The Transfer Operator
from Statistical Mechanics to Quantum Chaos

Cheng-Hung Chang[1]

Institut für Theoretische Physik, Technische Universität Clausthal

Arnold-Sommerfeld-Str. 6, 38678 Clausthal-Zellerfeld, Germany
e-mail: ch.chang@tu-clausthal.de

1. Einstein's problem

On May 11, 1917, when quantum mechanics was not yet fully developed and chaotic phenomena were hardly known in the physical community, A. Einstein addressed the question in his talk in Berlin [Ein17] how to understand the quantum behavior of classically non-integrable systems, i.e., where the number of conserved quantities, like energy, is smaller than the degrees of freedom of the systems. In a certain sense this question opened the history of quantum chaos.

The problem of Einstein was open for a long time until around 1970 M. Gutzwiller [Gut70], [Gut71] made the first breakthrough. By means of his semiclassical trace formula Gutzwiller related the energies of certain quantum systems to the periodic orbits of the corresponding classical dynamical systems and gave the first answer to the question of Einstein.

In the 90's it was recognized that another classical object, the transfer operator, is also able to determine the quantum energies of chaotic systems. More strikingly, this operator gives more information about the quantum world than the trace formula, namely it can determine the quantum states which are suppressed in the traditional approach by means of the trace formula.

The transfer operator is not new in the history of physics. It has appeared in different disciplines in physics and mathematics before its application in quantum chaos was found. It was originally invented to solve the partition function of the Ising model [Isi25]. Later in the 70's it was used to investigate the ergodic theory of chaotic dynamics [LM85]. And finally in the 90's it enters into the study of quantum chaos [May91], [BC93], [CM99].

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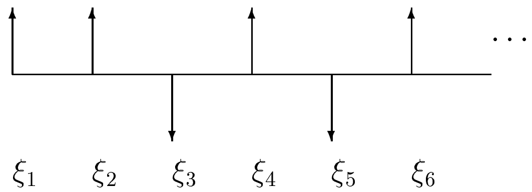
2. The transfer operator in statistical mechanics

In order to understand the phase transition of ferromagnetic materials around the Curie temperature E. Ising constructed a lattice spin model in his dissertation in 1924 (also in [Isi25]). In this model the ferromagnetic material is simply simulated by an one-dimensional lattice $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \cdots)$, where every site ξ_i in this lattice ξ can take the value $+1$ or -1, standing for the spin variable up respectively down. For example, the lattice spin configuration



will be denoted by $(+1, +1, -1, +1, -1, +1, \cdots) := \xi$. The configuration set S consists of all possible configurations on this lattice.

The shift operator $\tau : S \to S$ is defined as $(\tau\xi)_i = \xi_{i+1}$, $i \geq 1$. That means this operator replaces the value ξ_i on the i-th site by the value ξ_{i+1} on the $(i+1)$-th site and its action is just to shift the whole lattice one step to the left. This lattice spin system together with the shift operator τ is called a subshift of finite type [LM95]. If a configuration ξ is periodic, then there must exist some natural number m such that

$$\tau^m \xi = \xi. \tag{1}$$

Now we put this lattice in an external magnetic field B and suppose every spin has only interactions with its two nearest neighbors (ξ_1 has only one exceptionally). Due to the spin-magnetic field interaction and spin-spin interaction the total energy H of this model is equal to

$$H = \sum_{i=1}^{\infty} J\xi_i\xi_{i+1} + \sum_{i=1}^{\infty} \xi_i B,$$

with a real parameter J.

As known from statistical mechanics, the main physical thermodynamical quantities are encoded in the partition function of the system considered. The free energy for example can be obtained in the thermodynamical limit

$$f = \lim_{m\to\infty} \frac{1}{m} \log Z_m(A, \beta),$$

where $Z_m(A,\beta)$ stands for the partition function for all periodic configurations ξ with period m:

$$Z_m(A,\beta) = \sum_{\xi\in fix\tau^m} \exp\,\big(-\beta\sum_{k=0}^{m-1} A(\tau^k\xi)\big), \tag{2}$$

with β the reciprocal temperature and A the interaction function

$$A(\xi) = J\,\xi_0\xi_1 + B\,\xi_0. \tag{3}$$

Many techniques were devised to calculate $Z_m(A,\beta)$. An algebraical one giving an exact solution was invented by Ising when expressing (2) as

$$Z_m(A,\beta) = \sum_{\xi_1,\xi_2,\cdots,\xi_m\in\{\pm1\}} e(\xi_1,\xi_2)e(\xi_2,\xi_3)\cdots e(\xi_m,\xi_1)$$

with $e(\xi_i,\xi_j) = \exp(-\beta\,J\,\xi_i\,\xi_j - \beta\,B\,\xi_i)$, recognizing that this sum has the structure of the trace of some matrix L_β^m, i.e., $Z_m(A,\beta) = \operatorname{trace} L_\beta^m$. Indeed, for the system we considered, i.e., a one-dimensional lattice with two spin variables with nearest neighbor interaction, this matrix, called transfer matrix, is the two-dimensional matrix

$$L_\beta = \begin{pmatrix} e(+1,+1) & e(+1,-1) \\ e(-1,+1) & e(-1,-1) \end{pmatrix}.$$

However, for general systems with long range interactions, i.e., the interactions between the spins are not confined to the nearest neighbors, or the number of the spin variables is infinite the dimension of the corresponding transfer matrix can no longer be finite. For these cases D. Ruelle [Rue68] and H. Araki [Ara69] introduced the transfer operator $\mathcal{L}_\beta$ as a generalization of the transfer matrix L_β,

$$\mathcal{L}_\beta f(\xi) = \sum_{\eta\in\tau^{-1}\xi} \exp(-\beta\,A(\eta))f(\eta), \tag{4}$$

where the sum runs over the preimages $\tau^{-1}\xi := \{\eta\,|\tau\eta = \xi\}$ of ξ. With this generalization one gets again

$$Z_m(A,\beta) = \operatorname{trace}\mathcal{L}_\beta^m\,. \tag{5}$$

3. The transfer operator in ergodic theory

The transfer operator (4) in statistical mechanics is closely related to the Perron-Frobenius operator (PF operator) in ergodic theory as follows:

3.1 Discrete Time dynamical systems

Consider first a one-dimensional map $T : I \to I$ on the unit interval $I := [0, 1]$. Suppose T is piecewise analytical and fulfill the expanding property $|(T^n)'(x)| > 1$ for some $n \in \mathbb{N}$ and for all $x \in I$. The orbits, which consist of the iterated points $T^n\, x$, $n = 0,\ 1,\ 2,\ \cdots$, of such dynamics, are very sensitive to the initial points. That is, a slight deviation at the initial point can induce a totally different orbit after a few iterations. Such kind of maps are called chaotic.

To follow the correct orbit for such systems over long time is almost impossible. Giving up this ambition we can instead consider statistically a huge number of initial points separated in the interval I. If the number of these points is large enough and if they are distributed evenly enough, they can be approximately described by a smooth density function. While the map T describes the iteration of an individual point, the corresponding PF operator $\mathcal{L}_T$ describes the evolution of the density function. For the map T we considered above, the PF operator is defined as

$$\mathcal{L}_T f(x) = \sum_{y \in T^{-1}x} \frac{f(y)}{|T'(y)|}, \tag{6}$$

where the sum extends over the preimages $T^{-1}x := \{y \in I\,|Ty = x\}$. The leading eigenvalue of this operator is 1. The corresponding eigenfunction is the invariant density.

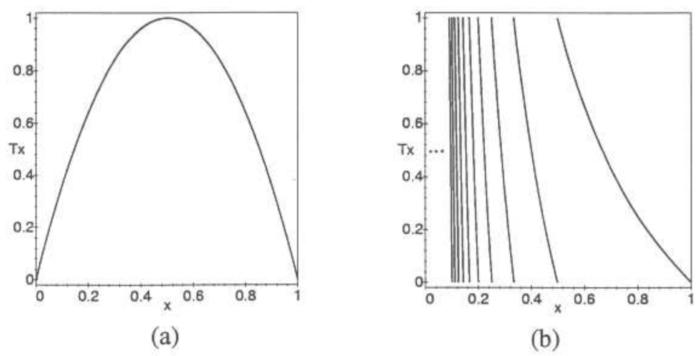


Figure 1: (a) quadratic map and (b) Gauss map

The transfer operator can however be defined not only for the above expanding maps, but also for the chaotic maps. An example is the quadratic map $Tx = 4x(1-x)$ in fig. 1 (a). This map doesn't fulfill the expanding property $|(T^n)'(x)| > 1$ for all x, but is chaotic due to its sensitivity to the initial points. Two slightly different initial points can depart from each other into totally different orbits after a few iterations [LM85]. According to the definition (6), the PF operator for this map reads

$$\mathcal{L}_T f(x) = \frac{1}{4\sqrt{1-x}} \left[f\Big(\frac{1-\sqrt{1-x}}{2}\Big) + f\Big(\frac{1+\sqrt{1-x}}{2}\Big) \right]. \tag{7}$$

If an initial density function f(x) is given, say f(x)=1, this function will be transformed by the PF operator $\mathcal{L}_T$ step by step as in fig. 2 (a), (b) and (c) and finally approaches the invariant density (the dashed line in the figures)

$$\lim_{n\to\infty} \mathcal{L}_T^n f(x) = f^*(x) = \frac{1}{\pi\sqrt{x(1-x)}},$$

which is the eigenfunction of the PF operator (7) with eigenvalue 1.

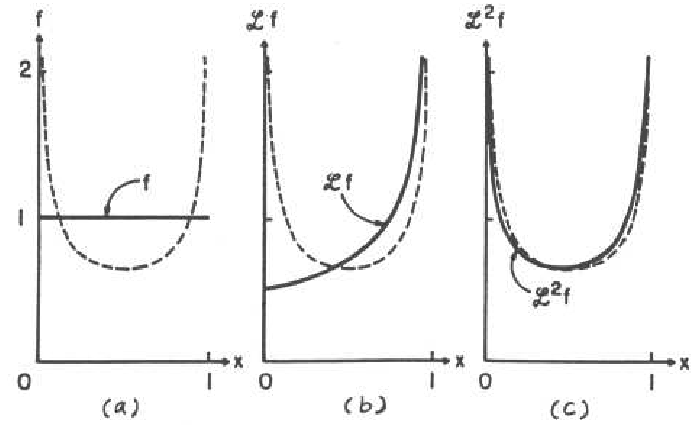


Figure 2: The evolution of the density $f(x) = 1$ for the quadratic map

The second interesting map is the Gauss map

$$T_G\, x = \begin{cases} \frac{1}{x} \bmod 1 & x \neq 0\,, \\ 0 & x = 0\,, \end{cases} \tag{8}$$

in fig. 1 (b). This map is chaotic and furthermore fulfills the expanding property $|(T^n)'(x)| > 1$. The corresponding PF operator reads

$$\mathcal{L}_{T_G} f(x) = \sum_{n=1}^{\infty} (\frac{1}{x+n})^2 f(\frac{1}{x+n}) \tag{9}$$

and the corresponding invariant density is the Gauss measure

$$f^*(x) = \frac{1}{\log 2} \frac{1}{x+1}.$$

The operator (9) will come up again in the discussion of quantum chaos later.

An interesting connection can be found between the one-dimensional discrete time dynamics and the one-dimensional Ising model. To illustrate it let us divide the interval I in fig. 1 (a) into two subintervals $I = A \cup B$ with $A := [0, \frac{1}{2})$ and $B := [\frac{1}{2}, 1]$. The orbit of T can be characterized by a series, defined by the subinterval A or B into which the point $T^n x$, $n = 0, 1, 2, \cdots$ falls. For example the orbit $\gamma := (A, A, B, A, B, A, \cdots)$ tells us that $x \in A$, $Tx \in A$, $T^2 x \in B$, etc. This structure is similar to the one-dimensional lattice spin $\xi = (+1, +1, -1, +1, -1, +1, \cdots)$ of paragraph 1.

The similarity between the structures of the lattice spin system and the discrete time dynamical system led to the development of the so-called thermodynamic formalism [Rue78] in ergodic theory by Y. Sinai, R. Bowen and D. Ruelle in the seventies which borrows ingredients and techniques from statistical mechanics to study chaos. The transfer operator is one of the important ingredients in this study, among other quantities like entropies, partition functions and free energies, etc.. Indeed, after furnishing the PF operator (6) with a parameter β,

$$\mathcal{L}_T f(x) = \sum_{y \in T^{-1}x} \frac{f(y)}{|T'(y)|^\beta} = \sum_{y \in T^{-1}x} \exp(-\beta \log |T'(y)|) f(y) \tag{10}$$

we find that this generalized PF operator (10) has the same structure as the transfer operator (4) in statistical mechanics.

3.2 Continuous time dynamical systems

So far we just discussed the chaotic dynamical systems for discrete maps. However, physicists are also interested in continuous time dynamical systems like Hamiltonian dynamics. For simplicity we consider a free particle on a two-dimensional compact surface M, i.e., the particle cannot escape to infinity. In the Hamiltonian formalism this system will be displayed in a four-dimensional phase space TM and can be reduced to a three-dimensional space $T_1 M$ since energy is conserved. We suppose the flow on $T_1 M$ is chaotic, in the sense that initial deviations will expand exponentially fast and arrive the diameter of T_1M in short time (positive Lyapunov exponent).

A well known method to study such chaotic systems is the investigation of the Poincaré section [LL92]. This section Σ is a hypersurface in T_1M, i.e., the dimension of Σ is one lower than that of T_1M, and must be chosen such that Σ cuts all trajectories in T_1M transversally. The Poincaré map $P : \Sigma \to \Sigma$ describes the next crossing point $P(x)$ of the trajectory through Σ (with the same orientation as the initial crossing point x) if the initial point $x \in \Sigma$ is given. The study of a continuous time dynamical system in T_1M is now reduced to the study of a discrete time dynamical system on Σ.

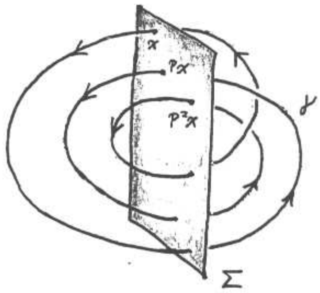


Figure 3: Poincaré map P and a periodic orbit with period $n = 3$

Suppose γ is a primitive periodic orbit in T_1M, i.e., it is not a repetition of another periodic orbit. Then there exists a natural number n with

$$P^n x = x \tag{11}$$

for the point $x \in \Sigma \cap \gamma$ on Σ. For example, γ in fig. 3 is a periodic orbit with $n = 3$. The recurrence time function $r(x)$ is defined as the length of the time interval the orbit needs between the two cutting points x and $P(x)$. The total period $l(\gamma)$ of the primitive periodic orbit γ is therefore

$$l(\gamma) = \sum_{i=0}^{n-1} r(P^i x). \tag{12}$$

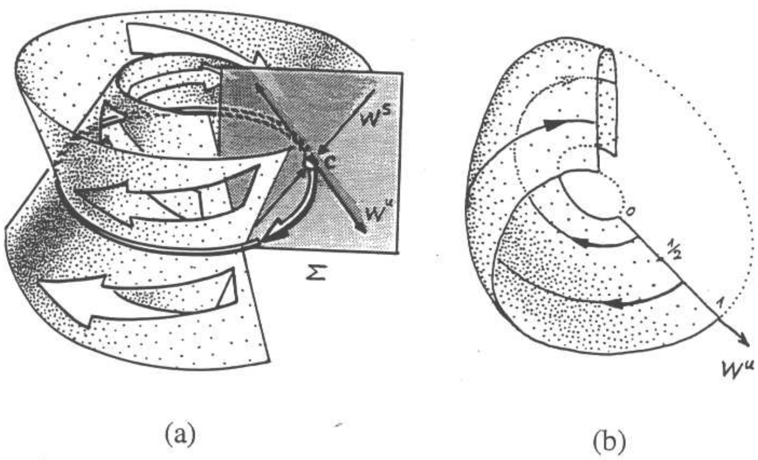


Figure 4: (a) A chaotic flow penetrates the Poincaré section Σ and (b) the expanding manifold W^u of the flow in (a)

A typical case of a chaotic flow is shown in fig. 4 (a). This flow consists of a stable manifold W^s and an unstable manifold W^u. These two manifolds cross each other at the curve C, where $C \cap \Sigma$ is the fixed point of the map P. Under the flow a point on Σ will be attracted to W^u and repelled from W^s. The manifold W^u has an expanding property and behaves like in fig. 4 (b), for example. Different initial points on $W^u \cap \Sigma$ will arrive at different points after they turn around along the flow and meet $W^u \cap \Sigma$ again. It induces the discrete map $P_u : W^u \cap \Sigma \to W^u \cap \Sigma$. In our example in fig. 4 (b), P_u is just the tent map $T_T : I \to I$ with

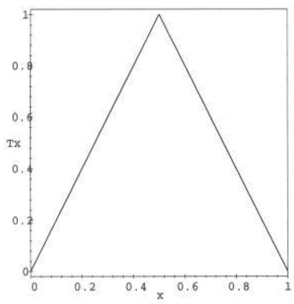


Figure 5: tent map

$$T_T x = \begin{cases} 2x & \text{for} x \leq \frac{1}{2}\,, \\ 2(1-x) & \text{for} x > \frac{1}{2}\,. \end{cases}$$

as shown in fig. 5. The PF operator for P_u is

$$\mathcal{L}_{P_u} f(x) = \sum_{y \in T^{-1}x} \frac{f(y)}{|P_u'(y)|}. \tag{13}$$

It determines the invariant density of the Poincaré map P on the expanding respectively the unstable manifold W^u.

Conversely, the manifold W^s has an contracting property and the corresponding discrete map is $P_s : W^s \cap \Sigma \to W^s \cap \Sigma$. The long time behavior of this map is attracting all points on $W^s \cap \Sigma$ to the fixed point $C \cap \Sigma$. It is not necessary to construct the PF operator, since we know the invariant density of the map P_s is the Dirac δ function concentrated on $W^s \cap \Sigma$.

3.3 The dynamical zeta function

The dynamical zeta function can be defined for both discrete and continuous time dynamics [Rue94]. Here we mention merely the continuous time case. The periodic orbits characterize important properties of dynamical systems. Now suppose we know the lengths, respectively the periods $l(\gamma)$, of all the periodic orbits γ of a dynamical system. One can ask if it is possible to use a generating function that carries all this information. Indeed, there are many candidates. An elegant one is the so-called dynamical zeta function introduced by D. Ruelle [Rue76] and S. Smale [Sma67]

$$\zeta_{RS}(\beta) = \prod_{\gamma} (1 - e^{-\beta\, l(\gamma)})^{-1}\,. \tag{14}$$

Due to (12) the zeta function (14) is equal to

$$\zeta_{RS}(\beta) = \prod_{\gamma} \left(1 - e^{-\beta \sum_{i=1}^{n(\gamma)-1} r(P^i x_\gamma)}\right)^{-1}$$

and can be rewritten as

$$\zeta_{RS}(\beta) = \exp \sum_{m=1}^{\infty} \frac{\mathcal{Z}_m(r,\beta)}{m}$$

with

$$\mathcal{Z}_m(r,\beta) = \sum_{x_\gamma \in fix\, P^m} \exp\Big(-\beta \sum_{k=0}^{m-1} r(P^k x_\gamma)\Big). \tag{15}$$

However, this function has the same structure as a partition function (2) in statistical mechanics. Since the sum (2) can be computed by means of $\mathcal{L}_\beta$ in (4), why not construct a transfer operator to compute (15)? Indeed, this operator is nothing but the PF operator for the Poincaré map on the expanding manifold (13), endowed with the reciprocal temperature β as in (10).

Surprisingly, this parameter β is related to quantum energies for some classically chaotic systems.

4. The transfer operator in quantum chaos

What is 'Quantum Chaos'? Around 1990 this modern terminology was still under debate. If quantum chaos is defined as sensitive dependence on initial conditions like in classical chaotic dynamics, then we will find that there is no chaos in quantum mechanics, because the Schrödinger equation is linear and the corresponding evolution of the wave function (for bound systems) is quasi-periodic. In contrast to this definition, we nowadays understand by 'quantum chaos' the quantum mechanical behavior of classically chaotic systems.

The main task of the study of quantum chaos is not to invent a new mechanics to replace the Schrödinger equation, or to device an easier technique to calculate its solutions. (To compute quantum energies by means of the methods developed in quantum chaos like the trace formula or the transfer operator is even more complicated than to let a computer solve the Schrödinger equation numerically.) Instead, following Einstein's intention mentioned in paragraph 1, the study of quantum chaos is to find out if there are some connections between classically chaotic dynamical systems and their corresponding quantum systems, above all regarding the quantum energies and the quantum states.

4.1 Classical and quantum properties for integrable and chaotic systems

Prime examples in the study of quantum chaos are chaotic billiards and hyperbolic surfaces of negative curvature. Let us consider first two billiards: The first one is the circle billiard in fig. 6 (a), namely a free particle sliding ballistically in a circle. The dynamics of this system is not chaotic because an initial deviation will not expand too fast. Now if we cut the circles into two semi-circles and combine these two separated semi-circles by two straight lines, then we get Bunimovich's stadium billiard [Bun74], [Bun79] as in fig. 6 (b). The dynamics of a freely sliding particle in this billiard is chaotic because a slight initial deviation can induce a totally different orbit.

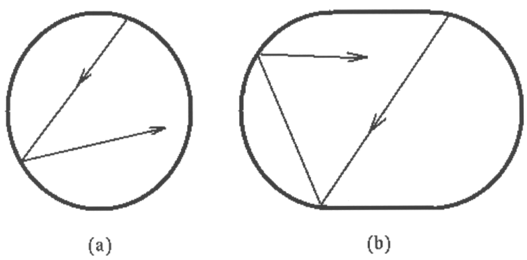


Figure 6: (a) circle billiard (b) stadium billiard

Now if the billiards mentioned are so small that they should be described by quantum mechanics, then they show different properties in quantum energies and quantum states. To give an impression, let us take a look at the probability density of finding the free particle in the billiards, calculated from the n-th quantum eigenstate in fig. 7. (Only eigenstates with odd-odd symmetry in the whole stadium billiard are considered. Due to up-down and right-left symmetry we just show one-fourth of the whole billiard). An extreme difference can be found between these two billiards. The densities of the circle billiard are quite regular and symmetric. In contrast, the densities for the stadium billiard

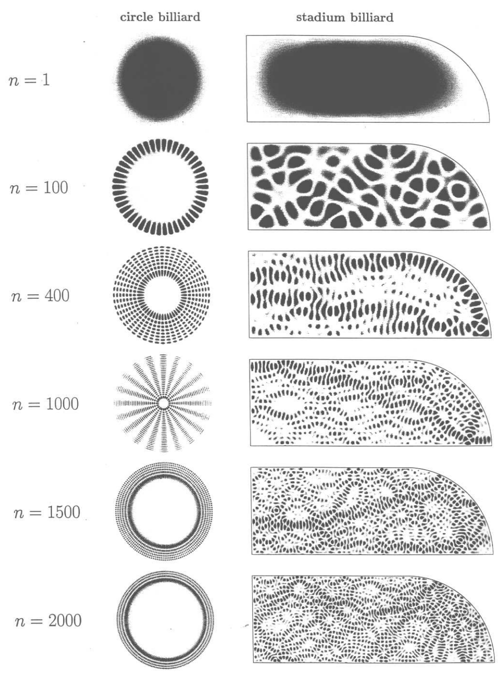


Figure 7: Probability densities [B$\ddot{9}$8]

look irregular. The quantum energies of these two billiards have also different properties [B98].

The second class of systems which appears very often in the study of quantum chaos are hyperbolic surfaces (surfaces with constant negative curvatures). We consider first a free particle sliding on a surface with a constant positive curvature, i.e., a sphere as in fig. 8 (a). The dynamics is not chaotic because it doesn't depend on the initial point sensitively. If two particles depart at the same time from the north pole with the same velocity but a slight deviation in the direction, then they will arrive at the south pole at the same time later. The deviation of these two particles is predictable at any time. Quantum mechanics on such sphere is well known. The quantum energies and quantum states are analytically solvable.

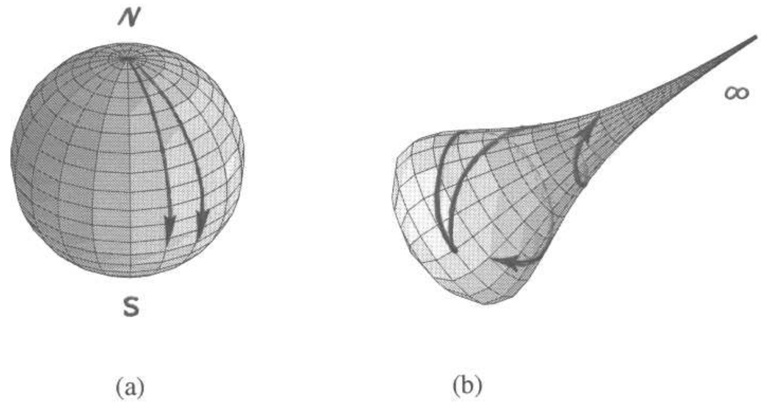


Figure 8: Dynamics of a free particle sliding on (a) a sphere and (b) the modular surface

As our next example we consider a free particle sliding on a hyperbolic surface, which is a Riemannian surface with constant negative Gaussian curvature. A prominent example is the so-called modular surface in fig. 8 (b) (definition see next paragraph) whose curvature is constant and every point on this surface is locally a saddle point. The dynamics of a sliding free particle on such surface is sensitive to initial conditions and unpredictable and therefore chaotic. The quantum energies and quantum states (Maass wave forms) are not known analytically.

In general the systems like circle billiards and spheres are called integrable systems, although the dynamics of billiards are not smooth in phase space and violates the conventional definition of the integrability of the classical smooth Hamiltonian flow. In contrast to such systems we call the stadium billiard and the modular surface chaotic.

4.2 Quantum chaos on the modular surface

Now let us illustrate quantum chaos on the modular surface. The modular surface is a quotient space $\mathbb{H}/PSL(2,\mathbb{Z})$, where $\mathbb{H}$ is the Poincaré upper half-plane, i.e., the upper complex half-plane $\{z = x + iy \,|\, x,\, y \in \mathbb{R},\ y > 0\}$ with the Poincaré metric (arc length) $d\,s^2 = \frac{d\,x^2 + d\,y^2}{y^2}$. The action of an element g of the modular group

$$PSL(2,\mathbb{Z}) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\, \det g = 1 \right\} / \{\pm 1\}$$

on $\mathbb{H}$ is realized by the Möbius transformation $gz = \frac{az+b}{cz+d}$, where z is a point in $\mathbb{H}$. The upper half-plane in fig. 9 can be tessellated by a fundamental region (shadowed region in fig. 9), which is a representative of the quotient space $\mathbb{H}/PSL(2,\mathbb{Z})$, and is topologically like a sphere in fig. 8 (b) with a cusp at infinity. It is analog to the construction of a torus as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$. For more details on this, see [Ter85].

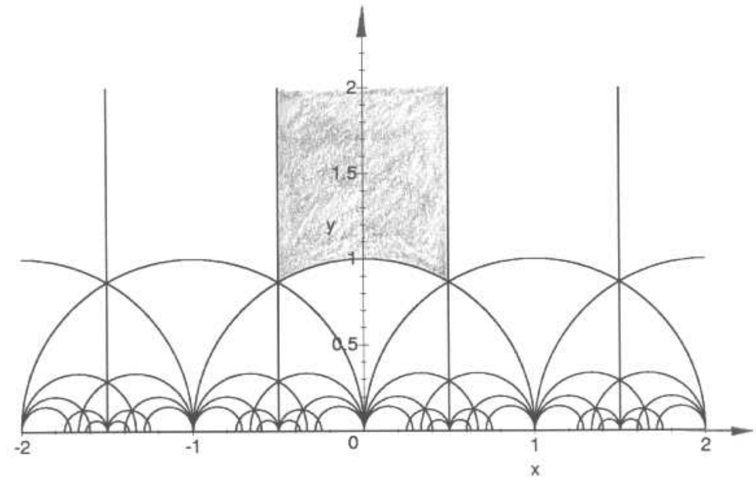


Figure 9: Poincaré upper-half plane tessellated by the fundamental region (shadowed region) by means of the elements of $PSL(2,\mathbb{Z})$

To discuss quantum chaos, as mentioned already, one should keep in mind that the goal is to find classical-quantum correspondences. Let us consider first the classical dynamics on the modular surface. The phase space of a free particle on the two-dimensional modular surface is of four dimension. Since the particle is free and the energy is conserved, the phase space is three-dimensional. After choosing a special Poincaré section [Ser85] we get a Poincaré map $P : I \times I \times \{\pm 1\} \to I \times I \times \{\pm 1\}$ with

$$P\,(x_1, x_2, e) = \left(T_G\, x_1, \frac{1}{[\frac{1}{x_1}] + x_2}, -e \right),$$

where T_G is the Gauss transformation in (8), $e = \pm 1$ and $[x]$ denotes the integer part of x. As discussed in paragraph 3.2 we are only interested in the expanding direction of the Poincaré map $P_u : (x_1, e) \to (T_G x_1, -e)$. The corresponding transfer operator reads

$$\tilde{\mathcal{L}}_\beta \underline{f}(x) = \begin{pmatrix} 0 & \mathcal{L}_\beta \\ \mathcal{L}_\beta & 0 \end{pmatrix} \underline{f}(x), \tag{16}$$

where

$$\mathcal{L}_\beta f(x) = \sum_{n=1}^{\infty} (\frac{1}{x+n})^{2\beta} f(\frac{1}{x+n}) \quad \text{and} \quad \underline{f}(x) = \begin{pmatrix} f_+(x) \\ f_-(x) \end{pmatrix}.$$

The dynamical zeta function for this system is explicitly known and can be represented by the transfer operator as

$$\zeta_{RS}(\beta) = \frac{\det(1 - \tilde{\mathcal{L}}_{\beta+1})}{\det(1 - \tilde{\mathcal{L}}_\beta)}. \tag{17}$$

So far we have collected some dynamical informations for this system. Next we want to discuss the quantum properties of a free particle on this surface. The Schrödinger operator of a free particle on this surface is nothing but the Laplace-Beltrami operator

$$-\triangle_{LB} = -y^2 (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}). \tag{18}$$

The first question is if the classical dynamics reveals some information about the quantum energies of this system, namely the eigenvalues of $-\triangle_{LB}$. In 1956, to study Riemann's hypothesis and without motivation from quantum chaos, A. Selberg devised his famous trace formula which determines the discrete spectrum of the operator $-\triangle_{LB}$ on the modular surface [Sel56]. Shortly after this discovery S. Smale and Y. Sinai recognized that this formula can be interpreted as a connection between the discrete spectrum of the operator $-\triangle_{LB}$ and the sum over all primitive periodic orbits of the classical dynamics. It's a key step. In mathematics it gives a connection between the fields of number theory, spectral theory and dynamical systems. In physics it relates quantum mechanics and classical mechanics, since the Laplace-Beltrami operator $-\triangle_{LB}$ describes the quantum world and the periodic orbits come from the classical world. This yields the first classical-quantum correspondence.

The content of the trace formula can be represented by another device of Selberg, today known as Selberg's zeta function [Sel56]

$$Z_S(\beta) = \prod_\gamma \prod_{k=0}^{\infty} \left(1 - e^{-(\beta+k)\, l(\gamma)}\right). \tag{19}$$

This function has so-called trivial zeros for negative integers β and non-trivial zeros on the line $\Re\beta = \frac{1}{2}$ and $\Re\beta = \frac{1}{4}$. Suppose some β with $\Re\beta = \frac{1}{2}$ is a zero point of $Z_S(\beta)$, then, due to the theory of Selberg, $\beta(1-\beta)$ is an eigenvalue of the Laplace-Beltrami operator, and vice versa. This yields the second classical-quantum correspondence.

But from the definition of the Selberg zeta function (19) we know that it is a special case of the Ruelle-Smale dynamical zeta function in (14), which is representable in terms of the transfer operator (17) for the modular surface. In this way we get the relation [May91]

$$Z_S(\beta) = \det(1 - \tilde{\mathcal{L}}_\beta).$$

If the transfer operator $\tilde{\mathcal{L}}_\beta$ at some point β on the line $\Re\beta = \frac{1}{2}$ has the eigenvalue 1, then the determinant $\det(1 - \tilde{\mathcal{L}}_\beta)$ and $Z_S(\beta)$ vanish. Consequently, due to the properties of $Z_S(\beta)$ mentioned above, $\beta(1-\beta)$ is an eigenvalue of the Laplace-Beltrami operator, and vice versa. That is the transfer operator $\tilde{\mathcal{L}}_\beta$, defined from classical dynamics, is also capable to determine the corresponding quantum energies and gives an approach different from the traditional one via Selberg's trace formula and the zeta function.

More strikingly, this operator gives even information on the quantum states which the trace formula and the zeta function are not able to determine. This discovery was achieved by J. Lewis [Lew97], without motivation from quantum chaos, in his theory: Under some slight restriction, the eigenfunctions $u_\beta(x, y)$ of the Laplace-Beltrami operator $-\triangle_{LB}$ for eigenvalue $\beta(1-\beta)$ are in one-to-one correspondence to the holomorphic solutions $f_\beta(z)$ of the functional equation

$$\lambda_\beta \, [f_\beta(z) - f_\beta(z+1)] - (z+1)^{-2\beta} \, f_\beta(\frac{1}{z+1}) = 0 \,, \tag{20}$$

where $\lambda_\beta = \pm 1$ and $f_\beta(0) = 0$. The functions $u_\beta(x, y)$ and $f_\beta(z)$ can be translated explicitly into each other by the Lewis transformation.

Shortly after this discovery, D. Zagier recognized [LZ96] that the functional equation (20) is nothing but the functional equation for the eigenfunctions $f_\beta(z)$ of Mayer's transfer operator $\mathcal{L}_\beta$ in (16). By means of the Lewis transformation the eigenfunctions of the Laplace-Beltrami operator $-\triangle_{LB}$ and the eigenfunctions of the transfer operator $\mathcal{L}_\beta$ are then explicitly related. Physically, it means that the eigenfunctions of the classical object $\mathcal{L}_\beta$ determine the quantum mechanical probability density of the free particle on the modular surface.

4.3 Answers to Einstein's question

We have seen a remarkable classical-quantum connection between the quantum energies, the quantum states and the classical dynamics by means of the transfer operator in the special case of modular surface. Now a natural question comes up: is this correspondence a special case or an universal property? If it is neither of both the next question is what kind of physical systems respectively what kinds of dynamical systems, e.g., ergodic, mixing, exact, etc. have such a connection?

Two recent results show that this classical-quantum connection on the modular surface is not a special case. One of them is the generalization of the results on the modular surface to its covering spaces [Cha99]. The other progress is that E. Bogomolny has invented a formal transfer operator $\mathcal{L}_B(E)$ for general physical systems in the semiclassical limit $\hbar \to 0$ [Bog92]. His operator also reveals the quantum energies E and quantum states, although mathematically it is still not rigorous. In case of the modular surface, Bogomolny's transfer operator is just the transfer operator (16) [BC93].

The Selberg trace formula and zeta function on the modular surface have also been generalized to general physical systems, in the semiclassical limit. Together with the transfer operator, they serve as methods to understand quantum chaos, summarized in the following table.

modular surface	general physical systems	computable quantities
exact connection	connection only in the semiclassical limit $\hbar \to 0$	
Selberg trace formula $\{E\} \leftrightarrow \{l(\gamma)\}$	Gutzwiller trace formula $\{E\} \overset{\hbar\to 0}{\longleftrightarrow} \{S(\gamma) = \int_\gamma p\, dq\}$	quantum energies
Selberg's zeta function $Z_S(\beta) = 0$ ($\Re(\beta) = \frac{1}{2}$)	generalized zeta function $\zeta(\beta) \overset{\hbar\to 0}{=} 0$	quantum energies
Ruelle transfer operator $\det(1 - \tilde{\mathcal{L}}_\beta) = 0$	Bogomolny transfer operator $\det(1 - \mathcal{L}_B(E)) \overset{\hbar\to 0}{=} 0$	quantum energies quantum states

Table 1 *Answers to Einstein's problem*

The Modular surface serves as a prime example in the study of quantum chaos where many relations between classical chaotic dynamics and its quantum mechanics can be exactly tested. It serves as an example to support Gutzwiller's trace formula [Gut90] and serves as a counter example for an exceptional class [Bol93], [BSS92] in Bohigas-Giannoni-Schmidt famous conjecture about the different energy statistics in integrable and chaotic systems [BGS84]. The modular surface for quantum chaos plays a role similar to other simple systems like the one-dimensional potential well, the harmonical oscillator and the hydrogen atom in the inception of the development of quantum mechanics.

5. Interaction between physics and mathematics

Today the quantum chaos people have often common workshops and conferences with mathematician from branches like number theory and dynamical systems. The dynamical zeta function is a typical example for the reason why the scientists from these three groups come together: The dynamical zeta function bears the information of a dynamical system, has the typical properties of number theoretical zeta functions and reveals the eigenvalues of the Schrödinger operator in quantum chaos.

In number theory a big problem is to understand the statistical distribution of the prime numbers. 1859 Riemann found that the precision of the distribution estimate for the number of prime numbers can be improved quite a lot, if his conjecture that all the so-called non-trivial zeros of the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

are located on the line $\Re s = \frac{1}{2}$, were true [Lan85]. This conjecture has been numerically tested up to the first 10^{20} zeros [Odl] without contradiction, but analytically it still is without proof. In the long struggle with this historical problem many techniques were invented. Today some of them serve as techniques to study quantum energy distributions in quantum chaos.

Conversely, the physical progress gives the mathematicians new ideas to solve problems from different aspects. One of them is the development of the transfer operator method from statistical mechanics. In the seventies it entered into the study of chaotic dynamical systems. Moreover, this operator is surmised to be the hermitian operator which D. Hilbert and G. Polya suggested to prove the Riemann's conjecture in the number theory. This idea was recently taken up again by A. Connes in his remarkable attack

[Con96] of the Riemann conjecture.

On the other hand the eigenfunctions of the transfer operator for the eigenvalue 1 on the modular surface have recently been recognized to be the period polynomials of the modular forms and the period functions of the Maass wave forms, which are well known in number theory, but up to now have not been known to have something to do with dynamical systems [CM99], [CM98], [LZ96].

One of the difficulties of the study on quantum chaos is that appropriate mathematical models are difficult to find. These models must be classically chaotic on the one hand, but on the other hand their quantum properties should not be too difficult to find. However these two wishes are not easy to fulfill simultaneously. The open questions in quantum chaos are often related to the open questions in dynamical systems and number theory. A prime example is the analogue of the distributions of quantum energies in quantum chaos and the distribution of zeros of the Riemann zeta function. Further improvement of the knowledge about number theory and dynamical systems will surely promote the ability to answer Einstein's question.

Interesting readings concerning this topic are [Bal97], [Cvi] and [Ste94]. Short introductions about D. Ruelle and A. Selberg can be found in [Gle87] and [Reg88].

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