1} f(\frac{1}{z+1}). \tag{4.46}$$

On the other hand assume f is a polynomial solution of degree $\leq k-1$ of Lewis equation (4.4) with $\lambda = -1$. Define $\psi(z) = f(z-1)$ which fulfills

$$\psi(z) = \psi(z+1) - z^{k-1}\psi(1+\frac{1}{z}) = (T^* - E^*T^*)\psi(z)\,.$$

Therefore

$$E^*\psi(z) = (E^*T^* - T^*)\psi(z) = -\psi(z)$$

and hence

$$\psi(z) = (T^* + E^*T^*E^*)\psi(z) = (S^*U^* + S^*{U^*}^2)\psi(z)\,.$$

From this it follows

$$(1+S^*)\psi(z) = (1+U^*+{U^*}^2)\psi\,. \tag{4.47}$$

Obviously

$$S^*(1+S^*)\psi = (1+S^*)\psi$$

and

$$U^*(1+U^*+{U^*}^2)\psi = (1+U^*+{U^*}^2)\psi\,,$$

which shows that the function $(1+S^*)\psi(z)$ is invariant under all transformations γ^* with $\gamma \in PSL(2,\mathbb{Z})$ and hence must be identical zero. This shows that $\psi \in W_{k-1}$ and hence is modulo the function $z^{k-1}-1$ the odd part of a period polynomial. ◻

We still have to show that the polynomials $p_k^\pm(z) = r^\pm_{\varphi_{k+1}}(z+1)$ are eigenfunctions of $\mathcal{L}_\beta$ at $\beta = \beta_k$ with eigenvalue $\lambda = \mp 1$. But this follows immediately from Lewis' equation

$$\lambda_\pm\, p_k^\pm(z) = \lambda_\pm\, p_k^\pm(z+1) + (z+1)^{k-1} p_k^\pm(\frac{1}{z+1})\,.$$

Inserting the Taylor expansion $p_k^\pm(\frac{1}{z+1}) = \sum_{n=0}^{k-1} c_n (\frac{1}{z+1})^n$ we get by using the functional equation for the Hurwitz zeta function $\zeta(s,z) - \zeta(s,z+1) = (z+1)^{-s}$:

$$\lambda_\pm\, p_k^\pm(z) = \lambda_\pm\, p_k^\pm(z+1) + \sum_{n=0}^{k-1} \frac{p_k^{\pm(n)}(0)}{n\,!}\Big[\zeta(1-k+n,z+1) - \zeta(1-k+n,z+2)\Big]$$

and therefore

$$\begin{aligned}\lambda_\pm\, p_k^\pm(z) &- \sum_{n=0}^{k-1} \frac{p_k^{\pm(n)}(0)}{n\,!}\zeta(2\beta_k+n,z+1)\\ &= \lambda_\pm\, p_k^\pm(z+1) - \sum_{n=0}^{k-1} \frac{p_k^{\pm(n)}(0)}{n\,!}\zeta(2\beta_k+n,z+2)\,.\end{aligned}$$

Hence the polynomial

$$\lambda_\pm \, p_k^\pm(z) - \sum_{n=0}^{k-1} \frac{p_k^{\pm(n)}(0)}{n\,!} \zeta(2\beta_k + n, z+1)$$

is periodic with period 1 in z. But the only bounded polynomial is the constant one and therefore

$$\lambda_\pm \, p_k^\pm(z) - \sum_{n=0}^{k-1} \frac{p_k^{\pm(n)}(0)}{n\,!} \zeta(2\beta_k + n, z+1) = c$$

and $p_k^\pm$ is a regular eigenfunction of $\mathcal{L}_\beta$ at $\beta = \beta_k$ with eigenvalue $\lambda = \lambda_\pm$ by Proposition 7.

Combining then all the above results about the regular and singular eigenvalues of the operator $\mathcal{L}_\beta$ for $\beta = \beta_k$ we get

Theorem 2. *The Selberg zeta function $Z_S(s)$ for $PSL(2,\mathbb{Z})$ has 'trivial' zeros of order $2\,dim\,S_{k+1} + 1$ at the points $s = \beta_k$ for $k = 3, 5, 7, \cdots$. It has simple poles at the points $s = \beta_k$ for $k = 0, 1, 2, 4, 6, \cdots$.*

Remark We are at the moment not able to prove by the transfer operator approach that the above zeros and poles are indeed the only ones in the half plane $\Re s \leq 0$, which indeed is true as one knows from the trace formula approach, since we are at the moment unable to prove that $\mathcal{L}_\beta$ has no other eigenvalues $\lambda = \pm 1$ in $\Re\beta \leq 0$ besides the ones at the β values $\beta = \beta_k$. One can show only that in the half plane $\Re\beta \geq 1$ there are no eigenvalues $\lambda = \pm 1$ for $\mathcal{L}_\beta$ besides $\lambda = 1$ for $\beta = 1$.

5. The nontrivial zeros of Selberg's zeta function. The nontrivial zeros of $Z_S(\beta)$ are the so called spectral zeros related to the eigenvalues of the Laplace-Beltrami operator and the zeros related to the nontrivial zeros of the Riemann zeta function.

5.1. The spectral zeros. We use some recent results of J. Lewis in [L1] [LZ] related to the Maass cusp forms for $PSL(2,\mathbb{Z})$. His main result is

Theorem 3. *If φ_s denotes a Maass cusp form for the group $PSL(2,\mathbb{Z})$ to the eigenvalue $\lambda = s(1-s)$ then there exists a function $f_s(z)$ holomorphic in the complex z-plane cut along $(-\infty, -1]$ with $f_s(0) = 0$ and vanishing at infinity, such that*

$$\pm\, f_s(z) = \pm f_s(z+1) + \left(\frac{1}{z+1}\right)^{2s} f_s\left(\frac{1}{z+1}\right), \tag{5.1}$$

where $\pm$ refers to the parity of the Maass cusp form under the reflection $z \to -\bar{z}$ of the Poincaré upper half plane. On the other hand, to every such function $f_s(z)$ obeying this functional equation for $\Re s > 0$ there exists a Maass cusp form φ_s such that $-\triangle_{LB}\,\varphi_s = s(1-s)\,\varphi_s$.

Remark For even Maass cusp forms φ_s one has indeed [L1]

$$(5.2) \qquad f_s(z) = (z+1) \int_0^\infty \varphi_s(i\,y)\,(\,(z+1)^2 + y^2\,)^{-s-1} y^s\,dy\,,$$

respectively for odd φ_s one has [L2]

$$(5.3) \qquad f_s(z) = \int_0^\infty \frac{\partial}{\partial x}\varphi_s(i\,y)\,(\,(z+1)^2 + y^2\,)^{-s}\,y^s\,dy\,.$$

For the constant eigenfunction $\varphi = c$ belonging to the eigenvalue $\lambda = 0$ of $-\triangle_{LB}$ one finds applying the above transformation (5.2)

$$f(z) = \frac{1}{z+1}\,.$$

The function is holomorphic away from the point $z = -1$, vanishes at infinity and fulfills the functional equation

$$(5.4) \qquad f(z) = f(z+1) + (\frac{1}{z+1})^{2s} f(\frac{1}{z+1})\,,$$

with $s = 1$. This shows that also this eigenfunction of $-\triangle_{LB}$ for $PSL(2,\mathbb{Z})$ falls under Lewis' Theorem when the vanishing condition of f at $z = 0$ is suppressed.

Combining the result of Lewis and Proposition 7 on the relation between eigenfunctions of the operator $\mathcal{L}_\beta$ and functions fulfilling Lewis functional equation we get

Theorem 4. *Any eigenfunction $f \in B(D)$ of the operator $\mathcal{L}_\beta$ with eigenvalue $\lambda = +1$ or $\lambda = -1$ for $\Re\beta > 0$ vanishing at infinity determines an even respectively odd eigenfunction φ_β of $-\triangle_{LB}$ with eigenvalue $\rho = \beta(1-\beta)$ and vice versa.*

Proof. Any eigenfunction f_β of $\mathcal{L}_\beta$ for $0 < \Re\beta \le \frac{1}{2}$ vanishing at infinity must vanish also at the point $z = 0$. This follows immediately from the eigenfunction equation

$$\lambda_\beta\, f_\beta(z) = \mathcal{L}_\beta^{(0)}\, f_\beta(z) + f_\beta(0)\,\zeta(2\beta, z+1) \overset{z\to\infty}{\longrightarrow} 0\,.$$

But $\lim_{z\to\infty} \mathcal{L}_\beta^{(0)} f_\beta(z) = 0$ and hence $f_\beta(0) \lim_{z\to\infty} \zeta(2\beta, z+1) = 0$ if $f_\beta(z)$ vanishes at infinity. But for $0 < \Re\beta \le \frac{1}{2}$ the limit of $\zeta(2\beta, z+1)$ for $z \to \infty$ does not vanish and hence $f_\beta(0)$ must vanish. For $\Re\beta > \frac{1}{2}$ on the other hand any eigenfunction of $\mathcal{L}_\beta$ vanishes at infinity independently of $f_\beta(0)$ vanishing or not. This shows that Lewis' Theorem is fulfilled and hence f_β is the Lewis' transform of some Maass cusp form to the eigenvalue $\rho = \beta(1-\beta)$. On the other hand every Maass cusp form determines by

Lewis' Theorem a solution f_β of Lewis functional equation with $\rho = \beta(1-\beta)$ such that $f_\beta(\infty) = f_\beta(0) = 0$. But then trivially

$$\lim_{N\to\infty}\Big[f_\beta(z+N) - f_\beta(0)\zeta(2\beta, z+1+N)\Big] = 0$$

and f_β is an eigenfunction of $\mathcal{L}_\beta$ with eigenvalue $\lambda = \pm 1$. The eigenfunction $\varphi_0 = c$ of $-\triangle_{LB}$ with eigenvalue $\rho = 0$ determines via the Lewis transform the function $f_1(z) = \frac{1}{z+1}$ as we have mentioned already. It fulfills trivially Lewis equation for $\beta = 1$ and $\lambda = 1$. Since it vanishes at infinity it is also an eigenfunction of $\mathcal{L}_\beta$ for $\beta = 1$ with eigenvalue $\lambda = 1$. Indeed, since $\mathcal{L}_1$ is the Perron- Frobenius operator for the Gauss map, f_1 is just the famous density of the invariant Gauss measure for the continued fraction map. □

From Theorem 4 we conclude

Theorem 5. *The Selberg zeta function has in $\Re\beta \geq \frac{1}{2}$ zeros at the points β such that $\beta(1-\beta)$ is an eigenvalue of $-\triangle_{BL}$ for $PSL(2,\mathbb{Z})$.*

Remark The eigenfunctions $f_\beta(z)$ of $\mathcal{L}_\beta$ with eigenvalue $\lambda = 1$ or $\lambda = -1$ are called 'period functions' [LZ], generalizing the period polynomials of the holomorphic modular cusp forms of even weight respectively period functions of the holomorphic non cusp forms (Eisenstein series) of even weight for the group $PSL(2,\mathbb{Z})$. The above period functions are related via the Lewis transforms (5.2) and (5.3) to the nonholomorphic automorphic Maass cusp forms.

5.2. The Riemann zeros. Next we want to discuss the nontrivial zeros of the Selberg zeta function related to the nontrivial zeros of Riemanns zeta function. To this end consider once more the function

$$h_\beta(z) = \sum_{n,m\geq 1}\Big(\frac{1}{m(z+1)+n}\Big)^{2\beta} + \frac{1}{2}\zeta(2\beta) + \frac{1}{2}\zeta(2\beta)\Big(\frac{1}{z+1}\Big)^{2\beta}.$$

If $\Re\beta > 1$ we obviously find $h_\beta(\infty) = \frac{1}{2}\,\zeta(2\beta)$, on the other hand the function fulfills Lewis equation with $\lambda = 1$

$$h_\beta(z) = h_\beta(z+1) + \Big(\frac{1}{z+1}\Big)^{2\beta}\, h_\beta\Big(\frac{1}{z+1}\Big)$$

and hence

$$h_\beta(z) = h_\beta(z+N) + \sum_{r=1}^{N}\Big(\frac{1}{z+r}\Big)^{2\beta}\, h_\beta\Big(\frac{1}{z+r}\Big).$$

For $\Re\beta > 1$ we can perform the limit $N \to \infty$ and get

$$h_\beta(z) = \frac{1}{2}\zeta(2\beta) + \mathcal{L}_\beta h_\beta(z)\,. \tag{5.5}$$

Since the left hand side has a meromorphic extension to the entire β-plane with $h_\beta \in B(D)$ also the right hand side has this property and equation (5.5) holds true in the entire β-plane. This shows immediately that the function h_β is an eigenfunction of $\mathcal{L}_\beta$ for those β-values for which $\zeta(2\beta) = 0$. For the trivial zeros $2\beta = -2, -4, \cdots$ the corresponding functions have been discussed already in Proposition 6. The analytic extension of h_β for β-values which correspond to the nontrivial zeros of Riemanns zeta function and hence fulfill $0 < \Re\beta < \frac{1}{2}$ is the following function:

$$\begin{aligned} h_\beta(z) &= \frac{1}{\Gamma(2\beta)} \Big[\frac{c_{-2}(z)}{2\beta - 2} + \frac{c_{-1}(z)}{2\beta - 1} \Big] \\ &\quad + \frac{1}{\Gamma(2\beta)} \int_0^1 \Big[F_z(t) - c_{-2}(z)\, t^{-2} - c_{-1}(z)\, t^{-1} \Big]\, t^{2\beta-1}\, dt \\ &\quad + \frac{1}{\Gamma(2\beta)} \int_1^\infty F_z(t)\, t^{2\beta-1}\, dt\,, \end{aligned}$$

with

$$F_z(t) = \frac{1}{e^t - 1}\,\frac{1}{e^{t(z+1)} - 1}\,,\ c_{-2}(z) = \frac{1}{z+1} \text{ and } c_{-1}(z) = B_1\,(1 + \frac{1}{z+1})\,.$$

Since these functions are eigenfunctions of $\mathcal{L}_\beta$ for these β-values with $\zeta(2\beta) = 0$ with eigenvalue $\lambda = 1$ according to Proposition 7 they must fulfill the relation

$$\lim_{N\to\infty} \Big[h_\beta(z+N) - h_\beta(0)\,\zeta(2\beta, z+1+N) \Big] = 0\,, \tag{5.6}$$

since $\Re\beta > 0$ for 2β a nontrivial zero of Riemann. From the definition of h_β it follows immediately that $h_\beta(0) = \zeta(2\beta - 1)$ for all $\beta \in \mathbb{C}$. To get the asymptotic behaviour of $h_\beta(z)$ respectively $\zeta(2\beta, z)$ for $z \to \infty$ we use the Mellin-transformation method once more:

$$\zeta(2\beta, z) = \frac{1}{\Gamma(2\beta)} \int_0^\infty \frac{e^{\,t(1-z)}}{e^{\,t} - 1}\, t^{2\beta-1}\, dt \quad \text{for } \Re\beta > 1\,.$$

But [MOS]

$$\frac{e^{\,t}}{e^{\,t} - 1} = \sum_{r=0}^{\infty} \frac{B_k(1)}{k\,!}\, t^{\,k-1}\,,$$

where $B_k(1) = B_k$ for $k \neq 1$ and $B_1(1) = \frac{1}{2} = -B_1$. Hence

$$\begin{aligned} \zeta(2\beta, z) &= \frac{1}{\Gamma(2\beta)} \Big[\frac{\Gamma(2\beta - 1)}{z^{\,2\beta-1}} + \frac{1}{2}\,\frac{\Gamma(2\beta)}{z^{\,2\beta}} \Big] \\ &\quad + \frac{1}{\Gamma(2\beta)} \int_0^\infty (\frac{e^{\,t}}{e^{\,t} - 1} - \frac{1}{t} - \frac{1}{2})\, e^{-tz}\, t^{2\beta-1}\, dt\,. \end{aligned}$$

This equality holds for all $\Re\beta > -\frac{1}{2}$. But this shows that for large N

$$\begin{aligned}\zeta(2\beta, z+1+N) \quad \sim \quad & \frac{1}{2\beta-1}(z+1+N)^{1-2\beta} + \frac{1}{2}\left[\frac{1}{z+1+N}\right]^{2\beta} \\ & + O((z+1+N)^{-(2\beta+1)})\,.\end{aligned}$$

The function $h_\beta(z)$ on the other can be written as

$$h_\beta(z) = \frac{1}{\Gamma(2\beta)} \int_0^\infty F_z(t)\, t^{2\beta-1}\, dt\,, \text{ for } \Re\beta > 1 \tag{5.7}$$

with

$$F_z(t) = \frac{1}{e^t-1}\, \frac{e^{t(z+1)}}{e^{t(z+1)}-1} e^{-t(z+1)}\,.$$

But

$$\frac{e^{t(z+1)}}{e^{t(z+1)}-1} = \sum_{r=0}^{\infty} \frac{B_r(1)}{r\,!}(\,t(z+1)\,)^{r-1}$$

and therefore

$$\frac{1}{e^t-1}\, \frac{e^{t(z+1)}}{e^{t(z+1)}-1} = \sum_{r=0}^{\infty} t^{r-2} \sum_{l=0}^{r} \frac{B_{r-l}}{(r-l)\,!}\frac{B_l(1)}{l\,!}\,(z+1)^{l-1}\,.$$

Inserting this into (5.7) we get

$$\begin{aligned} h_\beta(z) \;=\; & \frac{1}{\Gamma(2\beta)} \int_0^\infty \Big[\, \frac{1}{e^t-1}\, \frac{e^{t(z+1)}}{e^{t(z+1)}-1} - \sum_{r=0}^{m} t^{\,r-2} c_r(z) \Big]\, e^{-t(z+1)}\, t^{2\beta-1}\, dt \\ & + \frac{1}{\Gamma(2\beta)} \sum_{r=0}^{m} c_r(z) \frac{\Gamma(2\beta+r-2)}{(z+1)^{2\beta+r-2}} \end{aligned} \tag{5.8}$$

with

$$c_r(z) = \sum_{l=0}^{r} \frac{B_{r-l}\, B_l(1)}{(r-l)\,!\, l\,!}(z+1)^{l-1}\,.$$

The representation (5.8) obviously makes sense for $\Re\beta > \frac{1-m}{2}$. For z large $c_r(z) \sim (z+1)^{r-1}$ and hence

$$\sum_{r=0}^{m} c_r(z) \frac{\Gamma(2\beta+r-2)}{(z+1)^{2\beta+r-2}} \sim \sum_{r=0}^{m} \frac{\Gamma(2\beta+r-2)}{(z+1)^{2\beta-1}}\,.$$

This shows that the function $h_\beta(z+N)$ behaves for $N \to \infty$ as

$$\begin{aligned} h_\beta(z+N) \;\sim\; & \sum_{r=0}^{\infty} \frac{B_r(1)}{r\,!}\, \frac{\Gamma(2\beta+r-2)}{\Gamma(2\beta)}(z+1+N)^{1-2\beta} \\ & + o(\,(z+1+N)^{-2\beta}\,)\,. \end{aligned} \tag{5.9}$$

Relation (5.6) hence leads to the following identity for the nontrivial zeros of Riemanns zeta function

$$\frac{\zeta(2\beta-1)}{2\beta-1} - \sum_{r=0}^{\infty} \frac{B_r(1)}{r\,!} \frac{\Gamma(2\beta+r-2)}{\Gamma(2\beta)} = 0$$

respectively

$$\zeta(2\beta-1) - \sum_{r=0}^{\infty} \frac{B_r(1)}{r\,!} \frac{\Gamma(2\beta+r-2)}{\Gamma(2\beta-1)} = 0\,,$$

a relation well known in the literature [MOS]. We have seen that the functions $h_\beta(z)$ for the trivial zeros $2\beta = -2m, m \in I\!N$ of Riemann's zeta function are just the odd parts of the period functions (argument shifted by 1) of the holomorphic Eisenstein series of integer weight. Obviously one expects the functions $h_\beta(z)$ with β a nontrivial zero of Riemann, to be related to the nonholomorphic Eisenstein series for these β-values. Presumably Lewis transformation in (5.2) when regularized in an appropriate way will give the explicit connection between the two functions. [2]

Summary : We have shown how the transfer operator method can be used in case of the modular group to derive the analytic properties of Selbergs zeta function for this group. The real advantage of this method when compared to the standard approach via the trace formula is that also the eigenfunctions of the transfer operator are closely related to the modular forms, especially the Maass cusp forms for this group. This should be of some interest also for the problem of quantum ergodicity where the spatial structure of these Maass forms are investigated. One can expect that the same transfer operator method applies also for congruence subgroups of $PSL(2,\mathbb{Z})$, partial results have indeed been obtained recently.

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